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2-Rowed Plane Overpartitions Modulo 8 and 16

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Abstract
In a recent study, a special type of plane overpartitions known as k-rowed plane overpartitions has been studied. The function $\overline{pl}_k(n)$ denotes the number of plane overpartitions of *n* with a number of rows at most *k*. In this paper, we prove two identities modulo 8 and 16 for $\overline{\mathrm{pl}}_2(\mathrm{n})$, the plane overpartitions with at most two rows. We completely specify the $\overline{\mathrm{pl}}_2(\mathrm{n})$ modulo 8. Our technique is based on expanding each term of the infinite product of the generating function of the $\overline{\mathrm{pl}}_2(\mathrm{n})$ modulus 8 and 16 and in which the proofs of the key results are dominated by an intriguing relationship between the overpartitions and the sum of divisors, which reveals a considerable link among these functions modulo powers of 2.

Keywords: Partitions, Overpartitions, Plane overpartitions, Congruences, Sum of divisors function.

التجزئات المستويه الفوقيه ذات الصفين من القياس 8 و16

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الخلاصة
في دراسة حديثة ، تمت دراسة نوع خاص من دوال التجزئه الفوقية المستويه المعروفة باسم التجزئات الفوقيه المستويه ذات الصفوف المحدده k . تشير الداله $\overline{\mathrm{pl}}_k(\mathrm{n})$ الى عدد التفسيمات الفوقيه المستويه ل nبعدد صفوف لا يتجاوز k. في هذا البحث اثبتنا متطابقتين من القياس 8 و 16 لدوال التجزئه الفوقيه المستويه $\overline{pl}_2(n)$ بعدد صفوف على الاكثر اثنان. حددنا بصوره كامله دوال التجزئه الفوقيه المستويه ذات المقياس 8 للدالة $\overline{pl}_2(n)$. اعتمدنا اسلوب توسيع كل حد من المضروب اللامنتهي للدالة المولده لدوال التجزئه الفوقيه المستويه ذات الصفين $\overline{pl}_2(n)$ بقياس 8 و16 حيث تهيمن على البراهين للنتائج الرئيسية علاقة مثيرة للاهتمام بين دالة التجزئة الفوقيه ودالة مجموع القواسم ، مما يكشف عن وجود صلة كبيرة بين هذه الدوال بقوى معيارية 2 .

1. Introduction

A partition of a positive integer n is a non-increasing sequence of integers which are called parts that sum to n [1]. As an example, there are seven partitions of *n*=5 denoted by

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(5),(4,1),(3,2),(3,1,1),(2,2,1),(2,1,1,1),(1,1,1,1,1).

It is possible to impose restrictions on the parts of a partition. In the given example, there are three partitions of n=5 with only odd parts given by

(5),(3,1,1),(1,1,1,1,1).

The concept of partition is generalized to overpartitions. An overpartition of a positive integer n is a partition of n such that the first occurrence of a part may be overlined [2]. The overpartition function of n is denoted by $\overline{p}(n)$ and $\overline{p}(0) = 1$. As an example, for *n*=3, there are eight overpartitions that are identified by

$$(3), (\overline{3}), (2,1), (\overline{2},1), (2,\overline{1}), (\overline{2},\overline{1}), (1,1,1), (1,1,\overline{1}).$$

Partitions with all parts overlined are partitions with distinct parts, whereas partitions with no overlined parts are ordinary partitions. As a result, the generating function of overpartitions has the following algebraic formula:

$$\overline{P}(w) = 1 + \sum_{t\in \mathrm{N}} \overline{p}(t) w^t = \prod_{t\in N} \frac{1+w^t}{1-w^t} = 1 + 2w + 4w^2 + 8w^3 + 14w^4 + \cdots \qquad (1)$$

The one row partition was extended by MacMahon [1] to a 2-dimensional array partition known as a plane partition. A plane partition $\mu_{i,j}$ of a positive integer t is a 2-dimensional array with weakly decreasing entries

$$\mu_{i+1,j} \leq \mu_{i,j}, \mu_{i,j+1} \leq \mu_{i,j},$$

and

$$\sum_{i,j} \mu_{i,j} = t.$$

The concept of plane overpartitions was introduced as a natural extension of the overpartitions and plane partitions [3]. In [4], the author defined the concept of *k*-rowed plane overpartitions, which is plane overpartitions with the number of rows at most *k*, as a restricted variant of plane overpartitions for a fixed number *k* of rows. The *k*-rowed plane overpartitions generating function is represented by

$$\overline{PL}_k(w) := 1 + \sum\nolimits_{t\in N} \overline{pl}_k(t) w^t = \prod_{t\in N} \frac{(1+w^t)^{\min\{k,t\}}}{(1-w^t)^{\min\{k,t\}}}. \qquad (2)$$

In [5], various identities modulo 4 and 8 for restricted plane overpartitions were established and proved. Congruences modulo 4 between overpartitions and plane overpartitions have also been demonstrated. As a result, we are motivated to investigate additional modulo 8 and 16 identities for the 2-rowed plane overpartitions, and perhaps the same work can be pursued for larger powers of 2. One may notice that $\overline{PL}_2(w)$ has a close structure to the generating function of overpartition pairs $\overline{PP}(w)$ since

$$\overline{PL}_2(w) = \frac{(1-w)}{(1+w)} \overline{P}^2(w) = \frac{(1-w)}{(1+w)} \overline{PP}(w). \qquad (3)$$

2. Preliminaries

The sum of divisors function, denoted by $\sigma(t)$ is an arithmetic function that sums all positive divisors of an integer *t*, including 1 and *t* itself. Thus,

$$\sigma(\mathrm{t}) = \sum_{\mathrm{d|t}} d$$

which is generated by

$$\sum_{t \in N} \sigma(t) w^t = \sum_{t \in N} \frac{t w^t}{1 - w^t}. \quad (4)$$

Note that

$$\frac{1}{(1-w)^2} = \frac{d}{dw}\left(\frac{1}{1-w}\right) = \frac{d}{dw}\left(\sum_{t \geq 0} w^t\right) = 1 + 2w + 3w^2 + 4w^3 + \cdots,$$

that leads to the following useful lemma.

Lemma 2.1. ([6])

$$\sum_{t \in N} \sigma(t) w^t = \sum_{t \in N} \frac{w^t}{(1 - w^t)^2}.$$

Note that the only divisors of a prime *p* are 1 and *p* itself which implies $\sigma(p) = 1 + p$ and

$$\sigma(p^\alpha) = \sum_{j=0}^{\alpha} p^j = \frac{p^{\alpha+1} - 1}{p - 1}. \quad (5)$$

Two integers are called coprime if one is the only positive divisor for both numbers. For any coprime integers *n* and *m*, $\sigma(nm) = \sigma(n)\sigma(m)$. Thus, for a positive integer n with prime factorization

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

and we find by the multiplicity of $\sigma(n)$,

$$\sigma(n) = \prod_{i=1}^{k} \sigma\left(p_i^{\alpha_i}\right) = \prod_{i=1}^{k} \left(1 + p_i + \cdots + p_i^{\alpha_i}\right) \quad (6)$$

If *n* is a square, all powers $\alpha_1, \alpha_2, \ldots, \alpha_k$ must be even. Replacing these powers by $2\alpha_i's$, we find,

$$\sigma\left(p_i^{\alpha_i}\right) \equiv 1 \; (mod\, 2).$$

Since the finite product of odd numbers is odd, thus

$$\sigma(n) \equiv 1 \; (mod\, 2).$$

The same argument can be demonstrated if *n* is twice a square. As a result, the product in (6) is odd as long as $n = m^2$ or $n = 2m^2$ for some integer *m*.

The following two lemmas are fairly basic, and the proofs are provided for the sake of completeness.

Lemma 2.2. Let *n* be a nonnegative integer. The following holds for any integer $m \geq 1$,

$$(4n + 3)^{2m} \equiv 1 \; (mod\, 8). \quad (7)$$

Proof. By induction on m. Clearly, (7) is true for *m*=1. Suppose (7) holds for *m*=*k*-1 which provides

$$(4n+3)^{2(k-1)} = 8t+1,$$

for some integer t. Then,

$$(4n+3)^{2k} - 1 = (4n+3)^{2(k-1)}(4n+3)^2 - 1 = (8t+1)(16n^2+24n+9) - 1$$
$$= 8(16n^2t + 2n^2 + 24nt + 3n + 9t + 1) \equiv 0 \ (mod\ 8).$$

Lemma 2.3. For all $t \geq 0$,

$$\sigma(4t+3) \equiv 0 \ \ (mod\ 4).$$

Proof. Let p be a prime of the form 4*t*+3 and α an odd integer. By (5),

$$\sigma(p^{\alpha}) = \frac{p^{\alpha+1}-1}{p-1} = \frac{(4t+3)^{\alpha+1}-1}{4t+2}.$$

Since

$$\alpha + 1 \equiv 0 \ (mod\ 2),$$

by Lemma 2.2, $(4t+3)^{\alpha+1} - 1$ is divisible by 8. Also, 2 is the highest power of 2 dividing $p-1$. Therefore, $\sigma(p^{\alpha})$ is divisible by 4. It is known that any integer of the form (4*t*+ 3) has at least one prime divisor congruent to 3 modulo 4 with an odd power. Let now *n* be an integer of the form $4t+3$. With no loss of generality, consider the prime factorization of *n* is given by

$$n = p^{\alpha} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where p is congruent to 3 modulo 4 and α is odd. Since α is multiplicative, we obtain

$$\sigma(n) = \sigma(p^{\alpha})\sigma\left(p_2^{\alpha_2}\right) \cdots \sigma\left(p_k^{\alpha_k}\right),$$

which is divisible by 4 since 4 divides $\sigma(p^{\alpha})$ and the proof is completed.

3. Main Results and Proofs

In this section, we apply a method that is based on expanding the infinite product modulo 8 and 16 for the 2-rowed plane overpartitions along with calling Lemma 2.1. Two results for the 2-rowed plane overpartitions modulo 8 and 16 will be stated and proved. The first result completely specifies all 2-rowed plane overpartitions modulo 8 as follows:

Theorem 3.1. For all positive integers *t*,

$$\overline{pl}_2(t) = \begin{cases} 2 \ \ (mod\ 8) & if t = (2j+1)^2 or\ t = 2j \neq i^{2,} 2i^2 for\ some\ \ i, j, \\ 6 \ (mod\ 8) & otherwise. \end{cases}$$

Proof. Recalling the generating function for 2-rowed plane overpartitions,

$$\overline{PL}_2(w) = \left(\frac{1-w}{1+w}\right)\left(\prod_{t\in N}\frac{1+w^t}{1-w^t}\right)^2$$
$$= \left(1 + 2\sum_{t=1}^{\infty}(-1)^{t+1} w^t\right)\prod_{t\in N}\left(1 + \frac{2w^t}{1-w^t}\right)^2$$
$$\equiv \left(1 + 2\sum_{t\in N}(-1)^{t+1} w^t\right)\left(1 + \sum_{t\in N}\frac{4w^t}{1-w^t} + \frac{4w^{2t}}{(1-w^t)^2}\right)(mod\ 8)$$
$$= 1 + 2\sum_{t\in N}(-1)^{t+1} w^t + 4\sum_{t\in N}\frac{w^t}{1-w^t} + \frac{w^{2t}}{(1-w^t)^2}$$
$$= 1 + 2\sum_{t\in N}(-1)^{n+1} w^t + 4\sum_{t\in N}\frac{w^t}{(1-w^t)^2}$$
$$\equiv 1 + 6\sum_{t\in N} w^{2t-1} + 2\sum_{t\in N} w^{2t} + 4\sum_{t\in N}\sigma(t)w^t \ (mod\ 8).$$

Note that $\sigma(t)$ is odd as long as $t = j^2$ or $t = 2j^2$ for some *j*. Reducing the last equivalence modulo 8, we obtain

$$\overline{PL}_2(w) \equiv 1 + 6\sum_{t\in N} w^{2t-1} + 2\sum_{t\in N} w^{2t} + 4\sum_{t=j^2,2j^2\geq 1} w^t \quad (mod\, 8) \cdots (8)$$

To finish the proof, we check the exponents of *w* in the right side of (8) as follows:

Case 1: if *t* is odd but not a square, then the coefficient of w^t is obtained only from the series

$$6\sum_{t\in N} w^{2t-1},$$

and hence $\overline{pl}_2(t) \equiv 6\ (mod\, 8)$.

Case 2: if *t* is odd and a square then the coefficient of w^t is obtained from the series

$$6\sum_{t\in N} w^{2t-1}, 4\sum_{t=j^2,2j^2\geq 1} w^t,$$

and so $\overline{pl}_2(t) \equiv 6 + 4 \equiv 2\ (mod\, 8)$.

Case 3: if *t* is even but not a square or twice a square, then the coefficient of w^t is obtained only from the series

$$2\sum_{t\in N} w^{2t},$$

which implies $\overline{pl}_2(t) \equiv 2\ (mod\, 8)$.

Case 4: if *t* is even and a square or twice a square, then the coefficient of w^t is obtained from the series

$$2\sum_{t\in N} w^{2t}, 4\sum_{t=j^2,2j^2\geq 1} w^t,$$

and so $\overline{pl}_2(t) \equiv 2 + 4 = 6\ (mod\, 8)$.

By Case 2 and Case 3,

$$\overline{pl}_2(t) \equiv 2\ (mod\, 8).$$

Otherwise, by Case 1 and Case 4,

$$\overline{pl}_2(t) \equiv 6\ (mod\, 8),$$

Therefore, we get the results.

Theorem 3.2. For all positive integers *t*,

$$\overline{pl}_2(t) \equiv \begin{cases} 2 + 4\sigma(t) + 8\sum_{j=1}^{t}\sigma(j) \quad (mod\, 16) \quad if\ t\ is\ an\ odd\ number, \\ 14 + 4\sigma(t) + 8\sum_{j=1}^{t}\sigma(j) \quad (mod\, 16) \quad if\ t\ is\ an\ even\ number. \end{cases}$$

Proof. Using a similar argument in the proof of Theorem 3.1,

$$\overline{PL}_2(w) = \left(\frac{1-w}{1+w}\right)\left(\prod_{t\geq 1}\frac{1+w^t}{1-w^t}\right)^2$$

$$\equiv \left(1+2\sum_{t\geq 1}(-1)^{t+1}w^t\right)\left(1+4\sum_{t\geq 1}\sigma(t)w^t\right)(mod\ 16)$$
$$\equiv 1+2\sum_{t\geq 1}(-1)^{t+1}w^t+4\sum_{t\geq 1}\sigma(t)w^t+8\sum_{t\geq 1}\left(\sum_{j=1}^{t}(-1)^{t-j+1}\right)\sigma(j)w^t\ (mod\ 16)$$
$$\equiv 1+2\sum_{t\geq 1}w^{2t-1}+14\sum_{t\geq 1}w^{2t}+4\sum_{t\geq 1}\sigma(t)w^t+8\sum_{t\geq 1}\sum_{j=1}^{t}\sigma(j)w^t\ \ (mod\ 16).$$

For odd integers $2t-1\geq 1$, we extract the terms w^{2t-1} and obtain modulus 16

$$\overline{pl}_2(2t-1)\equiv 2+4\sigma(2t-1)+8\sum_{j=1}^{2t-1}\sigma(j).$$

Otherwise, for even integers $2t\geq 2$, we extract the terms w^{2t} and obtain modulus 16

$$\overline{pl}_2(2t)\equiv 14+4\sigma(2t)+8\sum_{j=1}^{2t}\sigma(j),$$

which completes the proof.

Let $S_1(t), S_2(t)$ be the number of squares, double squares in the interval [1, t], respectively. Thus, we have the following result because of reducing Theorem 3.2.

Corollary 3.3. For all positive integers t,

$$\overline{pl}_2(4t+3)\equiv\begin{cases} 2\ \ (mod\ 16) & if\ S_1(t)+S_2(t)\equiv 0\ (mod\ 2),\\ 10\ (mod\ 16) & otherwise.\end{cases}$$

Proof. Note that

$$8\sum_{j=1}^{t}\sigma(j)=8\sum_{j\neq i^2,2i^2}^{t}\sigma(j)+8\sum_{j=i^2,2i^2}^{n}\sigma(j)\equiv 8\big(S_1(t)+S_2(t)\big)\quad (mod\ 16)$$

Since $4|\sigma(4t+3)$, thus we find by Theorem 3.2,

$$\overline{pl}_2(4t+3)\equiv 2+8\big(S_1(t)+S_2(t)\big)\ \ (mod\ 16),$$

as desired.

4. The Conclusion

Even though $\overline{PL}_2(t)$ appears to be quite like the generating function of overpartition pairs, we could not identify any congruences modulo 8 and 16 that are similar to those known for overpartition pairs. Also, greater powers of 2 can be used to obtain congruences modulo 32, 64, and so on, but this technique can be time-consuming when we work with higher powers of 2. Furthermore, if there exist any congruences for plane overpartitions modulo primes such as 3, 5, and 7, they have yet to be discovered. Plane overpartitions with no more than 3 or 4 rows may be considered; however, we recommend working on plane overpartitions with an odd number of rows when looking for congruences modulo primes, and plane overpartitions with an even number of rows when looking for congruences modulo even numbers or powers of 2. It is worth noting that there is a deep connection between overpartitions and the number of ways to represent an integer as a sum of two squares, three squares, and so on. Congruences for a higher power of 2 necessitate the investigation of a larger sum of squares. Such an idea can be generalized to consider other questions, such as representing integers, for example, as a sum of the forms $ax^2+by^2, ax^2+by^2+cz^2$ for fixed integers a, b and c.

Looking through the literature, one may find that many of these problems had been studied by famous mathematicians, such as Fermat [7], who investigated integers of the form $x^2 + y^2$. He also considered integers of the form $x^2 + 2y^2$ in which from a geometric standpoint, the case $x^2 + 2y^2 = x^2 + y^2 + y^2$ corresponds to asking what integers the sums of three squares are, where at least two of the squares are of the same size. By the 2-adic expansion series for $\overline{p}(n)$, we notice that various problems occur, which may need to look into arithmetic properties for the sum of squares. For example, $\overline{p}(n) \equiv 0 \ (mod\ 64)$ for $n \equiv 7(mod\ 8)$ which cannot be represented as a sum of three squares. Generally, Gauss (1801) [8] established for the first time that any positive integer that is not of the form $4^k(8n + 7)$ can be represented as a sum of three squares. Furthermore, there is a significant relationship between overpartition type functions and the sum of divisors functions, as well as having an interesting relationship with square integers. We believe that investigating such problems may lead to the discovery of congruences for overpartition-type functions modulo higher powers of 2. The readers may be interested in further partitioning concepts contained in [9] and [10].

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