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F-Approximately Regular Modules

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Abstract

We introduce in this paper the concept of an approximately pure submodule as a generalization of a pure submodule, that is defined by Anderson and Fuller. If every submodule of an R-module M is approximately pure, then M is called F-approximately regular. Further, many results about this concept are given.

Keywords: Approximately pure submodules, F-approximately regular modules, Pure submodules, regular modules.

المقاسات المنتظمة تقريبا

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الخلاصة

قدمنا في هذا البحث مفهوم المقاس الجزئي النقي تقريبا كأعمام للمقاس الجزئي النقي الذي عرفه أندرسون و فولر. اذا كان كل مقاس جزئي من المقاس M على R هو نقي تقريبا، فان M يسمى منتظم تقريبا من النمط – F. وأعطينا العديد من النتائج حول هذا المفهوم.

1. Introduction:

Let R be a commutative ring with identity and all modules are (left) unitary R- module. D. Fieldhouse [1] defined an R-module M to be regular if every submodule N of M is pure of M, where a submodule N of M is pure in M if $0 \longrightarrow L \otimes N \longrightarrow L \otimes M$ is exact for all R-modules L [2]. Anderson and Fuller in [3] called a submodule N of an R- module M is pure in M if $IM \cap N = IN$ for all ideal I of R. A submodule N of R- module M is called nearly pure in M if $IM \cap N = IN + J(M) \cap (IM \cap N)$, where $J(M)$ is the Jacobson radical of R [4]. In 2015, the concept of F - nearly regular submitted, where an R-module M is said to be F -nearly regular if every submodule N of M is nearly pure of M [5]. In (2019) Rafid M-Al-Shaibani and Nuhad S. Al-Mothafar gave the concept of regularity in another way, where an R-module M is called J-regular if every submodule of M is J-pure, where a submodule N of an R-module M is called J-pure if for each ideal I of R $IJ(M) \cap N = IN$ [6]. This idea leads to introduce the concept of approximately pure where a submodule N of M is called approximately pure in M if $IM \cap N = IN + Soc(M) \cap (IM \cap N)$, and $Soc(M)$ is the intersection of all essential submodule of M, where a non-zero submodule N of M is called essential (notational, $N \leq_e M$) if $N \cap W \neq 0$ for all submodules W of M [3]. By using the

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definition that is given in [5], we introduce another generalization of regular modules as follows: An R-module M is called an F-approximately regular module if every submodule N of M is approximately pure of M.

The main purpose of this work is to develop the properties of F-approximately regular modules and add some new definitions and results.

2. Approximately Pure Submodules.

Throughout this section, we introduce the concept of approximately pure submodules as a generalization of pure submodules, we study some of its basic properties.

Definition (2.1):

A submodule N of an R-module M is called an approximately pure submodule of M if $IM \cap N = IN + Soc(M) \cap (IM \cap N)$.

An ideal I of a ring R is called an approximately pure ideal of R if it is an approximately pure submodule of an R-module R.

Recall that a submodule N of an R-module M is called pure submodule of M if $IM \cap N = IN$ [3].

Remarks and Examples (2.2):

1. It is clear that every pure submodule is an approximately pure submodule, but the converse is not true in general.

For example **:** the submodule $\langle\bar{2}\rangle = \{\bar{0},\bar{2}\}$ of the Z-module Z_4 is an approximately pure submodule Since $I\,Z_4 \cap \langle\bar{2}\rangle = I\,\langle\bar{2}\rangle + Soc(Z_4) \cap (\,IZ_4 \cap \langle\bar{2}\rangle\,)$ for every ideal $I = nZ$ of Z where n is any positive integer and $Soc(Z_4) = \langle\bar{2}\rangle$ but it is not pure. Since if $I = 2Z$ then $I\,Z_4 \cap \{\bar{0},\bar{2}\} = 2\,Z_4 \cap \{\bar{0},\bar{2}\} = \{\bar{0},\bar{2}\}$, but $I\,\{\bar{0},\bar{2}\} = 2\,\{\bar{0},\bar{2}\} = \{\bar{0}\}$.

2. In any R-module M, the submodules M and $\{0\}$ are always approximately pure submodules in M .

3. The module Z as Z-module. The only approximately pure submodules are Z and $\{0\}$. To show this, for every submodule $n\,Z$ of Z, $n = n.1 \in \langle n\rangle\,Z \cap nZ$, but$n \notin \langle n\rangle(nZ) + Soc(Z) \cap (\,\langle n\rangle Z \cap nZ\,) = \langle n^2\rangle Z + Soc(Z) \cap (\,\langle n\rangle Z \cap nZ\,)$, where $Soc(Z) = \{0\}$.

4. Every nonzero cyclic submodule of the Z-module Q is not approximately pure submodule.

Proof:

Let N be a cyclic submodule of the Z-module , generated by an element $\frac{a}{b}$ where a and b are two nonzero element in Z . If we take an ideal $\langle n\rangle$ of Z where n is greater than one , then $\langle n\rangle.\frac{a}{b} = \langle\frac{na}{b}\rangle$.

Also , Q divisible because for any element $\frac{c}{d} \in Q$ we have $\frac{c}{d} = \frac{c}{nd}\,.n \in \langle n\rangle.Q$, thus $Q = \langle n\rangle.Q$. Therefore, $\langle n\rangle.Q \cap \langle\frac{a}{b}\rangle = \langle\frac{a}{b}\rangle$, but the other hand $\langle n\rangle.\langle\frac{a}{b}\rangle + Soc(Q) \cap (\langle n\rangle\,Q \cap \langle\frac{a}{b}\rangle) = \langle\frac{na}{b}\rangle$. Since $Soc(Q) = \{0\}$ implies that N is not approximately pure submodule of .

5. It is clear that every direct summand is an approximately pure submodule, since every direct summand is a pure submodule, hence by (1) is an approximately pure submodule, but the converse is not true.

For example **:** the submodule $\langle\bar{3}\rangle = \{\bar{0},\bar{3},\bar{6}\}$ of the Z-module Z_9 is an approximately pure submodule since $I\,Z_9 \cap \{\bar{0},\bar{3},\bar{6}\} = I\{\bar{0},\bar{3},\bar{6}\} + Soc(\,Z_9) \cap (I\,Z_9 \cap \langle\bar{3}\rangle)$ for each I of Z , where $Soc(\,Z_9) = \langle\bar{3}\rangle$ is not pure and hence not direct summand since if we take $I = 3Z$ of Z , then $3Z_9 \cap \langle\bar{3}\rangle = \langle\bar{3}\rangle = \{\bar{0},\bar{3},\bar{6}\}$ but $I\,\{\bar{0},\bar{3},\bar{6}\} = 3\,\{\bar{0},\bar{3},\bar{6}\} = \{\bar{0}\}$.

Proposition (2.3):

Let M be an R-module and let N be a pure submodule of M. If A is an approximately pure submodule of N, then A is approximately pure submodule in M.

Proof:

Let I be an ideal of a ring R. Since N is a pure submodule in M and A is an approximately pure submodule of N, then $IM \cap N = IN$ and $IN \cap A = IA + Soc(N) \cap (IN \cap A)$.

$IM \cap A = IM \cap (N \cap A)$

$= (IM \cap N) \cap A$

$= IN \cap A$ since N is pure in M

$= IA + Soc(N) \cap (IN \cap A)$

$\subseteq IA + (Soc(M) \cap N) \cap (IN \cap A)$ since $Soc(N) \subseteq Soc(M) \cap N$

$\subseteq IA + Soc(M) \cap (N \cap A) \cap IM$ since $IN \subseteq IM$

$= IA + Soc(M) \cap (IM \cap A)$ since $\cap A = A$.

Also, $IA + Soc(M) \cap (IM \cap A) \subseteq IM \cap A$

Thus, $IM \cap A = IA + Soc(M) \cap (IM \cap A)$

Proposition (2.4):

Let M be an R-module. If N is an approximately pure submodule of an R-module M and A is a submodule of M containing N such that $Soc(A) = Soc(M) \cap A$, then N is approximately pure submodule of A.

Proof:

Let I be an ideal of a ring R. Since N is an approximately pure submodule in M, then $IM \cap N = IN + Soc(M) \cap (IM \cap N)$ and $N \subseteq A \subseteq M$ then:

$IA \cap N = (IA \cap IM) \cap N$

$= IA \cap (IM \cap N)$

$= IA \cap [IN + Soc(M) \cap (IM \cap N)]$

$= IN + Soc(M) \cap (IA \cap N)$

$= IN + Soc(M) \cap IA \cap (A \cap N)$

$= IN + (Soc(M) \cap A) \cap (IA \cap N)$

$= IA + Soc(A) \cap (IA \cap N)$ since $(A) = Soc(M) \cap A$.

Proposition (2.5):

Let M be an R-module and N be an approximately pure submodule of M. If H is an approximately pure submodule of N, then H is an approximately pure submodule of M.

Proof:

Let I be an ideal of a ring R. Since N is approximately pure submodule of M and H is approximately pure submodule of N, then $IM \cap N = IN + Soc(M) \cap (IM \cap N)$ and $IN \cap H = IH + Soc(N) \cap (IN \cap H)$. Since $IM \cap H \subseteq IM \cap N = IN + Soc(M) \cap (IM \cap N)$, hence

$IM \cap H \subseteq [\, IN + Soc(M) \cap (IM \cap N)] \cap H$

Thus $= H \cap IN + Soc(M) \cap (IM \cap N \cap H)$

$= IH + Soc(N) \cap (IN \cap H) + Soc(M) \cap (IM \cap N \cap H)$ since H is approximately pure in N

$\subseteq IH + Soc(M) \cap (IM \cap H)$ Since $IH + Soc(M) \cap (IM \cap H) \subseteq IM \cap H$

Then $IM \cap H = IH + Soc(M) \cap (IM \cap H)$.

Proposition (2.6):

Let M be an R-module and N be an approximately pure submodule of M. If H is a submodule of N and $H \subseteq Soc(M)$, then $\frac{N}{H}$ is an approximately pure submodule of $\frac{M}{H}$.

Proof:

Let I be an ideal of a ring R. Since N is an approximately pure submodule in , then $IM \cap N = IN + Soc(M) \cap (IM \cap N)$ so

$$I\left(\frac{M}{H}\right) \cap \frac{N}{H} = \frac{IM+H}{H} \cap \frac{N}{H}$$
$$= \frac{(IM\cap N)+H}{H}$$
$$= \frac{IN+Soc(M)\cap(IM\cap N)+H}{H}$$
$$= \frac{IN+H}{H} + \frac{Soc(M)\cap(IM\cap N)}{H}$$
$$= I\left(\frac{N}{H}\right) + \frac{(Soc(M)+H)\cap(IM\cap N)}{H} \qquad \text{since } H \subseteq Soc(M)$$
$$= I\left(\frac{N}{H}\right) + \frac{Soc(M)+H}{H} \cap \frac{IM\cap N}{H}$$
$$= I\left(\frac{N}{H}\right) + \frac{Soc(M)}{H} \cap \left(\frac{IM}{H} \cap \frac{N}{H}\right) \qquad \text{since } Soc(M) = Soc(M) + H$$
$$\subseteq I\left(\frac{N}{H}\right) + Soc\left(\frac{M}{H}\right) \cap \left(I\left(\frac{M}{H}\right) \cap \frac{N}{H}\right)$$

by [Let $H \subseteq M$ and $\pi: M \to \frac{M}{H}$, π(Soc(M)) $\subseteq Soc(\frac{M}{H})$, then $\frac{Soc(M)}{H} \subseteq Soc\left(\frac{M}{H}\right)$] [3, proposition. 9.8, p. 119]

Since $I\left(\frac{N}{H}\right) + Soc(\frac{M}{H}) \cap (I(\frac{M}{H}) \cap \frac{N}{H}) \subseteq I(\frac{M}{H}) \cap \frac{N}{H}$

Thus $I\left(\frac{M}{H}\right) \cap \frac{N}{H} = I\left(\frac{N}{H}\right) + Soc(\frac{M}{H}) \cap (I(\frac{M}{H}) \cap \frac{N}{M})$.

Proposition (2.7);

Let M be an R-module and N is approximately pure submodule of M. If K is a submodule of M containing, then N is approximately pure of K.

Proof:

Let I be an ideal of a ring R. Since N is an approximately pure in M, then $IM \cap N = IN + Soc(M) \cap (IM \cap N)$. But $K \subseteq M$, therefore $IK \cap N \subseteq IM \cap N = IN + Soc(M) \cap (IM \cap N)$, hence $IK \cap N \subseteq [IN + Soc(M) \cap (IM \cap N)] \cap IK$

$$= IN + [Soc(M) \cap (IM \cap N \cap K) \cap IK) \qquad \text{(By modular law)}$$
$$= IN + [(Soc(M) \cap K) \cap (IM \cap IK \cap N)]$$
$$= IN + Soc(K) \cap (IK \cap N) \qquad \text{By [7, Exc 9, Page 29]}$$

Implies $IK \cap N \subseteq IN + Soc(K) \cap (IK \cap N)$

Since $IN + Soc(K) \cap (IK \cap N) \subseteq IK \cap N$

Then $IK \cap N = IN + Soc(K) \cap (IK \cap N)$.

Thus N is an approximately pure submodule of K.

Proposition (2.8):

If N is approximately pure in M and K is a submodule of M containing N such that $Soc(M) = Soc(K)$, then N is approximately pure in K.

Proof:

Since $IK \cap N \subseteq [IN + Soc(M) \cap (IM \cap N)] \cap IK$ By the same proof of proposition (2.7)

then $= IN + Soc(K) \cap (IK \cap N)$
But $IN + Soc(K) \cap (IK \cap N) \subseteq IK \cap N$
Thus $IK \cap N = IN + Soc(K) \cap (IK \cap N)$

3. F-approximately Regular Modules.

In this section, we first define approximately regular modules and study some of their properties. Next we consider some conditions to characterize approximately regular modules.

Definition (3.1):

An R-module M is called an F-approximately regular module if every submodule N of M is approximately pure of M.

Remarks and Examples (3.2):

1. It is clear that every F-regular module is an F-approximately regular module, however, the converse is not true in general, the following example explain that:
The Z-module Z_4 is an F-approximately regular module since every submodule of Z_4 is an approximately pure but not F-regular since the submodule $\langle\bar{2}\rangle = \{\bar{0}, \bar{2}\}$ of the Z-module Z_4 is not pure, see Remarks and Examples (2.2) (1).

2. The Z-module Z and Q are not F-approximately regular modules, see Remarks and Examples (2.2) (3) and (4).

3. The Z-module Z_9 is F-approximately regular since every submodule of Z_9 is approximately pure , but Z_9 is not F-regular the submodule since $\langle\bar{3}\rangle = \{\bar{0}, \bar{3}, \bar{6}\}$ is not pure , see remarks and examples (2.2)(5).

4. It is clear to check that every submodule of the Z-module Z_{12} is an approximately pure , hence Z_{12} is F-approximately regular but it is not F-regular since the submodule generated by $\langle\bar{2}\rangle$ is not pure since $6.\,Z_{12} \cap \langle\bar{2}\rangle = \langle\bar{6}\rangle \cap \langle\bar{2}\rangle = \langle\bar{6}\rangle$ but $6.\langle\bar{2}\rangle = \langle\bar{0}\rangle$.

5. The Z-module Z_8 is F-approximately regular module since every submodule of Z_8 is an approximately pure but not F-regular since the submodule $\langle\bar{2}\rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is not pure since $2.Z_8 \cap \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} = \langle\bar{2}\rangle$ but $2.\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} = \{\bar{0}, \bar{4}\}$.

The following theorem shows that the cyclic approximately pure submodule is enough to make the module approximately regular.

Theorem (3.3):

Let M be an R-module. Then M is an approximately regular if and only if every cyclic submodule of M is an approximately pure submodule.

Proof:

If M is approximately regular module, then by Definition (3.1), every submodule of M is approximately pure. Rather, every cyclic submodule of M is approximately pure.
Conversely, assume that every cyclic submodule of M is approximately pure. Let N be a submodule of M and I be an ideal of a ring R.
Let $x \in IM \cap N$, then $x \in IM$ and $x \in N$ implies $\langle x\rangle \subseteq N$. Therefore $x \in IM \cap \langle x\rangle = I\langle x\rangle + Soc(M) \cap (IM \cap \langle x\rangle)$, then $x \in I\langle x\rangle + Soc(M) \cap (IM \cap \langle x\rangle) \subseteq \text{IN} + Soc(M) \cap (\text{IM} \cap N)$. But $IN + Soc(M) \cap (IM \cap N) \subseteq IM \cap N$. Thus $IM \cap N = IN + Soc(M) \cap (IM \cap N)$ which implies that N is approximately pure in M.

Corollary (3.4):

Let M be an R-module. Then the following statements are equivalent:

(1) M is approximately regular R-module.
(2) Every cyclic submodule of M is approximately pure.
(3) Every finitely generated submodule of M is approximately pure.

Proof:
(1) $\Longrightarrow$ **(2)** It follows by Definition (3.1).
(2) $\Longrightarrow$ **(1)** by Theorem (3.3)
(1) $\Longrightarrow$ **(3)** It is follows by Definition (3.1).
(3) $\Longrightarrow$ **(2)** It is clear.

Proposition (3.5):

Let M be an R-module. Then M is an approximately regular if and only if $\frac{M}{N}$ is approximately regular for every submodule N of M.

Proof:

Suppose that M is an approximately regular R-module. Let N be a submodule of M and K be any submodule of M containing N. Since M is approximately regular then K is approximately pure in M. Thus by proposition (2.6) $\frac{K}{N}$ is approximately pure in M. Therefore, $\frac{M}{N}$ is approximately regular.
The converse is easily proved by taking $N = 0$.

Proposition (3.6):

Let M and M'be an R-modules and $f: M \to M'$ be an R-epimorphism. If M is F-approximately regular module, then M' is F-approximately regular.

Proof:

Since $f: M \to M'$ is an R-epimorphism. If M is F-approximately regular module. Then $\frac{M}{kerf}$ is F-approximately regular module by Proposition(3.5).
But $\frac{M}{kerf} \cong M'$ by the first isomorphism theorem. Thus, M' is F-approximately regular module.

Proposition (3.7):

If M is an approximately regular R-module and N is a submodule of M,Then N is an approximately regular module.

Proof:
Let K be a submodule in N and I is an ideal of R, then

$IN \cap K = (IM \cap IN) \cap K$
$= (IM \cap K) \cap IN$
$= [IK + Soc(M) \cap (IM \cap K)] \cap I$ since K is approximately pure in M
$= IK + Soc(M) \cap (IN \cap K)$ by Modular law
$= IK + Soc(M) \cap (IN \cap (N \cap K)$
$= IK + (Soc(M) \cap N) \cap (IN \cap K)$

$= IK + Soc(N) \cap (IN \cap K)$ since $Soc(N) = Soc(M) \cap N$ by [7, Exc 9, Page 29]

Thus K is approximately pure in N. That implies that N is approximately regular.

Recall that an R-module M is called F- regular , if for every $x \in M, r \in R,\ rx = rtrx$ for some $t \in R$, [5].

Proposition (3.8):

Let M be a module over principal ideal ring R. If for every $x \in M$ and $r \in R$, $rx - rtrx \in Soc(M)$ for some $t \in R$ implies M is F-approximately regular module.

Proof:

Let N be a submodule of an R-module M and I be an ideal of a ring R. First we prove $rM \cap N = rN + Soc(M) \cap (rM \cap N)$ for some $t \in R$. Let $x \in rM \cap N$ implies $x \in rM$, $x \in N$. Thus $x = rm$; $x \in M,\ r \in R,$ then by assumption $rm - rtrm \in Soc(M)$ for some $t \in R,$ then $rm - rtrm = s$, where $s \in Soc(M)$, implies $rm = rtrm + s$, then $x = rm = rtrm + s \in (rN + Soc(M)) \cap (rM \cap N)$ but $rN \subseteq rM \cap N,$ then $x \in rN + Soc(M) \cap (rM \cap N)$ (by Modular law) , then $rM \cap N \subseteq rN + Soc(M) \cap (rM \cap N)$. Since $rN + Soc(M)) \cap (rM \cap N) \subseteq rM \cap N$ then $rM \cap N = rN + Soc(M) \cap (rM \cap N)$. But R is principal ideal ring , then $IM \cap N = IN + Soc(M) \cap (IM \cap N)$, then N is an approximately pure submodule of M, and hence M is F- approximately regular.

Proposition (3.9):

Let M be a cyclic R-module. If for every element x of M and every element r of , $rx - rtrx \in Soc(M)$ for some $t \in R$, implies M is an F-approximately regular module.

Proof:

Let $M = Rm$ be a cyclic module for some $\in M$. Let N be a submodule of M and I is an ideal of a ring R. Let $x \in IM \cap N$ then $x \in IM$ and $x \in N$. Since M is cyclic, then $x = rm$; $\in M$, $r \in I \subseteq R$ then $rm - rtrm \in Soc(M)$ for some $t \in R$ by hypothesis , then $rm - rtrm = s$, where $s \in Soc(M)$, implies $rm = rtrm + s$. Thus $x = rm = rtrm + s \in (IN + Soc(M)) \cap (IM \cap N)$ but $IN \subseteq IM \cap N$, then $x \in IN + Soc(M) \cap (IM \cap N)$ (by Modular law) , then $IM \cap N \subseteq IN + Soc(M) \cap (IM \cap N)$. Since $IN + Soc(M) \cap (IM \cap N) \subseteq IM \cap N$. Thus $IM \cap N = IN + Soc(M) \cap (IM \cap N)$ implies that N is approximately puer submodule of , and hence M is F- approximately regular.

Proposition (3.10):

Let M be an F-approximately regular R-module, then for every $x \in M$ and $r \in R$, $rx - rtrx \in Soc(M)$ for some $t \in R$.

Proof:

Let x be an element of M and r is an element of R. Since $rx \in M$ and $rx \in \langle rx \rangle$ implies that $\in rM \cap \langle rx \rangle$. But M is F-approximately regular , then $rM \cap \langle rx \rangle = r\langle rx \rangle + Soc(M) \cap (rM \cap \langle rx \rangle)$. Thus $x \in r\langle rx \rangle + Soc(M) \cap (rM \cap \langle rx \rangle)$ implies that $rx = rtrx + s$ for some $\in R$; $s \in Soc(M) \cap (rM \cap \langle rx \rangle)$. Therefore $-rtrx \in Soc(M)$.

Corollary (3.11):

Let R be an approximately regular ring, then for every $r \in R$, $r - rtr \in Soc(R)$ for some $t \in R$. The converse is true if R is principal ideal ring.

Proof:

It follows from proposition (3.10) and (3.8).

Proposition (3.12):

Let $M_1$and M_2 be two R-modules. If $M = M_1 \oplus M_2$ is F-approximately regular module, then M_1 and M_2 are also F-approximately regular R-modules. The converse is true provided $\text{ann}(M_1) + \text{ann}(M_2) = R$, or if R is a principal ideal ring.

Proof:

Assume that $\text{ann}(M_1) + \text{ann}(M_2) = R$,and $M = M_1 \oplus M_2$ is an F-approximately regular R-module. Let $P_i{:}M \to M_i$ be then natural projection map of M onto M_i for each $i = 1,2$. Since P_i is an R-epimorphism, then by corollary (3.6), the epimorphic image of M is F-approximately regular, implying that M_i is F-approximately regular. Conversely, assume $M_1$and M_2 are F-approximately regular R-modules and $M = M_1 \oplus M_2$. Let N be a submodule of $M = M_1 \oplus M_2$. Since $\text{ann}(M_1) + \text{ann}(M_2) = R$, then by the same way of the proof [8, Proposition 4.2, Ch. 1], $N = N_1 \oplus N_2$ where N_1 is a submodule of M_1 and N_2 is a submodule of M_2. Let I be an ideal R. To prove $IM \cap N = IN + Soc(M) \cap (IM \cap N)$. Since $IM_1 \cap N_1 = IN_1 + Soc(M_1) \cap (IM_1 \cap N_1)$ and $IM_2 \cap N_2 = IN_2 + Soc(M_2) \cap (IM_2 \cap N_2)$ implies that $(IM_1 \cap N_1) \oplus (IM_2 \cap N_2) = [\, IN_1 + Soc(M_1) \cap (IM_1 \cap N_1)] \oplus [\, IN_2 + Soc(M_2) \cap (IM_2 \cap N_2)]$. Then $I(M_1 \cap M_2) \oplus (N_1 \cap N_2) = I(N_1 \oplus N_2) + Soc(M_1 \oplus M_2) \cap [\, I(M_1 \oplus M_2) \cap (N_1 \oplus N_2)$, therefore M is F-approximately regular module.

Case two:

Assume that R is a principal ideal ring. Let $(a_1, a_2) \in M = M_1 \oplus M_2$ then for each $r \in R$ there exists $t_1, t_2 \in R$ such that $ra_1 - rt_1 ra_1 \in Soc(M_1)$ and $ra_2 - rt_2 ra_2 \in Soc(M_2)$. If we choose $t = t_1 + t_2 - rt_1 t_2$ then it is easily to see that $ra_1 - rt_1 ra_1 \in Soc(M_1)$ and $ra_2 - rt_2 ra_2 \in Soc(M_2)$. Therefore $r(a_1, a_2) - rtr(a_1, a_2) \in Soc(M_1 \oplus M_2)$ since $(M_1 \oplus M_2) = Soc(M_1) \oplus Soc(M_2)$. Hence by Proposition (3.8) $M = M_1 \oplus M_2$ is F-approximately regular module.

Conclusions:

In this work, a generalization of a pure submodule has been introduced which is called an approximately pure submodule. We also show that If every submodule of an R-module M is approximately pure, then M is called F-approximately regular. Moreover, many results and properties of this concept are given and discussed.

References:

[1] D. J. Fieldhouse, "Pure Theories, " *Math. Ann.*, vol.184, pp. 1-18, 1970.

[2] P. M. Cohn, "On Free Product of Associative Rings, II," *Math. Z.*, vol.71, pp.541-560, 1959.

[3] E. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer – Verlage, New York, 1992.

[4] A. G. Naoum, and N.S. Al-Mothafar, “Nearly Regular Rings and Nearly Regular Modules,” *Mu’tah Journal for Research and Studies*,vol. 9, no.6, pp. 13-24, 1994.

[5] N.S. Al-Mothafar and M. B. Al-Hakeem, “Nearly Semiprime Submodules”, *Iraqi Journal of Science*, vol.56, no.4B, pp. 3210-3214, 2015.

[6] R.M.Al-Shaibani and N.S.Al-Mothafar, "Properties of J-Regular Modules", *Iraqi Journal of Science*, vol.3, no. 60, pp. 647-652, 2019.

[7] K.R.Goodearl, *Ring Theory, Nonsingular Ring and Module* , Marcel. Dekker. New York, 1976.

[8] M.S.Abbas, "On Fully Stable Modules", Ph.D. Thesis, University of Baghdad, Baghdad,1991.