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γ– algebra of Sets and Some of its Properties

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Abstract

The main objective of this work is to generalize the concept of fuzzy σ–algebra by introducing the notion of fuzzy γ–algebra. Characterization and examples of the proposed generalization are presented, as well as several different properties of fuzzy γ–algebra are proven. Furthermore, the relationship between fuzzy γ–algebra and fuzzy algebra is studied, where it is shown that the fuzzy γ–algebra is a generalization of fuzzy algebra too. In addition, the notion of restriction, as an important property in the study of measure theory, is studied as well. Many properties of restriction of a nonempty family of fuzzy subsets of fuzzy power set are investigated and it is shown that the restriction of fuzzy γ–algebra is fuzzy γ–algebra too.

Keywords: σ–algebra, Algebra, Measure, Fuzzy set, Outer measure.

دراسة الجبرا ضبابي من النمط–γ و بعض خصائصها

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الخلاصة

الهدف الرئيسي من هذا العمل هو تعميم المفهوم الجبرا ضبابي من النمط –σ من خلال تقديم مفهوم الجبرا الضبابي من النمط –γ . تم تقديم مميزات وأمثلة للتعميم المقترح ، بالإضافة إلى العديد من الخصائص المختلفة للجبرا الضبابي من النمط –γ تم برهانه. علاوة على ذلك ، تمت دراسة العلاقة بين أن الجبرا ضبابي من النمط –γ والجبرا ضبابي, حيث تبين أن الجبرا ضبابي من النمط –γ هو تعميم للجبر ضبابي ايضًا. بالإضافة إلى ذلك ، فإن فكرة التقييد ، باعتبارها خاصية مهمة في دراسة مجال نظرية القياس ، تمت

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دراستها أيضًا. تم التحقق في العديد من خصائص تقييد لعائلة غير خالية من مجموعات جزئية ضبابية لمجموعة القوة المضببة و تم اثبات أن تقييد الجبرا الضبابي من النمط γ– هو أيضًا الجبرا ضبابي من النمط γ– .

1. Introduction and Basic concepts

Wang [1] in 2009 studied some types of collections of sets which are generalizations of σ–algebra such as ring, σ–ring and proved some important results of these concepts, where a nonempty class $\mathcal{H}$ is called a ring iff E,F $\in \mathcal{H}$, then E-F and E∪F $\in \mathcal{H}$. In 2019 Ahmed and Ebrahim [2] introduced some generalizations of σ–algebra and σ–ring. Many other authors are interested in studying σ–algebra and σ– ring, for example, see [3], [4] and [5]. Zadeh [6] in 1965 first introduced the concept of the fuzzy set where $\mathcal{X}$ is a nonempty set, then a fuzzy set F in $\mathcal{X}$ is defined as a set of ordered pairs $\{(\omega, \mathcal{v}_{\mathrm{F}}(\omega)) : \omega \in \mathcal{X}\}$ where $\mathcal{v}_{\mathrm{F}} : \mathcal{X} \rightarrow [0 , 1]$ is a function such that for every $\omega \in \mathcal{X}$, $\mathcal{v}_{\mathrm{F}}(\omega)$ represents the degree of membership of ω in F. Brown [7] studied some types of fuzzy sets such as fuzzy power set, empty fuzzy set, universal fuzzy set, the complement of a fuzzy set, the union of two fuzzy sets and intersection of two fuzzy. Ahmed et al. [8-11] first introduced the concept of fuzzy σ −algebra and fuzzy algebra where $\mathcal{X}$ is a nonempty set and $\mathcal{P}^*(\mathcal{X})$ be a fuzzy power set of $\mathcal{X}$. A nonempty collection $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ is said to be a fuzzy σ–algebra of sets over a fuzzy set $\mathcal{X}^* = \{(\omega, 1) : \forall \omega \in \mathcal{X}\}$, if the following conditions are satisfied:

1. $\emptyset^* \in \mathcal{H}^*$, where $\emptyset^* = \{(\omega, 0) : \forall \omega \in \mathcal{X}\}$.
2. If $\mathrm{E} \in \mathcal{H}^*$, then $\mathrm{E}^c \in \mathcal{H}^*$.
3. If $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}^*$, then $\bigcap_{k=1}^{\infty} \mathrm{E}_k \in \mathcal{H}^*$.

If condition 3 is satisfied only for finite sets, then $\mathcal{H}^*$ is said to be a fuzzy algebra over a fuzzy set $\mathcal{X}^*$.

Another generalization of the fuzzy σ–algebra introduced in this paper, which is a fuzzy γ–algebra. The main aim of this chapter is to study this generalization and introduce some of its basic properties, examples and some characterizations of them.

Definition (1.1) [4]

Let $\mathcal{X} \neq \emptyset$. A collection $\mathcal{H}$ is called σ– ring if and only if the following conditions hold:

1. If $\mathrm{F}, \mathrm{E} \in \mathcal{H}$, then $\mathrm{F} \setminus \mathrm{E} \in \mathcal{H}$.
2. If $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}$, then $\bigcup_{k=1}^{\infty} \mathrm{E}_k \in \mathcal{H}$.

Definition (1.2) [8]

Let $\mathcal{X} \neq \emptyset$. A collection $\mathcal{H}$ is called σ– field if and only if the following conditions hold::

1. $\mathcal{X} \in \mathcal{H}$.
2. If $\mathrm{F} \in \mathcal{H}$, then $\mathrm{F}^c \in \mathcal{H}$.
3. If $\mathrm{E}_1, \mathrm{E}_2, \ldots \in \mathcal{H}$, then $\bigcup_{k=1}^{\infty} \mathrm{E}_k \in \mathcal{H}$.

Proposition (1.3) [1]

Every σ–field is a σ– ring.

Definition (1.4) [9]

Let $\mathcal{X}$ be a nonempty set. Then the union of the two fuzzy sets F and E in $\mathcal{X}$ with respective membership functions $\mathcal{v}_{\mathrm{F}}(\omega)$ and $\mathcal{v}_{\mathrm{E}}(\omega)$ is a fuzzy set G in $\mathcal{X}$ whose membership function is related to those of F and E by $\mathcal{v}_{\mathrm{G}}(\omega) = \max_{\omega \in \mathcal{X}} \{\mathcal{v}_{\mathrm{F}}(\omega) , \mathcal{v}_{\mathrm{E}}(\omega)\}$, In symbols:
$\mathrm{G} = \mathrm{F} \cup \mathrm{E} \Leftrightarrow \mathrm{G} = \{(\omega, \max_{\omega \in \mathcal{X}} \{\mathcal{v}_{\mathrm{F}}(\omega) , \mathcal{v}_{\mathrm{E}}(\omega)\}) : \omega \in \mathcal{X}\}$.

Definition (1.5) [10]

Let $\mathcal{X}$ be a nonempty set. Then the intersection of two fuzzy sets F and E in $\mathcal{X}$ with respective membership functions $\upsilon_{\mathrm{F}}(\omega)$ and $\upsilon_{\mathrm{E}}(\omega)$ is a fuzzy set G in $\mathcal{X}$ whose membership function is related to those of F and E by $\upsilon_{\mathrm{G}}(\omega) = \min_{\omega\in\mathcal{X}} \{\upsilon_{\mathrm{F}}(\omega), \upsilon_{\mathrm{E}}(\omega)\}$, In symbols:
$\mathrm{G} = \mathrm{F}\cap\mathrm{E} \Leftrightarrow \mathrm{G} = \{(\omega, \min_{\omega\in\mathcal{X}} \{\upsilon_{\mathrm{F}}(\omega), \upsilon_{\mathrm{E}}(\omega)\}) : \omega \in \mathcal{X}\}$.

Definition (1.6) [6]

Let $\mathcal{X}$ be a nonempty set and F is a fuzzy sets in $\mathcal{X}$. Then the complement of a fuzzy set F is denoted by F^{c} and defined as: $\mathrm{F}^{\mathrm{c}} = \{(\omega, 1 - \upsilon_{\mathrm{F}}(\omega)) : \in \mathcal{X}\}$.

Proposition (1.7) [11]

Every fuzzy σ−algebra is a fuzzy algebra .

2. The main results:

In this section, we introduce the concept of fuzzy γ−algebra which is a generalization for the concept of the fuzzy σ−algebra and fuzzy algebra. Also, we present many properties of fuzzy γ−algebra.

Definition (2.1):

Let $\mathcal{X}$ be a nonempty set. A nonempty collection $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ is said to be a fuzzy γ−algebra of sets (γ−field) over a fuzzy set $\mathcal{X}^*$, if the following conditions are satisfied:

1. $\emptyset^*, \mathcal{X}^* \in \mathcal{H}^*$, where $\emptyset^* = \{(\omega, 0) : \forall\omega \in \mathcal{X}\}$ and $\mathcal{X}^* = \{(\omega, 1) : \forall\omega \in \mathcal{X}\}$.
2. If $\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_{\mathrm{n}} \in \mathcal{H}^*$, then $\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_{\mathrm{k}} \in \mathcal{H}^*$.

Definition(2.2):

Let $\mathcal{X}$ be a nonempty set and $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ be a fuzzy γ−algebra (γ−field) over a fuzzy set $\mathcal{X}^*$. Then the pair ($\mathcal{X}^*$, $\mathcal{H}^*$) is said to be fuzzy measurable space relatively to fuzzy γ−algebra.

Example(2.3):

Let $\mathcal{X} = \{a, b\}$ and $\mathcal{H}^* = \left\{\begin{matrix} \emptyset^*, \{(a, 0.1), (b, 0.6)\}, \\ \{(a, 0.3), (b, 0.5)\}, \\ \{(a, 0.3), (b, 0.6)\}, \mathcal{X}^* \end{matrix}\right\}$. Then the pair ($\mathcal{X}^*$, $\mathcal{H}^*$) is said to be fuzzy measurable space relatively to fuzzy γ−algebra.

Example(2.4):

Let $\mathcal{X} = \{a, b\}$ and $\mathcal{H}^* = \left\{\begin{matrix} \emptyset^*, \{(a, 0.4), (b, 0.2)\}, \\ \\ \{(a, 0.3), (b, 0.5)\}, \mathcal{X}^* \end{matrix}\right\}$.

Then $\mathcal{H}^*$ is not a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$, because $\{(a,0.4),(b,0.2)\} \in \mathcal{H}^*$ and $\{(a,0.3),(b,0.5)\} \in \mathcal{H}^*$, but
$\{(a,0.4),(b,0.2)\} \cup \{(a,0.3),(b,0.5)\} = \{(a, Max\{0.4,0.3\}), (b, Max\{0.2,0.5\})\}$
$= \{(a,0.4),(b,0.5)\} \notin \mathcal{H}^*$.

Proposition(2.5):

Let $\mathcal{X}$ be an infinite set and $\mathcal{H}^* = \{\emptyset^*, \mathcal{X}^*$, all $\mathrm{E} \subset \mathcal{X}^*$ s.t E^{c} is finite$\}$. Then ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proof:

From the definition of $\mathcal{H}^*$, we get $\emptyset^*$, $\mathcal{X}^* \in \mathcal{H}^*$. Let $\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_\mathrm{n} \in \mathcal{H}^*$. Then $\mathrm{E}_\mathrm{k}^\mathrm{c}$ is finite for all k =1, 2, …, n. Hence, $\bigcap_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k}^\mathrm{c}$ is finite. Now, since

$$\begin{aligned}\bigcap_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k}^\mathrm{c} &= \{(\omega, \min_{\mathrm{k}=1,2,\dots,\mathrm{n}} \{1 - \upsilon_{\mathrm{E}_\mathrm{k}}(\omega)\}) : \omega \in \mathcal{X}\} \\ &= 1 - \{(\omega, \max_{\mathrm{k}=1,2,\dots,\mathrm{n}} \{\upsilon_{\mathrm{E}_\mathrm{k}}(\omega)\}) : \omega \in \mathcal{X}\} \\ &= 1 - (\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k}) \\ &= (\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k})^\mathrm{c}\end{aligned}$$

Then $\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k} \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$ and ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proposition(2.6):

Let $\mathcal{X}$ be a nonempty set and F be a fuzzy set such that $\emptyset^* \neq \mathrm{F} \subset \mathcal{X}^*$ and let $\mathrm{E} \subseteq \mathcal{X}^*$ denote to $\upsilon_\mathrm{E} \leq \upsilon_{\mathcal{X}^*}$. If $\mathcal{H}^* = \emptyset^* \cup \{\mathrm{E} \subseteq \mathcal{X}^* : \mathrm{F} \subset \mathrm{E}\}$. Then ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proof:

From the definition of $\mathcal{H}^*$, we get $\emptyset^* \in \mathcal{H}^*$. Since $\mathcal{X}^* \subseteq \mathcal{X}^*$ and $\mathrm{F} \subset \mathcal{X}^*$, then $\mathcal{X}^* \in \mathcal{H}^*$. Let $\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_\mathrm{n} \in \mathcal{H}^*$. Then $\mathrm{F} \subset \mathrm{E}_\mathrm{k}$ for all k=1,2,…, n and hence $\mathrm{F} \subset \bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k}$. Thus $\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k} \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$. Consequentially, ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proposition(2.7):

Let ($\mathcal{X}^*$, $\mathcal{H}^*$) be a fuzzy measurable space relatively to fuzzy γ−algebra and let $\mathrm{E} \subset \mathcal{X}^*$. Define $\mathcal{H}_1^* = \{ \mathrm{F} \subseteq \mathcal{X}^* : \mathrm{F} \cup \mathrm{E} \in \mathcal{H}^* \}$, then ($\mathcal{X}^*$, $\mathcal{H}_1^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proof:

Since $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$, then $\mathcal{X}^* \in \mathcal{H}^*$. By hypothesis $\mathrm{E} \subset \mathcal{X}^*$ implies that $\mathcal{X}^* = \mathcal{X}^* \cup \mathrm{E}$ and hence $\mathcal{X}^* \in \mathcal{H}_1^*$. Consider $\mathrm{E} = \emptyset^*$, then $\emptyset^* \subset \mathcal{X}^*$ and $\emptyset^* \cup \emptyset^* = \emptyset^* \in \mathcal{H}^*$, hence $\emptyset^* \in \mathcal{H}_1^*$. Let $\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_\mathrm{n} \in \mathcal{H}_1^*$. Then by definition of $\mathcal{H}_1^*$ we have, $\mathrm{E}_\mathrm{k} \cup \mathrm{E} \in \mathcal{H}^*$ for all k=1, 2, …, n. Hence, $\bigcup_{\mathrm{k}=1}^{\mathrm{n}} (\mathrm{E}_\mathrm{k} \cup \mathrm{E}) \in \mathcal{H}^*$ because $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$ this implies that $((\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k}) \cup \mathrm{E}) \in \mathcal{H}^*$. Thus $\bigcup_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{E}_\mathrm{k} \in \mathcal{H}_1^*$ by definition of $\mathcal{H}_1^*$. Therefore, $\mathcal{H}_1^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$ and ($\mathcal{X}^*$, $\mathcal{H}_1^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proposition(2.8):

Let ($\mathcal{X}^*$, $\mathcal{H}^*$) be a fuzzy measurable space relatively to fuzzy γ−algebra and $\mathrm{F} \subset \mathcal{X}^*$. If $\mathcal{H}_1^* = \{ \mathrm{G} \subseteq \mathcal{X}^* : \mathrm{F} \cap \mathrm{E} \subseteq \mathrm{G} \text{ for some } \mathrm{E} \in \mathcal{H}^* \}$ Then ($\mathcal{X}^*$, $\mathcal{H}_1^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra.

Proof:

Since ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to fuzzy γ−algebra, then $\emptyset^*, \mathcal{X}^* \in \mathcal{H}^*$. By hypothesis $\mathrm{F} \subset \mathcal{X}^*$ implies that $\mathrm{F} \cap \mathcal{X}^* \subseteq \mathcal{X}^*$, hence by definition of $\mathcal{H}_1^*$ we have, $\mathcal{X}^* \in \mathcal{H}_1^*$. Now, $\mathrm{F} \cap \emptyset^* = \emptyset^* \subseteq \emptyset^*$, hence by definition of $\mathcal{H}_1^*$ we have, $\emptyset^* \in \mathcal{H}_1^*$. Let

$G_1, G_2, \dots, G_n \in \mathcal{H}_1^*$. Then there is $E_k \in \mathcal{H}$ such that $F \cap E_k \subseteq G_k$ where k=1,2,…, n. So, we have $\bigcup_{k=1}^{n} G_k \supseteq \bigcup_{k=1}^{n} (F \cap E_k) = (F \cap E_1) \cup (F \cap E_2) \cup \dots \cup (F \cap E_n)$

$$= F \cap (E_1 \cup E_2 \cup \dots \cup E_n) = F \cap (\bigcup_{k=1}^{n} E_k).$$

But $E_k \in \mathcal{H}^*$ and $\mathcal{H}^*$ is a fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$, then $\bigcup_{k=1}^{n} E_k \in \mathcal{H}^*$, hence by definition of $\mathcal{H}_1^*$ we have, $\bigcup_{k=1}^{n} G_k \in \mathcal{H}_1^*$. Therefore, $\mathcal{H}_1^*$ is a fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$ and ($\mathcal{X}^*$, $\mathcal{H}_1^*$) is a fuzzy measurable space relatively to fuzzy γ–algebra.

Proposition(2.9):

Let $\mathcal{X}$ be an infinite set and let $\mathcal{X}^* =\{(\omega, 1) : \forall \omega \in \mathcal{X} \}$. If $\mathcal{H}^*=\{E \subseteq \mathcal{X}^* :$E is infinite fuzzy set$\}$. Then ($\mathcal{X}^*$, $\mathcal{H}^*$) is a fuzzy measurable space relatively to fuzzy γ–algebra.

Proof:

Since $\mathcal{X}$ be an infinite set, then each of $\emptyset^*, \mathcal{X}^*$ be an infinite fuzzy set, but $\mathcal{X}^* \subseteq \mathcal{X}^*$ and $\emptyset^* \subseteq \mathcal{X}^*$, then by definition of $\mathcal{H}^*$ we have, $\emptyset^*, \mathcal{X}^* \in \mathcal{H}^*$. Let $E_1, E_2, \dots, E_n \in \mathcal{H}^*$. Then E_k is an infinite fuzzy set for every k=1,2,…, n and hence $\bigcup_{k=1}^{n} E_k$ is an infinite fuzzy set. Therefore, $\bigcup_{k=1}^{n} E_k \in \mathcal{H}^*$ and $\mathcal{H}^*$ is a fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$.

Proposition (2.10):

Let $\{\mathcal{H}_i^*\}_{i \in I}$ be a collection of fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$. Then $\bigcap_{i \in I} \mathcal{H}_i^*$ is a fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$.

Proof:

Since $\mathcal{H}_i^*$ is a fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$, $\forall i \in I$, then $\emptyset^*, \mathcal{X}^* \in \mathcal{H}_i^*$ $\forall i \in I$. Hence $\emptyset^* \neq \bigcap_{i \in I} \mathcal{H}_i$ and $\emptyset^*, \mathcal{X}^* \in \bigcap_{i \in I} \mathcal{H}_i$. Let $E_1, E_2, \dots, E_n \in \bigcap_{i \in I} \mathcal{H}_i^*$. Then $E_1, E_2, \dots, E_n \in \mathcal{H}_i^*, \forall i \in I$. Since $\mathcal{H}_i^*$ is a fuzzy $\gamma-$ algebra over a fuzzy set $\mathcal{X}^*$ $\forall i \in I$, then $\bigcup_{k=1}^{n} E_k \in \mathcal{H}_i^*$, $\forall i \in I$. Hence, $\bigcup_{k=1}^{n} E_k \in \bigcap_{i \in I} \mathcal{H}_i^*$, therefore $\bigcap_{i \in I} \mathcal{H}_i^*$ is a fuzzy $\gamma-$ algebra over $\mathcal{X}^*$.

Definition(2.11):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then the intersection of all fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$, which includes $\mathfrak{T}^*$, is said to be the fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$ that is generated by $\mathfrak{T}^*$ and denoted by $\gamma(\mathfrak{T}^*)$, that is,

$$\gamma(\mathfrak{T}^*) = \bigcap \left\{ \begin{array}{c} \mathcal{H}_i^* : \mathcal{H}_i^* \text{ is a fuzzy } \gamma- \text{algebra over a fuzzy set } \mathcal{X}^* \\ \text{and } \mathcal{H}_i^* \supseteq \mathfrak{T}^*, \forall i \in I \end{array} \right\}$$

Proposition (2.12):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\gamma(\mathfrak{T}^*)$ is the smallest fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proof:

From the definition of $\gamma(\mathfrak{T}^*)$, we have:

$$\gamma(\mathfrak{T}^*) = \bigcap \left\{ \begin{array}{c} \mathcal{H}_i^* : \mathcal{H}_i^* \text{ is a fuzzy } \gamma- \text{algebra over a fuzzy set } \mathcal{X}^* \\ \text{and } \mathcal{H}_i^* \supseteq \mathfrak{T}^*, \forall i \in I \end{array} \right\}.$$ Hence $\gamma(\mathfrak{T}^*)$ is fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$. To prove $\gamma(\mathfrak{T}^*) \supseteq \mathfrak{T}^*$. For each $i \in I$ let $\mathcal{H}_i^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$. Then $\mathfrak{T}^* \subseteq \bigcap_{i \in I} \mathcal{H}_i^*$, thus $\mathfrak{T}^* \subseteq \gamma(\mathfrak{T}^*)$. Now, let $\mathcal{H}^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$. Then

$\cap\left\{\begin{matrix}\mathcal{H}_i^*:\mathcal{H}_i^* \text{ is a fuzzy } \gamma-\text{algebra over a fuzzy set} \\ \mathcal{X}^* \text{ and } \mathcal{H}_i^* \supseteq \mathfrak{T}^*, \forall i \in I\end{matrix}\right\} \subseteq \mathcal{H}^*$, hence $\gamma(\mathfrak{T}^*) \subseteq \mathcal{H}^*$. Therefore, $\gamma(\mathfrak{T}^*)$ is the smallest fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Example (2.13):

Let $\mathcal{X}=\{a, b, c\}$ and
$\mathfrak{T}^*=\{\{(a,0.4),(b,0.2),(c,0.3)\},\{(a,0.2),(b,0.3),(c,0.4)\}\}$. Then
$\gamma(\mathfrak{T}^*)=\{\emptyset^*,\{(a,0.4),(b,0.2),(c,0.3)\},\{(a,0.2),(b,0.3),(c,0.4)\},\{(a,0.4),(b,0.3),(c,0.4)\},$
$\mathcal{X}^*=\{(a,1),(b,1),(c,1)\}\}$. Therefore, $\gamma(\mathfrak{T}^*)$ is the smallest fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proposition (2.14):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\gamma(\mathfrak{T}^*)=\mathfrak{T}^*$ if and only if $\mathfrak{T}^*$ is fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$.

Proof:

Let $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\gamma(\mathfrak{T}^*)=\mathfrak{T}^*$. Since $\gamma(\mathfrak{T}^*)$ is a fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$, then $\mathfrak{T}^*$ is fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$.
Conversely, suppose that $\mathfrak{T}^*$ is a fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$. Since $\gamma(\mathfrak{T}^*)$ is a fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$ which includes $\mathfrak{T}^*$, then $\gamma(\mathfrak{T}^*) \supseteq \mathfrak{T}^*$. But $\mathfrak{T}^*$ is fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$ such that $\mathfrak{T}^* \supseteq \mathfrak{T}^*$ and $\gamma(\mathfrak{T}^*)$ is the smallest fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$, then $\gamma(\mathfrak{T}^*) \subseteq \mathfrak{T}^*$ and hence $\gamma(\mathfrak{T}^*)=\mathfrak{T}^*$.

Proposition (2.15):

Every fuzzy $\sigma-$algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$.

Proof:

Let $\mathcal{H}^*$ be a fuzzy $\sigma-$algebra over a fuzzy set $\mathcal{X}^*$. Then by definition of fuzzy $\sigma-$algebra, we have $\emptyset^* \in \mathcal{H}^*$and hence $\emptyset^{*c} \in \mathcal{H}^*$. Since $\emptyset^{*c}=\mathcal{X}^*$, then $\mathcal{X}^* \in \mathcal{H}^*$. Let $E_1, E_2, \ldots, E_n \in \mathcal{H}^*$. Consider, $E_m=\emptyset^*$ for all $m>n$, then we get $E_1, E_2, \ldots, E_n, E_{n+1}, E_{n+2}, \ldots \in \mathcal{H}^*$. Hence, from the definition of fuzzy $\sigma-$algebra we have, $E_1^c, E_2^c, \ldots, E_n^c, E_{n+1}^c, E_{n+2}^c, \ldots \in \mathcal{H}^*$ and $\bigcap_{k=1}^{\infty} E_k^c \in \mathcal{H}^*$, thus $(\bigcap_{k=1}^{\infty} E_k^c)^c \in \mathcal{H}^*$. But $(\bigcap_{k=1}^{\infty} E_k^c)^c=\bigcup_{k=1}^{\infty} E_k$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{H}^*$. So, we have:

$$\begin{aligned}\bigcup_{k=1}^{\infty} E_k &= \bigcup_{k=1}^{n} E_k \cup E_{n+1} \cup E_{n+1} \cup \ldots \\ &= \bigcup_{k=1}^{n} E_k \cup \emptyset^* \cup \emptyset^* \cup \ldots \\ &= \bigcup_{k=1}^{n} E_k\end{aligned}$$

Thus $\bigcup_{k=1}^{n} E_k \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$.

In general, the converse of Proposition (2.15) is not true as shown in the following example:

Example (2.16):

Let $\mathcal{X}=\{a, b, c\}$ and
$\mathcal{H}^*=\{\emptyset^*,\{(a,0.4),(b,0.2),(c,0.3)\},\{(a,0.2),(b,0.3),(c,0.4)\}, \{(a,0.4),(b,0.3),(c,0.4)\},$ $\mathcal{X}^*\}$. Then $\mathcal{H}^*$ is a fuzzy $\gamma-$algebra over a fuzzy set $\mathcal{X}^*$ but not fuzzy $\sigma-$algebra , because $\{(a,0.4),(b,0.2),(c,0.3)\} \in \mathcal{H}^*$, but
$\{(a,0.4),(b,0.2),(c,0.3)\}^c=\{(a,0.6),(b,0.8),(c,0.7)\} \notin \mathcal{H}^*$.

Proposition (2.17):
Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\gamma(\mathfrak{T}^*) \subseteq \sigma(\mathfrak{T}^*)$, where $\sigma(\mathfrak{T}^*)$ is the smallest fuzzy σ−algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proof:
Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\sigma(\mathfrak{T}^*)$ is a fuzzy σ−algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$. Since every fuzzy σ−algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy γ−algebra over a fuzzy set $\mathcal{X}^*$, then $\sigma(\mathfrak{T}^*)$ is a fuzzy γ−algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$. But $\gamma(\mathfrak{T}^*)$ is the smallest fuzzy γ−algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$ this implies that $\gamma(\mathfrak{T}^*) \subseteq \sigma(\mathfrak{T}^*)$.

Remark (2.18):
Every fuzzy algebra over a fuzzy set $\mathcal{X}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$.

Proof:
The result follows from the definition of fuzzy algebra.
In general, the converse of the previous Remark is not true as shown in the following example:

Example(2.19):
Let $\mathcal{X} = \{a, b, c\}$ and $\mathcal{H}^* = \{\emptyset^*, \{(a,0.6),(b,0.3),(c,0.1)\}, \{(a,0.4),(b,0.2),(c,0.4)\}, \{(a,0.6),(b,0.3),(c,0.4)\}, \mathcal{X}^*\}$. Then $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$ but it is not fuzzy algebra , because $\{(a,0.6),(b,0.3),(c,0.1)\} \in \mathcal{H}^*$, but
$\{(a, 0.6), (b, 0.3), (c, 0.1)\}^c = \{(a,0.4),(b,0.7),(c,0.9)\} \notin \mathcal{H}^*$.

Proposition (2.20):
Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$. Then $\gamma(\mathfrak{T}^*) \subseteq \mathrm{AL}(\mathfrak{T}^*)$, where $\mathrm{AL}(\mathfrak{T}^*)$ is the smallest fuzzy algebra over a fuzzy set $\mathcal{X}^*$ that includes $\mathfrak{T}^*$.

Proof:
It is clear, so that it is omitted.

Definition(2.21) :
Let $\mathcal{H}^*$ be a nonempty collection of fuzzy subsets of fuzzy power set $\mathcal{P}^*(\mathcal{X})$ of a nonempty set $\mathcal{X}$ that is $\mathcal{H}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$ that is $\mathcal{T}^* \subseteq \mathcal{X}^*$. Then the restriction of $\mathcal{H}^*$ on $\mathcal{T}^*$ is denoted by $\mathcal{H}^*_{|_{\mathcal{T}^*}}$ which is defined as follows:
$\mathcal{H}^*_{|_{\mathcal{T}^*}} = \{ F : F = E \cap \mathcal{T}^*, \text{ for some } E \in \mathcal{H}^*\}$.

Example(2.22):
Let $\mathcal{X} = \{\omega_1, \omega_2\}$ and $\mathcal{H}^* = \left\{ \begin{matrix} \{(\omega_1, 0.6), (\omega_2, 0.2)\}, \\ \{(\omega_1, 0.3), (\omega_2, 0.5)\} \end{matrix} \right\}$. Consider
$\mathcal{T}^* = \{(\omega_1, 0.5), (\omega_2, 0.5)\}$ and $\mathcal{H}^*_{|_{\mathcal{T}^*}} = \left\{ \begin{matrix} \{(\omega_1, 0.5), (\omega_2, 0.2)\}, \\ \{(\omega_1, 0.3), (\omega_2, 0.5)\} \end{matrix} \right\}$. Put,
$E_1 = \{(\omega_1, 0.6), (\omega_2, 0.2)\}$, $E_2 = \{(\omega_1, 0.3), (\omega_2, 0.5)\}$, $F_1 = \{(\omega_1, 0.5), (\omega_2, 0.2)\}$,
$F_2 = \{(\omega_1, 0.3), (\omega_2, 0.5)\}$. Then $E_1, E_2 \in \mathcal{H}^*$ and $F_1, F_2 \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$.

Now, $E_1 \cap \mathcal{T}^* = \{(\omega_1, Min\{0.6,0.5\}), (\omega_2, Min\{0.2,0.5\})\}$
$= \{(\omega_1, 0.5), (\omega_2, 0.2)\} = F_1$.
$E_2 \cap \mathcal{T}^* = \{(\omega_1, Min\{0.3,0.5\}), (\omega_2, Min\{0.5,0.5\})\}$
$= \{(\omega_1, 0.3), (\omega_2, 0.5)\} = F_2$. This implies that for any $F \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$ there is $E \in \mathcal{H}^*$ such that $F = E \cap \mathcal{T}^*$.
Therefore, $\mathcal{H}^*_{|_{\mathcal{T}^*}}$ is the restriction of $\mathcal{H}^*$ on $\mathcal{T}^*$.

Proposition (2.23):

Let $\mathcal{H}^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$. Then $\mathcal{H}^*_{|_{\mathcal{T}^*}}$ is a fuzzy γ–algebra over a fuzzy set $\mathcal{T}^*$.

Proof:

Since $\mathcal{H}^*$ is a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$, then $\emptyset^*, \mathcal{X}^* \in \mathcal{H}^*$. Since $\mathcal{T}^* \subseteq \mathcal{X}^*$, then $\upsilon_{\mathcal{T}^*}(\omega) \leq \upsilon_{\mathcal{X}^*}(\omega)$ for all $\omega \in \mathcal{X}$ and hence
$\mathcal{X}^* \cap \mathcal{T}^* = \{(\omega, \inf\{\upsilon_{\mathcal{X}^*}(\omega), \upsilon_{\mathcal{T}^*}(\omega)\} : \forall\omega \in \mathcal{X})\} = \{(\omega, \upsilon_{\mathcal{T}^*}(\omega)) : \forall\omega \in \mathcal{X}\} = \mathcal{T}^*$
Therefore, $\mathcal{T}^* \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Now, $\emptyset^* \cap \mathcal{T}^* = \{(\omega, \inf\{\upsilon_{\emptyset^*}(\omega), \upsilon_{\mathcal{T}^*}(\omega)\} : \forall\omega \in \mathcal{X})\}$
$= \{(\omega, \upsilon_{\emptyset^*}(\omega)) : \forall\omega \in \mathcal{X}\} = \emptyset^*$
Then $\emptyset^* \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Let $F_1, F_2, \ldots, F_n \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Then there are $E_1, E_2, \ldots, E_n \in \mathcal{H}^*$ such that $F_k = E_k \cap \mathcal{T}^*$ for all k=1,2,…, n which implies that
$\bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n}(E_k \cap \mathcal{T}^*) = (\bigcup_{k=1}^{n} E_k) \cap \mathcal{T}^*$. Since $\mathcal{H}^*$ is a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$, then $\bigcup_{k=1}^{n} E_k \in \mathcal{H}^*$ and hence by definition of $\mathcal{H}^*_{|_{\mathcal{T}^*}}$ we get $\bigcup_{k=1}^{n} F_k \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Therefore, $\mathcal{H}^*_{|_{\mathcal{T}^*}}$ is a fuzzy γ–algebra over a fuzzy set $\mathcal{T}^*$.

Proposition(2.24):

Let $\mathcal{H}^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$and $E \subseteq \mathcal{T}^* \subseteq \mathcal{X}^*$. If $E \in \mathcal{H}^*$, then $E \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$.

Proof:

Let $\mathcal{H}^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$and let $E \subseteq \mathcal{T}^* \subseteq \mathcal{X}^*$.
Suppose that $E \in \mathcal{H}^*$, since $E \subseteq \mathcal{T}^*$, then $\upsilon_E(\omega) \leq \upsilon_{\mathcal{T}^*}(\omega)$ $\forall\omega \in \mathcal{X}$. So, we have
$E \cap \mathcal{T}^* = \{(\omega, \inf\{\upsilon_E(\omega), \upsilon_{\mathcal{T}^*}(\omega)\} : \forall\omega \in \mathcal{X})\}$
$= \{(\omega, \upsilon_E(\omega)) : \forall\omega \in \mathcal{X}\} = E$
Therefore, $E \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$.

Proposition (2.25):

Let $\mathcal{H}^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$ such that $\mathcal{T}^* \in \mathcal{H}^*$. Then $\{E \subseteq \mathcal{T}^* : E \in \mathcal{H}^*\} \subseteq \mathcal{H}^*_{|_{\mathcal{T}^*}}$.

Proof:

Let $F \in \{E \subseteq \mathcal{T}^* : E \in \mathcal{H}^*\}$. Then $F \subseteq \mathcal{T}^*$ and $F \in \mathcal{H}^*$, hence $\upsilon_F(\omega) \leq \upsilon_{\mathcal{T}^*}(\omega)$ $\forall\omega \in \mathcal{X}$. So, we have $F \cap \mathcal{T}^* = \{(\omega, \inf\{\upsilon_F(\omega), \upsilon_{\mathcal{T}^*}(\omega)\} : \forall\omega \in \mathcal{X})\}$
$= \{(\omega, \upsilon_F(\omega)) : \forall\omega \in \mathcal{X}\} = F$.
Which implies that $F \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Therefore, $\{E \subseteq \mathcal{T}^* : E \in \mathcal{H}^*\} \subseteq \mathcal{H}^*_{|_{\mathcal{T}^*}}$.

Proposition (2.26):

Let $\mathcal{H}^*$ be a fuzzy γ–algebra over a fuzzy set $\mathcal{X}^*$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$ such that $\mathcal{T}^* \in \mathcal{H}^*$ and $G \cap \mathcal{T}^* \in \mathcal{H}^*$ whenever $G \in \mathcal{H}^*$. Then $\mathcal{H}^*_{|_{\mathcal{T}^*}} = \{E \subseteq \mathcal{T}^*: E \in \mathcal{H}^*\}$.

Proof:

Let $F \in \{E \subseteq \mathcal{T}^*: E \in \mathcal{H}^*\}$. Then $F \subseteq \mathcal{T}^*$ and $F \in \mathcal{H}^*$. Hence, $F \cap \mathcal{T}^* = F$, which implies that $F \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Therefore, $\{E \subseteq \mathcal{T}^*: E \in \mathcal{H}^*\} \subseteq \mathcal{H}^*_{|_{\mathcal{T}^*}}$.

Let $F \in \mathcal{H}^*_{|_{\mathcal{T}^*}}$. Then there exists $E \in \mathcal{H}^*$ such that $F = E \cap \mathcal{T}^*$. Since $E, \mathcal{T}^* \in \mathcal{H}^*$, then $E \cap \mathcal{T}^* \in \mathcal{H}^*$, thus $F \in \mathcal{H}^*$. On the other hand,

$F = E \cap \mathcal{T}^*$ which implies that
$$\begin{aligned} \upsilon_F(\omega) &= \upsilon_{E \cap \mathcal{T}^*}(\omega), \forall \omega \in \mathcal{X} \\ &= \inf\{\upsilon_E(\omega), \upsilon_{\mathcal{T}^*}(\omega)\} \le \upsilon_{\mathcal{T}^*}(\omega), \forall \omega \in \mathcal{X}. \end{aligned}$$

Thus $F \subseteq \mathcal{T}^*$ **.** Hence, $F \in \{E \subseteq \mathcal{T}^*: E \in \mathcal{H}^*\}$ implies that $\mathcal{H}^*_{|_{\mathcal{T}^*}} \subseteq \{E \subseteq \mathcal{T}^*: E \in \mathcal{H}^*\}$. Therefore, $\mathcal{H}^*_{|_{\mathcal{T}^*}} = \{E \subseteq \mathcal{T}^*: E \in \mathcal{H}^*\}$.

Proposition(2.27):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$. Then $\gamma(\mathfrak{T}^*)_{|_{\mathcal{T}^*}}$ is a fuzzy γ–algebra over a fuzzy set $\mathcal{T}^*$.

Proof:

The result follows from Proposition(2.12) and Proposition(2.23).

Proposition(2.28):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$. Then $\gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right)$ is the smallest fuzzy γ–algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathfrak{T}^*_{|_{\mathcal{T}^*}}$, where

$$\gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right) = \cap \left\{ \begin{matrix} {\mathcal{H}_i^*}_{|_{\mathcal{T}^*}} : {\mathcal{H}_i^*}_{|_{\mathcal{T}^*}} \text{ is a fuzzy } \gamma\text{– algebra over a fuzzy} \\ \text{set } \mathcal{T}^* \text{ which includes } \mathfrak{T}^*_{|_{\mathcal{T}^*}}, \forall i \in I \end{matrix} \right\}$$

Proof:

From Proposition (2.10), we get $\gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right)$ is a fuzzy γ– algebra over a fuzzy set $\mathcal{T}^*$. To prove that $\gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right) \supseteq \mathfrak{T}^*_{|_{\mathcal{T}^*}}$, suppose that $\mathcal{H}^*_{|_{\mathcal{T}^*}}$ is a fuzzy γ– algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathfrak{T}^*_{|_{\mathcal{T}^*}}$, $\forall i \in I$, then $\mathfrak{T}^*_{|_{\mathcal{T}^*}} \subseteq \cap \left\{ \begin{matrix} {\mathcal{H}_i^*}_{|_{\mathcal{T}^*}} : {\mathcal{H}_i^*}_{|_{\mathcal{T}^*}} \text{ is a fuzzy } \gamma\text{– algebra over a} \\ \text{fuzzy set } \mathcal{T}^* \text{which includes } \mathfrak{T}^*_{|_{\mathcal{T}^*}}, \forall i \in I \end{matrix} \right\}$.

Hence, $\gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right) \supseteq \mathfrak{T}^*_{|_{\mathcal{T}^*}}$. Now, let ${\mathcal{H}^*}_{|_{\mathcal{T}^*}}$ is a fuzzy γ– algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathfrak{T}^*_{|_{\mathcal{T}^*}}$. Then ${\mathcal{H}^*}_{|_{\mathcal{T}^*}} \supseteq \gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right)$. Therefore $\gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right)$ is the smallest fuzzy γ–algebra over a fuzzy set $\mathcal{T}^*$ which contains $\mathfrak{T}^*_{|_{\mathcal{T}^*}}$.

Lemma(2.29):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{T}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$, define a collection $\mathcal{H}^*$ as:

$\mathcal{H}^* = \{E \subseteq \mathcal{X}^* : E \cap \mathcal{T}^* \in \gamma\left({\mathfrak{T}^*}_{|\mathcal{T}^*}\right)\}$. Then $\mathcal{H}^*$ is a fuzzy γ– algebra over a fuzzy set $\mathcal{X}^*$.

Proof:

Since $\gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$ is fuzzy γ−algebra over a fuzzy set $\mathcal{T}^*$, then $\emptyset^*, \mathcal{T}^* \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$. Now $\mathcal{T}^* \subseteq \mathcal{X}^*$, then $\mathcal{T}^* = \mathcal{X}^* \cap \mathcal{T}^*$ and $\mathcal{X}^* \in \mathcal{H}^*$. Also $\emptyset^* = \emptyset^* \cap \mathcal{T}^*$, then $\emptyset^* \in \mathcal{H}^*$. Let $\mathrm{E}_1, \mathrm{E}_2, \dots, \mathrm{E}_n \in \mathcal{H}^*$. Then $(\mathrm{E}_i \cap \mathcal{T}^*) \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$, for all i=1,2,…,n. Hence, $(\bigcup_{i=1}^{n} \mathrm{E}_i \cap \mathcal{T}^*) \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$, thus $\bigcup_{i=1}^{n} \mathrm{E}_i \in \mathcal{H}^*$. Therefore, $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$.

Theorem (2.30):

Let $\mathcal{X}$ be a nonempty set and $\mathfrak{X}^* \subseteq \mathcal{P}^*(\mathcal{X})$ and let $\mathcal{T}^*$ be a nonempty fuzzy subset of a fuzzy set $\mathcal{X}^*$. Then $\gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*}) = \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$.

Proof:

Let $\mathrm{F} \in \mathfrak{X}^*{}_{|\mathcal{T}^*}$, then by Definition(2.21) $\mathrm{F} = \mathrm{E} \cap \mathcal{T}^*$, for some $\mathrm{E} \in \mathfrak{X}^*$. But $\mathfrak{X}^* \subseteq \gamma(\mathfrak{X}^*)$, then $\mathrm{E} \in \gamma(\mathfrak{X}^*)$, thus $\mathrm{F} \in \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$, hence $\mathfrak{X}^*{}_{|\mathcal{T}^*} \subseteq \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$. By Proposition(2.28), we have $\gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$ is the smallest fuzzy γ−algebra over a fuzzy set $\mathcal{T}^*$ which includes $\mathfrak{X}^*_{|_{\mathcal{T}^*}}$. From Proposition(2.27), $\gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$ is fuzzy γ−algebra over a fuzzy set $\mathcal{T}^*$, then $\gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*}) \subseteq \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$. Now, define a collection $\mathcal{H}^*$ as: $\mathcal{H}^* = \{\mathrm{E} \subseteq \mathcal{X}^* : \mathrm{E} \cap \mathcal{T}^* \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})\}$. Then $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$ by Lemma (2.29). Let $\mathrm{G} \in \mathfrak{X}^*$, then $\mathrm{G} \cap \mathcal{T}^* \in \mathfrak{X}^*{}_{|\mathcal{T}^*}$, but $\mathfrak{X}^*{}_{|\mathcal{T}^*} \subseteq \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$ implies that $\mathrm{G} \cap \mathcal{T}^* \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$, hence by definition of $\mathcal{H}^*$ we get $\mathrm{G} \in \mathcal{H}^*$ and $\mathfrak{X}^* \subseteq \mathcal{H}^*$. Let $\mathrm{F} \in \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$. Then $\mathrm{F} = \mathrm{E} \cap \mathcal{T}^*$, for some $\mathrm{E} \in \gamma(\mathfrak{X}^*)$. $\gamma(\mathfrak{X}^*)$ is the smallest fuzzy γ−algebra over a fuzzy set $\mathcal{X}^*$ which includes $\mathfrak{X}^*$ by Proposition (2.12) and $\mathcal{H}^*$ is a fuzzy γ− algebra over a fuzzy set $\mathcal{X}^*$ which contains $\mathfrak{X}^*$, then $\gamma(\mathfrak{X}^*) \subseteq \mathcal{H}^*$ and $\mathrm{E} \in \mathcal{H}^*$, hence by definition of $\mathcal{H}^*$ we get $\mathrm{E} \cap \mathcal{T}^* \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$, thus $\mathrm{F} \in \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$, consequently $\gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}} \subseteq \gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*})$. Therefore, $\gamma(\mathfrak{X}^*{}_{|\mathcal{T}^*}) = \gamma(\mathfrak{X}^*)_{|_{\mathcal{T}^*}}$.

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