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Generalized Radical Lifting Modules

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Abstract
In this paper we introduce G-Rad-lifting module as aproper generalization of lifting module, some properties of this type of modules are investigated. We prove that if M is G-Rad- lifting and $M = M_1 \oplus M_2$, then M_1, and M_2 are G-Rad- lifting, hence we Conclude the direct summand of G-Rad- lifting is also G-Rad- lifting. Also we prove that if M is a duo module with $M = M_1 \oplus M_2$ and M_1, M_2 are G- Rad- lifting then M is G-Rad- lifting.

Keywords: G-Rad-Lifting, Lifting.

مقاسات الرفع المعممة من النمط Radical

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الخلاصة
في هذا البحث سوف ندرس مقاسات الرفع المعممة من النمط Radical كتعميم لمقاسات الرفع. ونبرهن بعض خواص هذا النوع من المقاسات حيث سنبرهن انه اذا كان M مقاس رفع من النمط G−Rad ، وكان $M = M_1 \oplus M_2$ فإن M_1, M_2 مقاسات رفع من النمط G−Rad ، وعليه يكون كل جمع مباشر من مقاس من النمط G−Rad هو ايضاً مقاس من النمط G−Rad ، ايضاً سوف نبرهن بشروط اضافية انه اذا كان $M = M_1 \oplus M_2$ وكان M_1, M_2 مقاسا رفع من النمط G−Rad فان M هو مقاس رفع من النمط G−Rad.

1. Introduction

Let R be an associative ring with identity and M be a left R-module A submodule N of M is called small in M denoted by N$\ll$ M, if for every submodule L of M with M= N+ L implies L= M [1].

Asubmodule N of an R – module M is Called Supplement of L in M if and only if M=N+L and $N \cap L \ll L$. and a module M is Called supplemented if every submodule of M has a supplement in M. [2]. Asubmodule N of an R – module M is Called weakly Supplement of L in M if and only if M=N+L and $N \cap L \ll M$, and a module M is Called weakly supplemented if every submodule of M has a weakly supplement in M [2].

The intersection of all maximal submodules of M is called the jacobson Radical of M and denoted by Rad (M). Equivalently Rad (M) is the sum of all small submodules of M. If M has no maximal submodules then Rad (M) = M. It is clear that for any submodule N of M $Rad(N) \leq$ Rad (M) $\cap N$, but if N is a supplement Submodule of M then Rad(N)=Rad(M) $\cap$ N. Let N be any submodule of M. If M = N+k where $k \leq M$ and $N \cap k \leq Rad(M)$. Then K is called a weakly generalized Rad – Supplement of N in M [3], and M denoted by(w.g.s)Since Rad (M) is the sum of all small submodules

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of M, every supplement submodules is a w. generalized Rad – supplement in M. And a module M is called w. generalized Rad – supplemented if every submodule of M has a w. generalized Rad – supplement in M. [4] and [5]. It is clear that every supplemented (weakly supplemented) is a weakly generalized Rad –supplemented module On the other hand, M is called ⊕ weakly generalized Rad – supplemented (briefly ⊕ w.g.s) if every submodules of M has a w. generalized Rad – supplement that is a direct summand of M. A module M is called lifting or satisfy (D1) if for any submodule N of M there exists a direct summand K of M, and $K \leq N$ Such that $M = K \oplus \acute{K}$, $\acute{K} \leq M$ and $N \cap \acute{K} \ll M$ [6]. Eqnivalentely every submodule N of M can be written as N= A⊕S, where A is a direct summand of M and $S \ll M$. Recall that a module M has the property (p^*), if for every submodule N of M, there exists a direct summand K of M such that $k \leq N$ and $\frac{N}{K} \leq \text{Rad}\left(\frac{M}{K}\right)$[7]. It is known that every lifting module is satisfies the property (p^*).

A module M is called radical lifting if for any submodule N of M there exists a direct summand K of M, and $K \leq N$ Such that $M = K \oplus \acute{K}$, $\acute{K} \leq M$ and $N \cap \acute{K} \leq \text{Rad}(M)$.

In this paper we introduce generalized- Radical lifting modules as a generalization of lifting module and we study some properties of this type of modules and its relation with lifting modules, modules with the (p^*) property and some of other module.

2. G – Rad- lifting Module:

In this section we introduce a generalization of radical lifting module, and study some of the properties of this type of modules.

Definition 2.1: Let M be an R-module, and let N be any submodule of M, with Rad $(M) \leq N$. M is called a generalized- Rad- lifting (Briefly G- Rad- lifting), if there exist submodules $K, \acute{K}$ of $M, K \leq N$ such that $M = K \oplus \acute{K}$ and $N \cap \acute{K} \leq \text{Rad}(M)$.

Theorem 2.2: Let M be an R-module then M is G. Rad- lifting if and only if every submodule N of M with Rad $(M) \leq N$ can be written as $N = A \oplus S$, where A is a direct summand of M and $S \leq \text{Rad}(M)$.

Proof (⇒) : Let M be a G-Rad- lifting and let $N \leq M$, such that Rad $(M) \leq N$. Then there exists $K \leq N$, with $M = K \oplus \acute{K}$ and $N \cap \acute{K} \leq \text{Rad}(M)$. Now $N = N \cap M = N \cap (K \oplus \acute{K}) = K \oplus N \cap \acute{K}$ by modular law take $A = K$ and $S = N \cap \acute{K}$■.

(⇐): let N≤ M such that Rad (M) ≤ N, then can be written as N= A⊕ S where A is a direct summand of M i.e: M= A⊕ L, L ≤ M, A ≤ N and N= A ⊕ $L \cap N$= A⊕S. thus $L \cap N \leq \text{Rad}(M)$■.

It is clear that the semi-simple modules and lifting modules are G-Rad- lifting modules. But the conversely in general is not true. For example Q as Z- module is not semi-simple and not lifting but G- Rad- lifting. Since the only submodules of Q which are contains Rad (Q) is Q which is a direct summand. But if Rad (M) ≪ M, we have the following:

Lemma 2.3: Let M be a G-Rad- lifting module. If Rad (M) ≪ M. then M is lifting.

Proof: let M be a G-Rad- lifting module and N be any submodule of M, then Rad (M) ≤ Rad (M) + N, since M is G-Rad- lifting, then there exist submodule K of Rad (M) + N and M = K ⊕Ḱ, Ḱ ≤ M with Ḱ ∩ **(** N + Rad (M)**)** ≤ Rad (M) ≪ M. But Ḱ ∩ N ≤ Ḱ ∩ (N + Rad (M). Hence Ḱ ∩ N ≪ M. Now M = Ḱ + Rad (M) + N and Rad (M) ≪ M, then M = Ḱ + N, hence M = Ḱ + N = Ḱ ⊕ K. Then K ≤ N.■

It is clear that every G-Rad- lifting module is ⊕w. g. s. The next example show that a ⊕w. g. s. module doesn't need to be G-Rad-lifting.

Example 2.4: Let M = $\frac{Z}{2Z} \oplus \frac{Z}{8Z}$. (See [8], Example 3.1). Since Z- modules $\frac{Z}{2Z}$ and $\frac{Z}{8Z}$ are local. M is ⊕ w.g.s modules according ([9], theorem 2.1) Note that M is finitely generated . It follows that Rad $(M) \ll M$. If M is G-Rad-lifting module then M is lifting by lemma(2.3). This is a contradiction since, if M=$\frac{Z}{2Z} \oplus \frac{Z}{8Z}$ and N=$\{(\bar{0},\bar{0}),(\bar{1},\bar{2}),(\bar{0},\bar{4}),(\bar{1},\bar{6}),(0,\bar{2}),(\bar{0},\bar{6})\}$. Then the only direct summand of M Contained in N is $\{\bar{0},\bar{0}\}$. if M is lifting, then $N = A \oplus S$, where A is a direct summand of M and $S \ll M$ if A=0, then S=N. Therefore N is not small in M . [Sine $N + Z(\bar{1},\bar{1}) = M$]. Hence M is not lifting.

Recall that an R- Module M is called coatomic if every proper submodule is contained in maximal submodule of M [10].

Proposition 2.5: Every coatomic module has small Radical.

Proof: let M be a coatomic module, and let M = Rad (M)+ L for some submodule L of M. Suppose $L \neq M$, since M is a coatomic module then L is contained in maximal submodule K of M, $L \leq K$, hence M= Rad (M)+K, but Rad $(M) \leq K$ [since K is a maximal]. Implies M=k. This is contradiction. Therefore Rad $(M) << K$ ■.

Using lemma 2.3. we obtain the following Corollaries.

Corollary 2.6: Every Coatomic G-Rad lifting module, is lifting module. It is known that every lifting module Satisfies the property (P*). The following is an example of a module wich is G-Rad lifting but does not Satisfied the property (P*).

Example 2.7: Let M be the left Z-Module $M = \pi_{p \in \Lambda} \left(\frac{Z}{p}\right)$, where Λ is a collection of maximal ideals of Z. Then Rad (M)=0. By [11, Lemma 2.9].

For some submodule N of M. we have $\frac{N}{Tor(M)} \cong Q$, where Tor (M) is the torsion submodule of M. N is G Rad lifting but does not have the property (p*)." According to([12], Example 2.2)

Proposition 2.8: The following statement are equivalent for a finitely generated R- Module.

1. M is G-Rad-lifting.
2. M is lifting.
3. M has the property (p*).

Proof: (1) $\Longrightarrow$ (2): Since M is finitely generated then Rad $(M) << M$ and by lemma 2.3 . M is lifting■.

Proof: (2) $\Longrightarrow$ (3): Let M be an module, and N be a submodule of M. Since M is lifting there exist submodules $K \leq N$ and $\acute{K} \leq M$, such $M + K \oplus \acute{K}$ and $N \cap \acute{K} \ll M. N \cap \acute{K} \cong \frac{N}{K} \ll \frac{M}{K}$ Therefore $\frac{N}{K} \leq \text{Rad}\left(\frac{M}{K}\right)$ ■

Proof: (3) $\Longrightarrow$ (1): Let M be an module and let N be a submodule of M. with Rad $(M) \leq N$. Since M has the property (p*)., then $M + K \oplus \acute{K}, K \leq N$, $\acute{K} \leq M$. And $\frac{N}{K} \ll \frac{M}{K}$. But $\frac{N}{K} \cong N \cap \acute{K}$, hence $N \cap \acute{K} \ll M$ thus $N \cap K^{\acute{}} \leq$ Rad (M).Therefore M is G-Rad- lifting ■.

Recall that a submodule of M is called fully invariant if $f(N) \leq N$ for every $f \in$ End (M). ([1], 6.4). And R-module M is called a duo module if every submodule of M is fully invariant [13].

Notice that a submodule of G-Rad-lifting need not to be G-Rad-lifting; For example Z is a submodule of Q as Z-module is not G-Rad–lifting.

However we have the following .

Proposition 2.9: Let M be a G-Rad-lifting Module. If N is a direct summand submodule of M then N is a G- Rad- lifting.

Proof: Let N be a direct summand of M. Let $K \leq N$ Such that Rad $(N) \leq K$ then Rad $(M) \leq K + Rad(M)$.Since M is G-Rad –lifting then by (theorem2.2) $K + Rad(M)$ can be written as $K + Rad(M) = A \oplus S$ where A is a direct summand of M and $S \leq$ Rad (M). Hence $(K + Rad(M)) \cap N = A \cap N \oplus S \cap N$. Thus $K + (Rad(M) \cap N) = A \cap N \oplus S \cap N$. Since N is a direct summand of M, then Rad (N)= $Rad\,(M) \cap N$. Therefore $K + Rad\,(N) = A \cap N \oplus S \cap N$. But $Rad\,(N) \leq K$, then K= $A \cap N \oplus S \cap N$. Now $M = A \oplus L$, $L \leq M$, then $N = M \cap N = A \cap N \oplus L \cap N$ i.e. $A \cap N$ is a direct summand of M and $S \cap N \leq Rad\,(M) \cap N = Rad\,(N)$ ■.

Recall that a ring is called a left V-Ring if every left ideal in R is an intersection of Maximal left ideals; Equivalently R is left V-Ring if and only if every left simple R-Module is a left injective if and only if Rad (M) = 0, for all left R-Module [14]. And it is known that every commutative regular ring is V-Ring [1].

Corollary 2.10: Let R be a V-ring if M be a non-Zero G-Rad –lifting module then every submodule of M is G-Rad –lifting.

Proof: Let N be a submodule of M. Let K be asubmodule of N. with Rad $(N) \leq K$ Since R is V-ring then Rad (M)=0 . Hence Rad (N)=0. Therefore $0 \leq K \leq N$ is a submodule of M. Since M is G-Rad-lifting there exist submodules **L ≤ M** and , $\acute{K} \leq K$ such that $M = L \oplus \acute{K}$ and $L \cap K \leq Rad\ (M) = 0.$

Hence $M = K \oplus L$ Now $N = L \cap N \oplus \acute{K} \cap N$, Therefore $\acute{K} \cap N$ is a direct summand of N, and $\acute{K} \cap N \leq K \cap N = K.$ Thus N is G-Rad –lifting■ .

Corollary 2.11: Let M be a commutative regular ring or (V- ring) and M be any R-Module Then M is G-Rad – lifting if and only if M is semi- simple.

Proof: $(\Leftarrow)$ **it is clear■**

$(\Rightarrow)$ Since Rad (M)= 0, then for all submodule N of M, there exist a direct summands K of M and $K \leq N$ such that $M = K \oplus L$ for some submodule L of M. with $L \cap N = 0$, Since M is G- Rad – lifting. Hence $M = L \oplus N$■.

3. Direct Sum of G- Rad – lifting modules:

In this section we prove that under certain condition the direct sum of G-Rad – lifting is a gain G-Rad lifting.

Proposition 3.1: Let $M = M_1 \oplus M_2$, if M is a G-Rad –lifting module then M_1 and M_2 are G-Rad –lifting.

Proof: $\forall i = 1,2$ Let $N_i \leq M_i$ such that Rad $(M_i) \leq N_i$ **.** Hence Rad $(M) \leq N_i + Rad\ (M)$, Since M is G-Rad-lifting then there exist submodules A of $M, S \leq M$ such that $N_i + Rad\ (M) = A \oplus S$,where A is a direct summand of M, and$S \leq Rad\ (M)$,by(theorem2.2)Hence$M = A \oplus L$where$L \leq M$.Then $M \cap M_i = A \cap M_i \oplus L \cap M_i$,$\forall i = 1{,}2 M_i = A \cap M_i \oplus L \cap M_i$..Now

$A_i \cap M_i \oplus S \cap M_i = (N_i + Rad\ (M)) \cap M_i = N_i + Rad\ (M) \cap M_i = N_i + Rad\ (M_i) = N_i$ [Since M_i is supplement] and $S \cap M_i \leq Rad\ (M) \cap M_i$, Since $\forall i = 1,2$ M_i is supplement, then $Rad\ (M) \cap M_i = Rad\ (M_i).$ Hence $S \cap M_i \leq Rad\ (M_i)$■

Corollary 3.2: Let M be a G-Rad –lifting module, Then for a direct summand N of M , $\frac{M}{N}$ is G-Rad –lifting module.

Proof: Clear by Prop. 3.1.■

Notice that Prop. 2.9 also follows directely from Prop. 3.1

Corollary 3.3: Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ be a G-Rad –lifting module then M_i is a G-Rad –lifting $\forall i = 1.2, \ldots, n$

Example 3.4: Let p be aprime integer, and Consider z- module $M = \frac{z}{pz} \oplus \frac{z}{p^3 z}$, where $\frac{z}{pz}$ and $\frac{z}{p^3 z}$ are hollow local modules. Hence $\frac{z}{pz}$ and $\frac{z}{p^3 z}$ are lifting and thus are G-Rad –lifting.

Let $L = 0 \oplus \frac{z}{p^3 z}$ and $N = Z\,(1 + PZ, P + P^3 Z)$, then $M = N + L$, and $N \cap L = 0 \oplus \frac{p^2 z}{p^3 z}$, Thus $N \cap L \ll M$, but N is not direct summand of M. Therefor M is not G-Rad –lifting.

The following proposition gives a certain condition to be adirect sum of two G-Rad- lifting is G-Rad-lifting.

Proposition 3.5: Let $M = M_1 \oplus M_2$ be a due module. If M_1 and M_2 are G-Rad –lifting . then M is G-Rad –lifting.

Proof: Let N be a submodule of M, with Rad $(M) \leq N$. then Rad $(M) \cap Mi \leq N \cap Mi$ for all i=1,2. Hence Rad $(Mi) \leq N \cap Mi$ f or all i=1,2. Then there exist direct summands ki of Mi such that $Mi = ki \oplus Li$ for all i=1,2 and $ki \leq Ni \cap Mi$, and $Li \cap (N \cap Mi) \leq Rad\ (Mi)$.

For all i=1,2. Therefore take $K = K_1 + K_2$, $K_1 + K_2 \leq N$ and

$M = K_1 + K_2 \oplus L_1 + L_2$, $L_1 + L_2 \cap N = L_1 + L_2 \cap ((N \cap M_1) + (N \cap M_2)) = L_1 \cap (N \cap M_1) + L_2 \cap (N \cap M_2) \leq Rad\ (M)$

Thus M is G-Rad –lifting■ .

Corollary 3.6: Let $M = M_1 \oplus M_2 \oplus \dots . \oplus M_n$ be a duo module . if M_i is a G-Rad-lifting for all i=1,2,…,n. Then M is G-Rad –lifting .
Proof: Clear by Proposition 3.5■

Proposition 3.7: Let M be a non- Zero module with Rad (M)=0. Then M is G-Rad –lifting if and only if M is semi –simple.
Proof: $(\Leftarrow)$ Cleary since every semi-simple is G-Rad –lifting.■
$(\Rightarrow)$Since Rad (M)=0,then for any submodule N of M, $0 \le N \le M$.
Since M is G-Rad –lifting , there exist submodules $K \le N$ and $L \le M$ such that $M = K \oplus L$ and $N \cap L$ Rad $(M) = 0$. Therefore $L \cap N$=0. Thus $M = L \oplus N$■.

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