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NILPOTENCY OF DERIVATIONS

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Abstract
In this paper we show the nilpotency of nilpotent derivation of simeprime Γ-ring with characteristic 2 must be a power of 2 and we show the nilpotency of a nilpotent derivation of simeprime Γ-ring is either odd or a power of 2 without torsion condition.

Keywords: Nilpotency , Nilpotent derivation ,Semiprime ,Γ-ring.

القوة المعدومة للمشتقات

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الخلاصة
في هذه البحث بينا ان القيمة الصفرية للمشتقة عديمة القوى المعرفة على حلقة شبه اولية من النمط كاما التي ممثلها يساوي 2 يجب ان تكون من مضاعفات العدد 2 . كذلك بينا ان القيمة الصفرية للمشتقة عديمة القوى المعرفة على الحلقة شبه الاولية من النمط كاما تكون اما من مضاعفات العدد 2 او عدد فردي بدون شرط الالتواء .

1. Introduction

Nobusawa in [1] presented the idea of a Γ-ring , the concept of Γ-ring is more general of the ring , Barnes in [2] the definition of the Γ-ring with less conditions . On the basis of these two definitions many researchers in pure mathematics have made working on Γ-ring sense Barnes and Nobusawa see [3],[4],[5],and [6] , which parallel results in the ring theory, Barnes in [2] defined it as following : suppose M and Γ be an additive abelian groups , if there exists a map from $M\times\Gamma\times M$ to M , for all a,b,c ∈ M and γ,δ ∈Γ satisfying the following conditions :

1. aγb ∈ M .
2. (a+b)γc=aγc+bγc , a(γ+δ)b=aγb+aδb and aγ(b+c)=aγb+aγc
3. (aγb)δc=aγ(bδc) .

Then M is called Γ-ring. Some preliminaries of Γ-rings was given by S.Kyuno [7] as following : "Let I be a non-zero subset of a Γ-ring M, then I is called a left (right) ideal , if I be an additive subgroup of M and M ΓI⊆I (IΓM⊆I), if I is a left and right ideal then I is called an ideal of M . M is called 2-torsion free if 2a=0 obtain a=0 , a∈N . A Γ-ring M is said to be prime if aΓ M Γb=(0) with a,b ∈ M , obtain a=0 or b=0 and it simeprime if aΓ M Γa=(0), with a∈ M, obtain a=0 . A Γ-ring M is called commutative if aγb=bγa , for all a,b ∈Γ and γ∈Γ.

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The subset Z(M)={a∈ M | aγb=bγa, for all a∈ M and γ∈Γ } of a Γ-ring M is called center of M". An additive mapping d:M→M is called a derivation if d(xαy)=d(x)αy+xαd(y), for all x,y∈M and α∈Γ .Let M be a semiprime Γ-ring and d be a nilpotent derivation of M (d is derivation of M and $d^n(M)$=(0) for some positive integer n) the smallest n is called nilpotency of d this definition of nilpotent derivation is given by sameer in [8] which proved the nilpotency of d is odd number if M is 2-tosion free semiprime Γ-ring . In this paper extended the results of Chung and Luh [9] they proved that for semiprime ring of characteristic 2 the nilpotency of nilpotent derivation was a power of 2 and was odd or a power of 2 when the ring without torsion condition .

2. The results :

Theorem 2.1. Let d be a derivation of a semiprime Γ-ring M of characteristic 2 and $d^n(M) = (0)$ where $2^k < n \leq 2^{k+1} - 1$ for some positive integer k. Then $d^{2^k}(M) = (0)$.

Proof. we will prove the theory by induction on k . if k =1 then n=3 and $d^3(M) = (0)$. For all $x,y \in M$ and $\alpha \in \Gamma$, $0=d^3(d(x)\alpha y) = \sum_{j=0}^{3} \binom{3}{j} d^j\big(d(x)\big)\alpha d^{3-j}(y)$)= $d^2(x)\alpha d^2(y)$. By replacing y by yβx for all $\beta \in \Gamma$ and note that d is derivation on M because M of characteristic 2 , we have 0=$d^2(x)\alpha d^2(y\beta x) = d^2(x)\alpha(d^2(y)\beta x + y\beta d^2(x))$= $d^2(x)\alpha y\beta d^2(x)$, but M is semiprime then $d^2(x)$ =0 , for all x∈M . Assume the theory is true on 2^k , i.e. $d^{2^k}(x) = (0)$, for all $x \in M$. We now assume $k > 1$ where $2^k < n \leq 2^{k+1} - 1$, we want prove that $d^{2^{k+1}}(x) = 0$ for all x∈M , the prove will be by two cases as following :

<u>Case one</u> : Suppose $n < 2^{k+1} - 1$, then $n = \sum_{i=1}^{k} a_i 2^i$, where a_i is either zero or one and at least one of a_i's is zero , pick $k=0$, $a_0 = a_1 = 0$ and $a_2 = 1$

we have $\sum_{i=0}^{2} a_i 2^i = 2^2 < 2^3 - 1$. Let i be the smallest one with $a_i = 0$, then

If $i=0$ and $a_0 = 0$ then $n = \sum_{i=1}^{k} a_i 2^i = 2\sum_{i=1}^{k} a_i 2^{i-1} = 2n_0$, where n_0 $=\sum_{i=1}^{k} a_i 2^{i-1}$, but $2^k < n \leq 2^{k+1} - 1$ then $2^{k-1} < n_0 \leq 2^k - 1$, let $d_1 = d^2$ then d_1 is derivation on M and ${d_1}^{n_0} = 0$ becouse $2^{k-1} < n_0 \leq 2^k - 1$ by the induction hypothesis corresponding to the derivation d, then $0 = {d_1}^{2^{k-1}} = (d^2)^{2^{k-1}} = d^{2^k}$

If $i > 0$, then $a_0 = 1$ and $n + 1 = \sum_{i=1}^{k} a_i 2^i + 2^0 = 2^0 + 2^0 + \sum_{i=1}^{k} a_i 2^i = 2^1 + \sum_{i=1}^{k} a_i 2^i$

$= 2^1 + 2^1 + \sum_{i=2}^{k} a_i 2^i$, where $a_1 = 1$

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$=2^j + \sum_{i=j+1}^{k} a_i 2^i$=$2^j(1 + \sum_{i=j+1}^{k} a_i 2^{i-j})$

Therefore $n + 1 = 2^j(1 + \sum_{i=j+1}^{k} a_i 2^{i-j}$)= $2^j s$, where $s = 1 + \sum_{i=j+1}^{k} a_i 2^{i-j}$.

Let $d_1 = d^{2^j}$, then d_1 is derivation on M and $2^{k-i} < s \leq 2^k - 1$, also by the induction hypothesis corresponding to the derivation d then ${d_1}^{2^{k-i}}$=0 or d^{2^k} =0 .

<u>Case two</u>: Suppose $n = 2^{k+1} - 1$, then $n - 1 < 2^{k+1} - 1$ in view of case one we need only to prove that $d^{n-1} = 0$. Assume the contrary that $d^{n-1} \neq 0$ then for any x,y ∈M and α∈ Γ we have

0=$d^n(d^{n-2}$(x)αy)=$\sum_{i=0}^{n}\binom{n}{i} d^i\big(d^{n-2}(x)\big)\alpha d^{n-i}(y)$

= $d^{n-2}(x)\alpha d^n(y) + \binom{n}{1} d^{n-1}(x)\alpha d^{n-1}(y) + \binom{n}{2} d^n(x)\alpha d^{n-2}(y) + \binom{n}{3} d^{n+1}(x)\alpha d^{n-3}(y)$ + … + $\binom{n}{n} d^n(d^{n-2}(x)\alpha y$ $= nd^{n-1}(x)\alpha d^{n-1}(y)$

But $n = 2^{k+1} - 1$ then 0=$2^{k+1}d^{n-1}(x)\alpha d^{n-1}(y)$ - $d^{n-1}(x)\alpha d^{n-1}(y)$ and $d^{n-1}(x)\alpha d^{n-1}(y)$= $2^{k+1}d^{n-1}(x)\alpha d^{n-1}(y)$ =2($2^k d^{n-1}(x)\alpha d^{n-1}(y)$) =0 therefore

$d^{n-1}(x)\alpha d^{n-1}(y)$=0 (1)

Since $d^{n-1} \neq 0$ then there exist a non-zero element a∈ d^{n-1}(M) , d(a)=0 and by eq. 1 then $d^{n-1}(x)\alpha$a=0 and $a\alpha d^{n-1}(y)$=0 .

Let I=∩ {J | J $is\ an\ ideal\ of\ M\ and$ d^{n-1}(M) ⊆ J} then I is an ideal of M generated by d^{n-1}(M). And let $H = \{(s,t) |\ s,t \in \mathbb{Z}^+\ such\ that\ there\ exist\ anon-zero\ element\ b \in I\ with\ d(b) = 0\ , d^s(M)\alpha b = b\alpha d^t(M) = (0)\ \}$, then

1. H≠ ∅ , since $(n-1, n-1)$ ∈H .
2. H subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ and is partial order

Suppose (p,q) be a minimal element in H and let c be a non-zero in I such that d(c)=0 and $d^p(M)\alpha c = c\alpha d^q(M)$

Let p or q less than or equal of 2^k say q then $c\alpha d^{2^k}(x\beta y) = c\alpha d^q\left(d^{2^k-q}(x\beta y)\right) = 0$

Therefore 0= $c\alpha(x\beta d^{2^k} + d^{2^k}(x)\beta y)$= $c\alpha x\beta d^{2^k}$ for all x,y ∈M and β∈Γ and consequently cαI=(0) . but M is semiprime Γ-ring that means c=0 , a contradiction .

If both p and q great than 2^k , then for any x,y ∈M and α,β∈Γ , then

0=$d^n(d^{p-2^k-1}(x)\alpha d^{q-2^k}(y) = d^{2^k-1}(d^{2^k}(d^{p-2^k-1}(x)\alpha c\beta d^{q-2^k}(y)))$

= $d^{2^k-1}(d^{2^k}(d^{p-2^k-1}(x)\alpha c))\beta d^{q-2^k}(y) + d^{p-2^k-1}(x)\alpha c\beta d^{2^k}(d^{q-2^k}(y))$

=$d^{2^k-1}((d^{p-1}(x)\alpha c$+$d^{p-2^k-1}(x)\alpha d^{2^k}(c)\beta d^{q-2^k}(y))$

=$d^{2^k-1}(d^{p-1}(x)\alpha c\beta d^{q-2^k}(y))$

=$d^{2^k-2}(d(d^{p-1}(x)\alpha c\beta d^{q-2^k}(y))$

= $d^{2^k-2}(d^{p-1}(x)\alpha c\beta d^{q-2^k+1}(y))$

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0=$d^{p-1}(x)\alpha c\beta d^{q-1}(y)$

From the above equation if $d^{q-1}(y) = 0$, then (p,q-1)∈ H a contradiction with minimal of (p,q) in H .

Suppose that $0 \neq c_0$= $c\beta d^{q-1}(y_0)$, for some y_0 ∈M , then it is clear that

1. c_0 ∈I
2. $d(c_0) = 0$
3. $c_0\beta d^q(y) = 0$
4. $d^{p-1}(x)\alpha c_0$= $d^p(x)\alpha c\beta d^{q-1}(y_0)$ =0

Then from the point 4 we have a contradiction with minimal of (p,q) in H, therefore $d^{n-1} = 0$.

Theorem 2.2. Let d be a nilpotent derivation of a semiprime Γ-ring M .Then the nil potency of d is either a power of 2 or an odd number.

Proof. Let M_2={$x \in M$| $2x = 0$} , if M_2=(0) , then M is 2-torsion free and by [8] the nil- potency of d is odd number .

If $M_2 \neq$(0) , then $\frac{M}{M_2}$ is a 2-torsion free semiprime Γ-ring , define the derivation map as following: $\bar{d}: \frac{M}{M_2} \to \frac{M}{M_2}$ by $\bar{d}(x + M_2) = d(x) + M_2$ now by [8] then the nilpotency of $\bar{d}$ is an odd number say 2n+1 (where n is positive integer) i.e. $\bar{d}^{2n+1}(\frac{M}{M_2})$= M_2 which means $d^{2n+1}(M) \subseteq M_2$ (1)

If $d^{2n}(M) \subseteq M_2$ then $d^{2n}(M) + M_2 = M_2$ or $\bar{d}^{2n}(\frac{M}{M_2})$= M_2 contradiction , therefore $d^{2n}(M) \nsubseteq M_2$. also it is clear that M_2 is 2-torsion free simeprime Γ-ring then by Theorem2.1

$d^{2^k}(M_2) = 0$ (2)

Clime that $d^{2^k}(M) \cap M_2 = (0)$

Let a∈ M_2 then $x\alpha a \in M_2$ and from eq. 2 we have

0=$d^{2^k}(x\alpha a)$=$\sum_{i=0}^{2^k}\binom{2^k}{i} d^i(x)\alpha d^{2^k-i}(a) = d^{2^k}(x)\alpha a + x\alpha d^{2^k}(a)$

0=$d^{2^k}(x)\alpha a$ (3)

From eq.3 then $d^{2^k}(x)$ belong to left annihilator of M_2 which means $d^{2^k}(x) = 0$.

If $2n + 1 > 2^k$, then $d^{2n+1}(M) \subseteq d^{2^k}(M)$ and by eq. 1 $d^{2n+1}(M) \subseteq d^{2^k}(M) \cap M_2 = (0)$. it follows that $d^{2n+1}(M) = (0)$ with $d^{2n}(M) \neq (0)$.

If $2n + 1 < 2^k$, then $d^{2^k}(M) \subseteq d^{2n+1}(M) \subseteq M_2$, but (0)= $d^{2^k}(M) \cap M_2 = d^{2^k}(M)$ and since (0)≠ $d^{2^k}(M_2) \subseteq d^{2^k}(M)$, then the nilpotency of d is a power of 2.

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