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Solvability for Optimal Classical Continuous Control Problem Controlling by Quaternary Hyperbolic Boundary Value Problem

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Abstract

This work is concerned with studying the solvability for optimal classical continuous control quaternary vector problem that controls by quaternary linear hyperbolic boundary value problem. The existence of the unique quaternary state vector solution for the quaternary linear hyperbolic boundary value problem is studied and demonstrated by employing the method of Galerkin, where the classical continuous control quaternary vector is Known. Also, the existence theorem of an optimal classical continuous control quaternary vector related to the quaternary linear hyperbolic boundary value problem is demonstrated. The existence of a unique solution to the adjoint quaternary linear hyperbolic boundary value problem associated with the quaternary linear hyperbolic boundary value problem is formulated and studied. The directional derivative for the cost functional is derived. Finally, the necessary optimality theorem for the optimal classical continuous control quaternary vector is proved.

Keywords: Optimal classical continuous Quaternary control, Quaternary linear hyperbolic boundary value problem, necessary optimality theorem.

قابلية الحل لمسألة السيطرة الامثلية التقليدية المستمرة المسيطرة بمسالة قيم حدودية زائدية رباعية

جميل امير علي الهواسي ، ميادة عبد الستار حسن
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الخلاصه

يتناول هذا البحث دراسة قابلية الحل لمسالة السيطرة الامثلية التقليدية المستمرة المسيطرة بمسالة قيم حدودية زائدية خطية رباعية. تم ذكر نص وبرهان مبرهنة وجود متجه الحالة الرباعي لمسالة القيم الحدودية الخطية الرباعية عندما يكون متجه االسيطرة التقليدية المستمرة معلوما باستخدام طريقة كاليركن. كذلك تم برهان مبرهنة وجود متجه سيطرة امثلية تقليدية مستمرة. تمت دراسة وجود حل وحيد للمسألة المرافقة لمسألة القيم الحدودية الزائدية الخطية الرباعية. تم ايجاد مشتقة الاتجاهية لدالة الهدف و كذلك تم برهان مبرهنة الشروط الضرورية لوجود سيطرة امثلية للمسالة.

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1. Introduction

Different applications in the real life are classified as an optimal control problems. For example in robots, medicine, economic, engineering, pharmacy, chemistry, electromagnetic and many other fields [1- 7]. In the field of applied mathematics, many researchers [8-3] studied optimal control problems that are controlled by ODEs or PDEs of parabolic, hyperbolic and elliptic type, couple of these types, while some of them investigated optimal control problems that controlled by triple linear PDEs of the mentioned three types[14-16].

In general, the study of the study for the optimal classical continuous control vector problem controlled by triple linear PDEs of the hyperbolic type [15] encourages us to generalize this problem to a new proposed optimal classical continuous control quaternary vector problem controlling by quaternary linear hyperbolic boundary value problem.

In this paper, we investigate the state and proof the existence theorem of a unique solution quaternary state vector solution for the quaternary linear hyperbolic boundary value problem employing the method of Galerkin when the classical continuous control quaternary vector is known. Further, the existence theorem for an optimal classical continuous control quaternary vector is stated and proved. In addition, the existence of a unique solution of the adjoint quaternary linear hyperbolic boundary value problem related to the quaternary linear hyperbolic boundary value problem is formulated. We also derive the directional derivative for the cost functional. Finally, the necessary optimality theorem for the problem is proved.

2. Problem Description:

Let $\Omega \subset \mathbb{R}^2$, $x = (x_1, x_2)$, $Q = I \times \Omega, I = [0, T]$, $\Gamma = \partial\Omega$ and $\Sigma = \Gamma \times I$. The optimal classical continuous control quaternary vector problem includes of the quaternary state equations that means the following quaternary linear hyperbolic boundary value problem:

$$y_{1tt} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x,t) + u_1, \text{ in } Q \tag{1}$$

$$y_{2tt} - \Delta y_2 + y_1 + y_2 - y_3 - y_4 = f_2(x,t) + u_2, \text{ in } Q \tag{2}$$

$$y_{3tt} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x,t) + u_3, \text{ in } Q \tag{3}$$

$$y_{4tt} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x,t) + u_4, \text{ in } Q \tag{4}$$

with the following boundary conditions and the initial conditions:

$$y_1(x,t) = 0, \text{ on } \Sigma \tag{5}$$

$$y_2(x,t) = 0, \text{ on } \Sigma \tag{6}$$

$$y_3(x,t) = 0, \text{ on } \Sigma \tag{7}$$

$$y_4(x,t) = 0, \text{ on } \Sigma \tag{8}$$

$$y_1(x,0) = y_1^0(x), \text{and } y_{1t}(x,0) = y_1^1(x), \text{ in } \Omega \tag{9}$$

$$y_2(x,0) = y_2^0(x), \text{and } y_{2t}(x,0) = y_2^1(x), \text{ in } \Omega \tag{10}$$

$$y_3(x,0) = y_3^0(x), \text{and } y_{3t}(x,0) = y_3^1(x), \text{ in } \Omega \tag{11}$$

$y_4(x,0) = y_4^0(x)$,and $y_{4t}(x,0) = y_4^1(x)$, in Ω (12)

where $(f_1, f_2, f_3, f_4) \in L^2(\mathrm{Q})=(L^2(\mathrm{Q}))^4$ is a function given vector for each $(x_1, x_2) \in \Omega$, $\vec{u} = (u_1, u_2, u_3, u_4) \in L^2(\mathrm{Q})$ is a given classical continuous control quaternary vector and the corresponding quaternary state vector solution is $\vec{y} = (y_1, y_2, y_3, y_4) \in H^2(\Omega) = (H^2(\Omega))^4$.

The set of admissible classical continuous control quaternary vector is:
$\overrightarrow{W_A} = \{ \vec{u} \in L^2(\mathrm{Q}) | \vec{u} \in \vec{\mathrm{U}} = (U_1 \times U_2 \times U_3 \times U_4) \subset \mathbb{R}^4,$ a.e. in Q$\}$, $\vec{U}$is a convex and compact set
The cost functional will be given by

$$G_0(\vec{u}) = \frac{1}{2} \sum_{i=1}^{4} \left(\| y_i - y_{id} \|_Q^2 + \frac{\beta}{2} \| u_i \|_Q^2 \right), \beta > 0 \quad (13)$$

Let$V = H_0^1(\Omega)$, $\vec{\mathrm{V}} = (V)^4 = \{\vec{v}: \vec{v} = (v_1, v_2, v_3, v_4) \in H^1(\Omega), v_1 = v_2 = v_3 = v_4 =$ 0 on $\partial\Omega\}$. The weak form of ((1)-(12)) is given by

$$(y_{1tt}, v_1) - (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1 + u_1, v_1) \quad (14a)$$

$$(y_1^0, v_1) = (y_1(0), v_1), \text{ and } (y_{1t}^1, v_1) = (y_{1t}(0), v_1) \quad (14b)$$

$$(y_{2tt}, v_2) - (\Delta y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2 + u_2, v_2) \quad (15a)$$

$$(y_2^0, v_2) = (y_2(0), v_2), \text{ and } (y_{2t}^1, v_2) = (y_{2t}(0), v_2) \quad (15b)$$

$$(y_{3tt}, v_3) - (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3 + u_3, v_3) \quad (16a)$$

$$(y_3^0, v_3) = (y_3(0), v_3), \text{ and } (y_{3t}^1, v_3) = (y_{3t}(0), v_3) \quad (16b)$$

$$(y_{4tt}, v_4) - (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4 + u_4, v_4) \quad (17a)$$

$$(y_4^0, v_2) = (y_4(0), v_4), \text{ and } (y_{4t}^1, v_4) = (y_{4t}(0), v_4) \quad (17b)$$

Assumption 2.1: The function f_i $(\forall i = 1,2,3,4)$ is satisfied the following condition: $| f_i | \leq \eta_i(x,t), \forall (x,t) \in Q, \eta_i \in L^2(\mathrm{Q}, \mathbb{R})$

3. The weak form Solution:

Theorem 3.1 (Existence of a Unique Solution): With assumption 2.1 for each given classical continuous control quaternary vector, $\vec{u} \in L^2(\mathrm{Q})$, the weak form ((14)-(17)) has a unique solution $\vec{y} = (y_1, y_2, y_3, y_4) \in L^2(\mathrm{I} \times \mathrm{V})$ and $\overrightarrow{y_t} = (y_{1t}, y_{2t}, y_{3t}, y_{4t}) \in L^2(\mathrm{I} \times \mathrm{V}^*)$.

Proof: Let $\overrightarrow{V_n} = (V_n)^4 \subset \vec{V}$ be a set of piecewise affine function in Ω *and* $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of $\vec{V}$ such that for all $\vec{v} = (v_1, v_2, v_3, v_4) \in \vec{V}$, there is a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n}) \in \vec{V}_n$, $\forall n$ and $\vec{v}_n \to \vec{v}$ (strongly) in $\vec{V}$,then $\vec{v}_n \to \vec{v}$ (strongly) in $(L^2(\Omega))^4$. Let $\{\vec{v_j} = (v_{1j}, v_{2j}, v_{3j}, v_{4j}): j = 1,2, \dots, M(n)\}$ be a finite basis of $\overrightarrow{V_n}$ (where $\vec{v_j}$ is piecewise affine function in Ω, with $\vec{v_j}(x) = 0$ on the boundary Γ) and let $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})$ be the Galerkin approximate solution to the exact solution $\vec{y} = (y_1, y_2, y_3, y_4)$ such that

$$y_{in} = \sum_{j=1}^{n} c_{ij}(t) v_{ij}(x) \quad (18a)$$

$$z_{in} = \sum_{j=1}^{n} d_{ij}(t) v_{ij}(x) \quad (18b)$$

Where $c_{ij}(t)$ and $d_{ij}(t)$ are unknown function of t, for all $i = 1,2,3,4$, $j = 1,2, \dots, n$. The method of Galerkin is utilized to approximate the weak form ((14)-(17)) w.r.t. x, they become after substituting $y_{int} = z_{in}$:

$(z_{1nt}, v_1) - (\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) + (y_{3n}, v_1) + (y_{4n}, v_1)$
$= (f_1 + u_1, v_1), \forall v_1 \in V_n$ (19a)
$(y_{1n}^0, v_1) = (y_1^0, v_1)$, and $(z_{1n}^1, v_1) = (y_1^1, v_1)$ (19b)
$(z_{2nt}, v_2) - (\nabla y_{2n}, \nabla v_2) + (y_{1n}, v_2) + (y_{2n}, v_2) - (y_{3n}, v_2) - (y_{4n}, v_2)$
$= (f_2 + u_2, v_2)$, $\forall v_2 \in V_n$ (20a)
$(y_{2n}^0, v_2) = (y_2^0, v_2)$, and $(z_{2n}^1, v_2) = (y_2^1, v_2)$ (20b)
$(z_{3nt}, v_3) - (\nabla y_{3n}, \nabla v_3) - (y_{1n}, v_3) + (y_{2n}, v_3) + (y_{3n}, v_3) + (y_{4n}, v_3)$
$= (f_3 + u_3, v_3)$, $\forall v_3 \in V_n$ (21a)
$(y_{3n}^0, v_3) = (y_3^0, v_3)$, and $(z_{3n}^1, v_3) = ((y_3^1, v_3)$ (21b)
$(z_{4nt}, v_4) - (\nabla y_{4n}, \nabla v_4) - (y_{1n}, v_4) + (y_{2n}, v_4) - (y_{3n}, v_4) + (y_{4n}, v_4)$
$= (f_4 + u_4, v_4), \quad \forall v_4 \in V_n$ (22a)
$(y_{4n}^0, v_2) = (y_4^0, v_4)$, and $(z_{4n}^1, v_4) = ((y_4^1, v_4)$ (22b)

Where $y_{in}^0 = y_{in}^0(x) = y_{in}(x, 0) \in V_n$ ($z_{in}^0 = y_{in}^1 = y_{in}^1(x) = y_{int}(x, 0) \in L^2(\Omega)$) is the projection of y_i^0 onto V (the projection of $y_i^1 = y_{it}$ on to $L^2(\Omega)$) $\forall i = 1,2,3,4$, i.e.,
$y_{in}^0 \to y_i^0$ ST in V, with $\| \vec{y}_n^0 \| \leq b_0$ (23)
$y_{in}^1 \to y_i^1$ ST in $L^2(\Omega)$, with $\| \vec{y}_n^1 \| \leq b_1$ (24)

Substituting (18a and b) with $i = 1,2,3,4$) in ((19)-(22)) and setting $v_i = v_{il}$,$\forall l = 1,2, \dots, n$, the obtained equations are equivalent to the following linear system of first order ODEs with initial conditions , which has a unique solution, that means

$A_1 D_1(t) + B_1 C_1(t) - E C_2(t) + F C_3(t) + K C_4(t) = b_1$ (25a)
$A_1 C_1(0) = b_1^0$ and $A_1 \overline{D_1}(0) = b_1^1$ (25b)
$A_2 D_2(t) + B_2 C_2(t) + H C_1(t) - G C_3(t) + D C_4(t) = b_2$ (26a)
$A_2 C_2(0) = b_2^0$ and $A_2 \overline{D_2}(0) = b_2^1$ (26b)
$A_3 D_3(t) + B_3 C_3(t) - R C_1(t) + W C_2(t) + Z C_4(t) = b_3$ (27a)
$A_3 C_3(0) = b_3^0$ and $A_3 \overline{D_3}(0) = b_3^1$ (27b)
$A_4 D_4(t) + B_4 C_4(t) - T C_1(t) + M C_2(t) - N C_3(t) = b_4$ (28a)
$A_4 C_4(0) = b_4^0$ and $A_4 \overline{D_4}(0) = b_4^1$ (28b)

where $A_i = (a_{ilj})_{n\times n}$, $a_{ilj} = (v_{ij}, v_{il})$, $B_i = (b_{ilj})_{n\times n}$, $b_{ilj} = (\nabla v_{ij}, \nabla v_{il}) + (v_{ij}, v_{il})$, $E = (e_{lj})_{n\times n}$,$e_{lj} = (v_{2j}, v_{1l})$, $F = (f_{lj})_{n\times n}, f_{lj} = (v_{3j}, v_{1l})$, $G = (g_{lj})_{n\times n}, g_{lj} = (v_{3j}, v_{2l})$, $H = (h_{lj})_{n\times n}, h_{lj} = (v_{1j}, v_{2l})$, $R = (r_{lj})_{n\times n}, r_{lj} = (v_{1j}, v_{3l})$, $W = (w_{lj})_{n\times n}, w_{lj} = (v_{2j}, v_{3l})$, $K = (k_{lj})_{n\times n}, k_{lj} = (v_{4j}, v_{1l})$, $D = (d_{lj})_{n\times n}, d_{lj} = (v_{4j}, v_{2l})$, $Z = (z_{lj})_{n\times n}$, $z_{lj} = (v_{4j}, v_{3l})$, $T = (t_{lj})_{n\times n}, t_{lj} = (v_{1j}, v_{4l})$, $M = (m_{lj})_{n\times n}, m_{lj} = (v_{2j}, v_{4l})$, $N = (n_{lj})_{n\times n}$, $n_{lj} = (v_{3j}, v_{4l})$, $b_{il}^0 = (y_i^0, v_{il})$, $b_i^0 = (b_{il}^0)$, $\overline{D_\iota}(0) = (\overline{D_{\iota j}}(0))_{n\times 1}$, $D(0) = (D_{ij}(0))_{n\times 1}$, $b_{il} = (f_i + u_i, v_{il})$, $D_i(t) = (D_{ij}(t))_{n\times 1}$.

Then corresponding to the sequence $\{\overrightarrow{V_n}\}$, the following problems hold that means for each $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n} , v_{4n}) \subset \overrightarrow{V_n}$, and $n = 1,2, \dots$
$(y_{1ntt}, v_{1n}) + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n})$
$= (f_1 + u_1, v_{1n})$ (29a)
$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n})$, and $(y_{1n}^1, v_{1n}) = (y_1^1, v_{1n})$ (29b)
$(y_{2ntt}, v_{2n}) + (\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_2) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n})$
$= (f_2 + u_2, v_{2n})$ (30a)
$(y_{2n}^0, v_{2n}) = (y_1^0, v_{2n})$, and $(y_{2n}^1, v_{2n}) = (y_1^1, v_{2n})$ (30b)
$(y_{3ntt}, v_{3n}) + (\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n})$

$= (f_3 + u_3, v_{3n})$ (31a)

$(y_{3n}^0, v_{3n}) = (y_3^0, v_{3n})$, and $(y_{3n}^1, v_3) = (y_3^1, v_{3n})$ (31b)

$(y_{4ntt}, v_{4n}) + (\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n})$
$= (f_4 + u_4, v_{4n})$ (32a)

$(y_{4n}^0, v_{4n}) = (y_4^0, v_{4n})$, and $(y_{4n}^1, v_{4n}) = (y_4^1, v_{4n})$ (32b)

Which has a sequence of unique solution $\{\vec{y}_n\}$. Substituting $v_{in} = y_{int}$, for $i = 1,2,3,4$ in ((29a- (32a)) and we use Lemma 1.2 in [17] for the first term of the LHS once get

$\frac{d}{dt} \| y_{nt} \|_0^2 + \frac{d}{dt} \| y_n \|_1^2 = 2[(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{3nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt}) + (f_1 + u_1, y_{1nt}) + (f_2 + u_2, y_{2nt}) + (f_3 + u_3, y_{3nt}) + (f_4 + u_4, y_{4nt})]$ (33)

Taking the absolute value, then it yields to:

$\frac{d}{dt}[\| y_{nt} \|_0^2 + \| y_n \|_1^2] \le 2[| (y_{2n}, y_{1nt} | + | (y_{3n}, y_{1nt}) | + | (y_{4n}, y_{1nt}) | + |(y_{1n}, y_{2nt})| + |(y_{3n}, y_{2nt})| + |(y_{4n}, y_{2nt})| + |(y_{1n}, y_{3nt})| + |(y_{2n}, y_{3nt})| + | (y_{4n}, y_{3nt}) | + | (y_{1n}, y_{4nt}) | + | (y_{2n}, y_{4nt}) | + | (y_{3n}, y_{4nt}) | + | (f_1 + u_1, y_{1nt}) | + | (f_2 + u_2, y_{2nt}) | + | (f_3 + u_3, y_{3nt}) | + | (f_4 + u_4, y_{4nt}) |]$ (34)

Using the Cauchy- Schwartz inequality for the R.H.L. of (34), integrating both sides on $[0, t]$, using
$\| y_{in} \|_0 \le \| y_{in} \|_1 \le \| \vec{y}_n \|_1$, $\| y_{int} \|_0 \le \| \vec{y}_n \|_0$, and Ass. 2.1, to get

$\int_0^t \frac{d}{dt}[\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2]dt \le 3 \int_0^t [\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2]dt + \int_0^t \sum_{i=1}^4 [\| y_i \|_0^2 + \| u_{ii} \|_0^2]dt$
$+ \int_0^t [\| \vec{y}_{nt} \| \; dt] \le \sum_{i=1}^4 \| y_i \|_0^2 + b_i^2 + \alpha_1 \int_0^t \frac{d}{dt}[\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2]dt$
$\le \alpha_2 + \alpha_1 \int_0^t \frac{d}{dt}[\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2]dt$ (35)

where $\alpha_2 = \sum_{i=1}^4 (b_1 + \widetilde{b_i})$, $\alpha_1 \ge 4$ with $\| y_i \|_Q^2 \le \overline{b_i}$, $\| u_i \|_Q \le \widetilde{b_i}$ for each $i = 1,2,3,4$. Since $\| \vec{y}_n^0 \|_1 \le b_1$ and $\| \vec{y}_n^1 \|_0 \le b_0$ with $\alpha_3 = b_0 + b_1 + \alpha_2$, inequality (35) becomes

$\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n(t) \|_1^2 \le \alpha_3 + \alpha_1 \int_0^t [\| \vec{y}_{nt} \|_0^2 + \| \vec{y}_n \|_1^2]dt$

Using the Belman-Gronwall inequality to get for all $t \in [0, t]$
$\| \vec{y}_{nt}(t) \|_0^2 + \| \vec{y}_n(t) \|_1^2 \le \alpha_3 e^{\alpha_1} = b^2(c) \Longrightarrow \| \vec{y}_{nt}(t) \|_Q^2 \le b^2(c)$ and $\| \vec{y}_n(t) \|_1^2 \le b^2(c)$,
they give that $\| \vec{y}_{nt}(t) \|_Q \le b_1(c)$ and $\| \vec{y}_n(t) \|_{L^2(I,V)} \le b(c)$.

Then, applying the Alaoglu's theorem, then there is a subsequence of $\{\vec{y}_n\}_{n \in N}$, for simplicity say $\{\vec{y}_n\}$ s.t.

$\vec{y}_{nt} \to \vec{y}$ (weakly) in $L^2(Q)$ and $\vec{y}_n \to \vec{y}$ (weakly) in $L^2(I, V)$ (36)

Now, multiplying both sides of ((29a)- (32a)) by $\phi_i(t) \in C^2[0, T]$ s.t. $\phi_i(T) = \phi_i'(T) = 0$, $\phi_i(0) \ne 0$, $\phi_i'(0) \ne 0$, $\forall i = 1,2,3,4$, integrating on $[0, T]$, finally integrating by parts twice the first term in each obtained equations yields to

$$-\int_0^T \frac{d}{dt}(y_{1n},v_{1n})\phi_1' dt+\int_0^T [(\nabla y_{1n},\nabla v_{1n})+(y_{1n},v_{1n})-(y_{2n},v_{1n})+(y_{3n},v_{1n})+(y_{4n},v_{1n})]\phi_1(t)dt=\int_0^T (f_1+u_1,v_{1n})\phi_1(t)dt+(y_{1n}',v_{1n})\,\phi_1(0) \quad (37)$$

$$\int_0^T (y_{1n},v_{1n})\phi_1'' dt+\int_0^T [(\nabla y_{1n},\nabla v_{1n})+(y_{1n},v_{1n})-(y_{2n},v_{1n})+(y_{3n},v_{1n})+(y_{4n},v_{1n})]\phi_1(t)dt=\int_0^T (f_1+u_1,v_{1n})\phi_1(t)dt+(y_{1n}',v_{1n})\phi_1(0)-(y_{1n}^0,v_{1n})\phi_1'(0) \quad (38)$$

$$-\int_0^T \frac{d}{dt}(y_{2n},v_{2n})\phi_2' dt+\int_0^T [(\nabla y_{2n},\nabla v_{2n})+(y_{1n},v_{2n})+(y_{2n},v_{2n})-(y_{3n},v_{2n})-(y_{4n},v_{2n})]\phi_2(t)dt=\int_0^T (f_2+u_2,v_{2n})\phi_2(t)dt+(y_{2n}',v_{2n})\,\phi_1(0) \quad (39)$$

$$\int_0^T (y_{2n},v_{2n})\phi_2''(t) dt+\int_0^T [(\nabla y_{2n},\nabla v_{2n})+(y_{1n},v_{2n})+(y_{2n},v_{2n})-(y_{3n},v_{2n})-(y_{4n},v_{2n})]\phi_2(t)dt=\int_0^T (f_2+u_2,v_{2n})\phi_2(t)dt+(y_{2n}',v_{2n})\phi_2(0)-(y_{2n}^0,v_{2n})\phi_2'(0) \quad (40)$$

$$-\int_0^T \frac{d}{dt}(y_{3n},v_{3n})\phi_3'(t) dt+\int_0^T [(\nabla y_{3n},\nabla v_{3n})-(y_{1n},v_{3n})+(y_{2n},v_{3n})+(y_{3n},v_{3n})+(y_{4n},v_{3n})]\phi_3(t)dt=\int_0^T (f_3+u_3,v_{3n})\phi_3(t)dt+(y_{3n}',v_{3n})\,\phi_3(0) \quad (41)$$

$$\int_0^T (y_{3n},v_{3n})\phi_3'' dt+\int_0^T [(\nabla y_{3n},\nabla v_{3n})-(y_{1n},v_{3n})+(y_{2n},v_{3n})+(y_{3n},v_{3n})+(y_{4n},v_{3n})]\phi_3(t)dt=\int_0^T (f_3+u_3,v_{3n})\phi_3(t)dt+(y_{3n}',v_{3n})\phi_3(0)-(y_{3n}^0,v_{3n})\phi_3'(0) \quad (42)$$

$$-\int_0^T \frac{d}{dt}(y_{4n},v_{4n})\phi_4'(t) dt+\int_0^T [(\nabla y_{4n},\nabla v_{4n})-(y_{1n},v_{4n})+(y_{2n},v_{4n})-(y_{3n},v_{4n})+(y_{4n},v_{4n})]\phi_4(t)dt=\int_0^T (f_4+u_4,v_{4n})\phi_4(t)dt+(y_{4n}',v_{4n})\,\phi_4(0) \quad (43)$$

$$\int_0^T (y_{4n},v_{4n})\phi_4'' dt+\int_0^T [(\nabla y_{4n},\nabla v_{4n})-(y_{1n},v_{4n})+(y_{2n},v_{4n})-(y_{3n},v_{4n})+(y_{4n},v_{4n})]\phi_4(t)dt=\int_0^T (f_4+u_4,v_{4n})\phi_4(t)dt+(y_{4n}',v_{4n})\phi_4(0)-(y_{4n}^0,v_{4n})\phi_4'(0) \quad (44)$$

First, since

$v_{in}\to v_i$ (strongly) in $V \Longrightarrow \begin{cases} v_{in}\phi_i(t)\to v_i\phi_i(t) \\ v_{in}\phi_i'(t)\to v_i\phi_i'(t)\end{cases}$ (strongly) in $L^2(\mathrm{I,V})$, and $v_{in}\phi_i(0)\to v_i\phi_i(0)$ (strongly) in $L^2(\Omega)$.

$v_{in}\to v_i$ (strongly) in $L^2(\Omega) \Longrightarrow \begin{cases} v_{in}\phi_i'(t)\to v_i\phi_i'(t) \\ v_{in}\phi_i''(t)\to v_i\phi_i''(t)\end{cases}$ (strongly) in $L^2(\mathrm{Q})$ and $v_{in}\phi_i'(0)\to v_i\phi_i'(0)$ ST in $L^2(\Omega)$ for $i=1,2,3,4$.

Second, $y_{int}\to y_{it}$ (weakly) in $L^2(\mathrm{Q})$ and $y_{int}\to y_{it}$ (weakly) in $L^2(\mathrm{I,V})$ and (strongly) in $L^2(\mathrm{Q})$.

Third, since $v_{in}\phi_i\to v_i\phi_i$ (weakly) in $L^2(\mathrm{I,V})$, then

$$\int_0^T (f_i+u_i,v_{in})\phi_i(t)dt\to\int_0^T (f_i+u_i,v_i)\phi_i(t)dt, \forall i=1,2,3,4$$

From these convergences, (23) and (24) we can passage the limits in ((37)-(44)), to get

$$-\int_0^T (y_{1t}, v_1)\phi_1'(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)]\phi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\phi_1(t)dt + (y_1', v_1)\,\phi_1(0) \quad (45)$$

$$\int_0^T (y_1, v_1)\phi_1''dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)]\phi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\phi_1(t)dt + (y_1', v_1)\phi_1(0) - (y_1^0, v_1)\phi_1'(0) \quad (46)$$

$$-\int_0^T (y_{2t}, v_2)\phi_2'dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)]\phi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\phi_2(t)dt + (y_2', v_2)\,\phi_1(0) \quad (47)$$

$$\int_0^T (y_2, v_2)\phi_2''(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)]\phi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\phi_2(t)dt + (y_2', v_2)\phi_2(0) - (y_2^0, v_2)\phi_2'(0) \quad (48)$$

$$-\int_0^T (y_{3t}, v_3)\phi_3'(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)]\phi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\phi_3(t)dt + (y_3', v_3)\,\phi_3(0) \quad (49)$$

$$\int_0^T (y_3, v_3)\phi_3''dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)]\phi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\phi_3(t)dt + (y_3', v_3)\phi_3(0) - (y_3^0, v_3)\phi_3'(0) \quad (50)$$

$$-\int_0^T (y_{4t}, v_4)\phi_4'(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)]\phi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\phi_4(t)dt + (y_4', v_4)\,\phi_4(0) \quad (51)$$

$$\int_0^T (y_4, v_4)\phi_4''dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)]\phi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\phi_4(t)dt + (y_4', v_4)\phi_4(0) - (y_4^0, v_4)\phi_4'(0) \quad (52)$$

Now, we have three cases

Case1: Choose $\phi_i \in C^2[0,T]$ s.t. $\phi_i(0) = \phi_i'(0) = \phi_i'(T) = \phi_i(T) = 0$, $\forall i = 1,2,3,4$ in (46), (48), (50), (52), integrating by parts twice the first terms in the LHS, i.e.

$$\int_0^T (y_{1tt}, v_1)\phi_1 dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1)]\phi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\phi_1(t)dt \quad (53)$$

$$\int_0^T (y_{2tt}, v_2)\phi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2)]\phi_2(t)dt$$
$$=\int_0^T (f_2 + u_2, v_2)\phi_2(t)dt) \quad (54)$$
$$\int_0^T (y_{3tt}, v_3)\phi_3 dt + \int_0^T [(\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3)]\phi_3(t)dt$$
$$=\int_0^T (f_3 + u_3, v_3)\phi_3(t)dt \quad (55)$$
$$\int_0^T (y_{4tt}, v_4)\phi_4 dt + \int_0^T [(\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4)]\phi_4(t)dt$$
$$= \int_0^T (f_4 + u_4, v_4)\phi_4(t)dt. \quad (56)$$

Hence, $\vec{y}$ is a solution of ((14a)-(17a)) which is almost everywhere on I

Case2: Choose $\phi_i \in C^2[0,T]$ s.t. $\phi_i(T) = 0, \& \phi_i(0) = 0,$ for all $i = 1,2,3,4$, multiplying both sides of (14a), (15a), (16a) and (17a) by $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, and $\phi_4(t)$, integrating on $[0,T]$, integrating by parts the first term in the LHS of each obtained equation, then subtracting each one of these obtained equation from (45),(47),(49) and (51) , once get

$(y_{it}(0), v_i)\phi_i(0) = (y_i'(0), v_i)\phi_i(0)$, for all $i = 1,2,3,4$.

Case 3: Choose $\phi_i \in C^2[0,T]$ such that $\phi_i(0) = \phi_i'(T) = \phi_i(T) = 0, \phi_i'(0) \neq 0$
, for all $i = 1,2,3,4$. W multiply both sides of (14a), (15a), (16a) and (17a) by $\phi_1(t)$, $\phi_2(t)$, $\phi_3(t)$, and $\phi_4(t)$, respectively. Then, integrating on $[0,T]$, integrating by parts twice the first term in the LHS of each obtained equations, then subtracting each one of these obtained equations from ((46)-(49)), once get
$(y_{it}(0), v_i)\phi_i'(0) = (y_i^0, v_i)\phi_i'(0), \forall i = 1,2,3,4.$

From the last two cases easily once get the initial conditions (14b), (15b), (16b) and (17b).
To prove that $\vec{y}_n \to \vec{y}$ ST in $L^2(I,V)$, we start by integrating (33) on $[0,T]$

$$\int_0^T \frac{d}{dt} \| y_n \|_0^2 \, dt + 2\int_0^T \| y_n \|_1^2 \, dt = (57a)+(57b) \quad (57)$$

$= 2[(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) +$
$(y_{4n}, y_{2nt}) + (y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{4nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt})$
$+(y_{3n}, y_{4nt})] + 2[(f_1 + u_1, y_{1nt}) + (f_2 + u_2, y_{2nt}) + (f_3 + u_3, y_{3nt}) + (f_4 + u_4, y_{4nt})]$
(57a)= $2[(y_{2n}, y_{1nt}) - (y_{3n}, y_{1nt}) - (y_{4n}, y_{1nt}) - (y_{1n}, y_{2nt}) + (y_{3n}, y_{2nt}) + (y_{4n}, y_{2nt})$
$+(y_{1n}, y_{3nt}) - (y_{2n}, y_{3nt}) - (y_{4n}, y_{4nt}) + (y_{1n}, y_{4nt}) - (y_{2n}, y_{4nt}) + (y_{3n}, y_{4nt})]$
(57b)=$2[(f_1 + u_1, y_{1nt}) + (f_2 + u_2, y_{2nt}) + (f_3 + u_3, y_{3nt}) + (f_4 + u_4, y_{4nt})]$.

By the same way that is applied to acquire (33) and (57), we can be used here to get

$$\| y_t(T) \|_0^2 - \| y_t(0) \|_0^2 + 2\int_0^T \| \vec{y}(t) \|_1^2 \, dt = (58a)+(58b) \quad (58)$$

(58a)= $2[(y_2, y_{1t}) - (y_3, y_{1t}) - (y_4, y_{1t}) - (y_1, y_{2t}) + (y_3, y_{2t}) + (y_4, y_{2t}) + (y_1, y_{3t}) -$
$(y_2, y_{3t}) - (y_4, y_{4t}) + (y_1, y_{4t}) - (y_2, y_{4t}) + (y_3, y_{4t})]$
(58b)=$2[(f_1 + u_1, y_{1t}) + (f_2 + u_2, y_{2t}) + (f_3 + u_3, y_{3t}) + (f_4 + u_4, y_{4t})]$

Since, $\| \vec{y}_{nt}(T) - \overrightarrow{y_t}(T) \|_0^2 - \| \vec{y}_{nt}(0) - \vec{y}_t(0) \|_0^2 + 2\int_0^T \| \vec{y}_n(t) - \overrightarrow{y}(t) \|_1^2 \, dt$

= (59a)-(59b)-(59c) (59)

(59a)= $\| \vec{y}_{nt}(T) \|_0^2 - \| \vec{y}_{nt}(0) \|_0^2 + 2\int_0^T \| \vec{y}_n(t) \|_1^2 \, dt$

(59b)= $\left(\vec{y}_{nt}(T), \vec{y}_t(T) - \left(\vec{y}_{nt}(0) - \vec{y}_t(0)\right)\right) + 2\int_0^T \left(\vec{y}_n(t), \vec{y}(t)\right)_1 dt$

(59c)= $(\vec{y}_t(T), \vec{y}_{nt}(T) - \vec{y_t}(T)) - (\vec{y}_t(0), \vec{y}_{nt}(0) - \vec{y}_t(0)) + 2\int_0^T \left(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t)\right)_1 dt$

Since $\vec{y}_n \to \vec{y}$ (strongly) in $L^2(Q)$, and $\vec{y}_{nt} \to \vec{y}$ (weakly) in $L^2(Q)$, then from (57) and the assumption 2.1, we get

(59a)=(57a)+(57b)→ (58a)+(58b)
In the same way that was employed to acquire (24), it used here to acquire
$\vec{y}_{nt}(T) \to \vec{y}(T)$ (strongly) in $L^2(\Omega)$.

On the other hand, since $\vec{y}_n \to \vec{y}$ in $L^2(\mathrm{I}, \mathrm{V})$, then from (24) and (60) with (59)=(58a)+(58b).
All the term in (59c) approach zero, so as the first two terms in the LHS of (59), hence (59) gives
$\int_0^T \| \vec{y}_n(t) - \vec{y}(t) \|_1^2 \, dt \to 0$ as $n \to \infty$, therefore $\vec{y}_n \to \vec{y}$ ST in $L^2(\mathrm{I}, \mathrm{V})$.

3.1Uniqueness of the Solution:

Let $\vec{y} = (y_1, y_2, y_3, y_4)$ and $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$ be two quaternary state vector solution of the weak form ((14)-(17)), subtracting each equation from the other and then replace $v_i = y_i - \bar{y}_i$ for $i = 1,2,3,4$. Therefore,
$((y_i, \bar{y}_i)_{tt}, y_i - \bar{y}_i) + \| y_i - \bar{y}_i \|_1^2 = 0$, $((y_i, \bar{y}_i)\,(0), (y_i - \bar{y}_i)(0)) = 0$ and for $v_i = (y_i - \bar{y}_{1i})_t$, we have $((y_i - \bar{y}_i)\,(0), (y_i - \bar{y}_i)(0)) = 0$.
Collecting the above equalities for $i = 1,2,3,4$. Using Lemma 1.2 in [17] for the first in LHS of each equation which will be positive, and we integrate both sides from 0 to , by employing the initial conditions and at last from the Belman-Gronwall inequality one has

$$\int_0^T \left[\frac{d}{dt} \| (\vec{y} - \vec{\bar{y}})_t(t) \|_0^2 + 2 \| (\vec{y} - \vec{\bar{y}})_t \|_1^2\right] dt \le 2\int_0^T \| (\vec{y} - \vec{\bar{y}}) \|_1^2 \, dt$$

$\| \left(\vec{y} - \vec{\bar{y}}\right)(t) \|_1^2 = 0, \forall t \in I \Longrightarrow \| \left(\vec{y} - \vec{\bar{y}}\right)(t) \|_{L^2(\mathrm{I},\mathrm{V})} = 0$. Therefore, the solution is unique.

4. Existence of a classical continuous optimal control quaternary vector :

Lemma 4.1: In addition to assumption 2.1, suppose that $\vec{y}$, $\vec{y} + \delta\vec{y}$ are the state quaternary vector solution corresponding to the classical continuous control quaternary vector $\vec{u}, \vec{u} + \delta\vec{u} \in L^2(Q)$ respectively, then
$\| \delta\vec{y}_\varepsilon \|_{L^\infty(I, L^2(\Omega))} \le \delta \| \delta\vec{u} \|_Q$, $\| \delta\vec{y}_\epsilon \|_{L^2(I,V)} \le \delta \| \delta\vec{u} \|_Q$ and $\| \delta\vec{y}_\varepsilon \|_Q \le \delta \| \delta\vec{u} \|_Q$ with $\delta \in \mathbb{R}^+$.

Proof: Let $\vec{u} = \vec{\bar{u}} - \vec{u}$, where $\vec{u} = (u_1, u_2, u_3, u_4)$, $\vec{\bar{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in L^2(Q)$, then $\vec{u}_\varepsilon = \vec{u} + \varepsilon\delta\vec{u} \in L^2(Q)$, for $\varepsilon > 0$, then from Theorem 3.1, we have $\vec{y} = \vec{y}_{\vec{u}} = (y_1, y_2, y_3, y_4)$ and $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon} = (y_{1\varepsilon}, y_{2\varepsilon}, y_{3\varepsilon}, y_{4\varepsilon})$ are the corresponding state quaternary vector solution which are satisfied the weak form ((14)-(17)), after substituting $\delta\vec{y}_\varepsilon = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}, \delta y_{3\varepsilon}, \delta y_{4\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, they give

$$(\delta y_{1\varepsilon tt}, v_1) + (\nabla \delta y_{1\varepsilon}, \nabla v_1) + (\delta y_{1\varepsilon}, v_1) - (\delta y_{2\varepsilon}, v_1) + (\delta y_{3\varepsilon}, v_1) + (\delta y_{4\varepsilon}, v_1) = (\varepsilon \delta u_1, v_1) \quad (61a)$$

$$\delta y_{1\varepsilon}(x,0) = 0 \text{ and } \delta y_{1\varepsilon t}(x,0) = 0 \quad (61b)$$

$$(\delta y_{2\varepsilon tt}, v_2) + (\Delta \delta y_{2\varepsilon}, \nabla v_2) + (\delta y_{1\varepsilon}, v_2) + (\delta y_{2\varepsilon}, v_2) - (\delta y_{3\varepsilon}, v_2) - (\delta y_{4\varepsilon}, v_2) = (\epsilon \delta u_2, v_2) \quad (62a)$$

$$\delta y_{2\varepsilon}(x,0) = 0 \text{ and } \delta y_{2\varepsilon t}(x,0) = 0 \quad (62b)$$

$$(\delta y_{3\varepsilon tt}, v_3) + (\nabla \delta y_{3\varepsilon}, \nabla v_3) - (\delta y_{1\varepsilon}, v_3) + (\delta y_{2\varepsilon}, v_3) + (\delta y_{3\varepsilon}, v_3) + (\delta y_{4\varepsilon}, v_3) = (\delta \varepsilon u_3, v_3) \quad (63a)$$

$$\delta y_{3\varepsilon}(x,0) = 0 \text{ and } \delta y_{3\varepsilon t}(x,0) = 0 \quad (63b)$$

$$(\delta y_{4\varepsilon tt}, v_4) + (\nabla \delta y_{4\varepsilon}, \nabla v_4) - (\delta y_{1\varepsilon}, v_4) + (\delta y_{2\varepsilon}, v_4) - (\delta y_{3\varepsilon}, v_4) + (\delta y_{4\varepsilon}, v_4) = (\delta \varepsilon u_4, v_4) \quad (64a)$$

$$\delta y_{4\varepsilon}(x,0) = 0 \text{ and } \delta y_{4\varepsilon t}(x,0) = 0 \quad (64b)$$

Using $v_i = \delta y_{i\varepsilon t}$ for $i = 1,2,3,4$ in (61a),(62a),(63a)and (64a), we collect the obtained equations and employing the same steps that are used to acquire (34), a similar equation can be acquired, however $\delta \vec{y}_\varepsilon$ is instead of $\vec{y}_n$, then we integrate both sides on $[0,t]$, this yields to

$$\int_0^t \frac{d}{dt}[\| \delta \vec{y}_{\varepsilon t}(t) \|_0^2 + \| \delta \vec{y}_\varepsilon \|_1^2] dt \le 2\int_0^t \frac{d}{dt}[| \delta y_{2\varepsilon} | + | \delta y_{3\varepsilon} | + | \delta y_{4\varepsilon} | + \varepsilon | \delta u_1 |] | \delta y_{1\varepsilon t} | dt$$
$$+2\int_0^t \frac{d}{dt}[| \delta y_{1\varepsilon} | + | \delta y_{3\varepsilon} | + | \delta y_{4\varepsilon} | + \varepsilon | \delta u_2 |] | \delta y_{2\varepsilon t} | dt +$$
$$2\int_0^t \frac{d}{dt}[| \delta y_{1\varepsilon} | + | \delta y_{2\varepsilon} | + | \delta y_{4\varepsilon} | + \varepsilon | \delta u_3 |] | \delta y_{3\varepsilon t} | dt$$
$$+2\int_0^t \frac{d}{dt}[| \delta y_{1\varepsilon} | + | \delta y_{2\varepsilon} | + | \delta y_{3\varepsilon} | + \varepsilon | \delta u_4 |] | \delta y_{4\varepsilon t} | dt .$$

Therefore, we get

$$\| \delta \vec{y}_{\varepsilon t} \|_0^2 + \| \delta \vec{y}_\varepsilon \|_1^2 \le 4\int_0^t [\| \delta \vec{y}_\varepsilon \|_0^2 + \| \delta \vec{y}_{\varepsilon t} \|_1^2] dt + \varepsilon \| \delta \vec{u} \|_Q^2$$

We apply the Belman-Gronwall inequality with $\delta^2 = \varepsilon e^{4t}$ to get
$\| \delta \vec{y}_{\varepsilon t} \|_0^2 + \| \delta \vec{y}_\varepsilon \|_1^2 \le \delta^2 \| \delta \vec{u} \|_Q^2$, for all $t \in I$. Hence, $\| \delta \vec{y}_\varepsilon \|_1^2 \le \delta^2 \| \delta \vec{u}(t) \|_Q^2$ for all $t \in I$.
$\| \delta \vec{y}_\varepsilon \|_{L^\infty(I,L^2(\Omega))} \le \delta \| \delta \vec{u} \|_Q$, $\| \delta \vec{y}_\epsilon \|_{L^2(I,V)} \le \delta \| \delta \vec{u} \|_Q$ and $\| \delta \vec{y}_\varepsilon \|_Q \le \delta \| \delta \vec{u} \|_Q$.

Lemma 4.2 : With assumption 2.1 $\vec{u} \to \vec{y}_{\vec{u}}$ is continuous from $L^2(Q)$ in to $L^\infty(I, L^2(\Omega))$ or to L^2(Q), or to $L^2(I,V)$.

Proof: Let $\delta \vec{u} = \vec{\bar{u}} - \vec{u}$ and $\delta \vec{y} = \vec{\bar{y}} - \vec{y}$, where $\vec{\bar{y}}$ and $\vec{y}$ are the corresponding state quaternary vector solution to the classical continuous control quaternary vector $\vec{\bar{u}}$, $\vec{u}$ and by the first result in Lemma 4.1, one has $\| \vec{\bar{y}} - \vec{y} \|_{L^\infty(I,L^2(\Omega))} \le M \| \vec{\bar{u}} - \vec{u} \|_Q$, if $\vec{\bar{u}} \underset{L^2(Q)}{\longrightarrow} \vec{u}$ then $\vec{\bar{y}} \underset{L^\infty(I,L^2(\Omega))}{\longrightarrow} \vec{y}$, thus the operator $\vec{u} \to \vec{y}_{\vec{u}}$ is Lipschitz continuous from $L^2(Q)$ and into $L^2(I, L^2(\Omega))$
Similarly, the operator is also Lipschitz continuous from $L^2(Q)$ into $L^2(Q)$ and into $L^2(I,V)$.

Lemma 4.3 [13]: The norm $\|\cdot\|_0$ is weakly lower semi continuous.

Lemma 4.4: The cost functional in (13) is weakly lower semi continuous.

Proof: From Lemma 4.3, $G_0(\vec{u})$ is weakly lower semi continuous.

Lemma 4.5 [13]: The norm $\|\cdot\|_0$ is strictly convex.

Theorem 4.1: Consider the cost functional (13), if $G_0(\vec{u})$ is coercive and the set $\vec{U}$ is convex, then there exist a classical continuous control quaternary vector.

Proof: From the hypotheses on $G_0(\vec{u})$, there is a minimizing sequence $\{\vec{u}_k\} = \{(u_{1k}, u_{2k}, u_{3k}, u_{4k})\} \in \vec{W}_A$, $\forall k$ such that $\lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}})$, and $\| \vec{u}_k \| \leq C$, then by Alaoglu's Theorem ,there is a subsequence of $\{\vec{u}_k\}$, for simplicity say$\{\vec{u}_k\}$, such that $\vec{u}_k \to \vec{u}$ weakly in $L^2(Q)$ as $k \to \infty$. From Theorem 3.1 and corresponding to the sequence $\{\vec{u}_k\}$, there is a sequence of a unique state quaternary vector solution $\{\vec{y}_k = \vec{y}_{u_k}\}$ and that $\| \vec{y}_k \|_{L^2(I,V)}$, $\| \vec{y}_{kt} \|_{L^2(Q)}$ are bounded, and then by Alaoglu's Theorem, there is subsequence of $\{\vec{y}_k\}$, let be $\{\vec{y}_k\}$ such that $\vec{y}_k \to \vec{y}$ weakly in $L^2(I,V)$, $\vec{y}_{kt} \to \vec{y}$ weakly in $L^2(Q)$. Now for each k, the state quaternary vector solution $\vec{y}_k$ satisfies, the weak form ((19)-(22)), multiplying both sides of each equation by $\phi_i(t)$, $\forall i = 1,2,3,4$ (with $\phi_i \in C^2[0,T]$, such that $\phi_i(T) = \phi_i'(T) = 0, \phi_i(0) \neq 0$, $\phi_i'(0) \neq 0$). Rewriting the first term in the LHS of each one, then we integrate both sides on $[0,T]$. Finally, we integrate by parts twice for their first term, same equations like ((37)-(40)) can be obtained with different that each $v_{in} = v_i$ and that the term

$$\int_0^t (f_i + u_i, v_{in})\phi_i(t)dt, \forall i = 1,2,3,4, \text{ and for all } k \tag{65}$$

Hence, the similar technique that employed in the proof of Theorem 3.1 can be applied here to passage the limit as $k \to \infty$ in both sides of the above indicated equations, except the new term (65) which converges to the following term (since $u_{ik} \to u_i$ is weakly in $L^2(Q)$)

$$\int_0^t (f_i + u_i, v_i)\phi_i(t)dt, \forall i = 1,2,3,4. \tag{66}$$

From these convergences, we get the weak form like (14a), (15a), (16a) and (17a). To passage the limits in the initial conditions and to get (28b), (15b) and (16b) the same steps that are used in the proof of Theorem 3.1 can be also used. Therefore, the limit point (y_1, y_2, y_3, y_4) is a solution of the state quaternary equations.

Finally, $G_0(\vec{u})$ is weakly lower semi continuous. From Lemma 4.1 and $\vec{u}_k \to \vec{\vec{u}}$ is weakly in $(L^2(\Omega))^4$, this implies that

$G_0(\vec{u}) \leq \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}})$, then

$G_0(\vec{u}) \leq \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}})$. This leads to $G_0(\vec{u}) \leq \min_{\vec{u}_k \in \vec{W}_A} G_0(\vec{\vec{u}})$, then $\vec{u}$ is classical continuous control quaternary vector.

5. The Necessary Conditions.

In order to state the necessary conditions for classical continuous optimal control, the Fréchet derivative of the cost functional (13) is derived and the theorem for the necessary conditions is proved

Theorem 5.1 : Consider the cost functional (13) and the adjoint quaternary linear hyperbolic boundary value problem of the state quaternary equations. ((1)-(12)) is:

$$Z_{1tt} - \Delta Z_1 + Z_1 + Z_2 - Z_3 - Z_4 = (y_1 - y_{1d}), \text{ on } Q \tag{67a}$$

$$Z_1 = 0 \text{ on } \Sigma, Z_1(x,T) = Z_{1t}(x,T) = 0 \text{ on } \Omega \quad (67b)$$
$$Z_{2tt} - \Delta Z_2 - Z_1 + Z_2 + Z_3 + Z_4 = (y_2 - y_{2d}), \text{ on } Q \quad (68a)$$
$$Z_2 = 0 \text{ on } \Sigma, Z_2(x,T) = Z_{2t}(x,T) = 0 \text{ on } \Omega \quad (68b)$$
$$Z_{3tt} - \Delta Z_3 + Z_1 - Z_2 + Z_3 - Z_4 = (y_3 - y_{3d}), \text{ on } Q \quad (69a)$$
$$Z_3 = 0 \text{ on } \Sigma, Z_3(x,T) = Z_{3t}(x,T) = 0 \text{ on } \Omega \quad (69b)$$
$$Z_{4tt} - \Delta Z_4 + Z_1 - Z_2 + Z_3 + Z_4 = (y_4 - y_{4d}), \text{ on } Q \quad (70a)$$
$$Z_4 = 0 \text{ on } \Sigma, Z_4(x,T) = Z_{4t}(x,T) = 0 \text{ on } \Omega \quad (70b)$$

And the Hamiltonian in this case is:

$$H(x,t,\vec{y},\vec{u},\vec{Z}) = \sum_{i=1}^{4} (Z_i(f_i(x,t) + u_i) + \frac{1}{2}\sum_{i=1}^{4} \left(\| y_i - y_{id} \|_Q^2 + \frac{\beta}{2} \| u_i \|_Q^2 \right).$$

Then for $\vec{u} \in \vec{W}$, the directional derivative of G is given by

$$DG(\vec{u}, \vec{\bar{u}} - \vec{u}) = \lim_{\varepsilon \to 0} \frac{G(\vec{u}+\varepsilon\delta\vec{u}) - G(\vec{u})}{\varepsilon} = \int_Q H_{\vec{u}}(x,t,\vec{y},\vec{u},\vec{Z})(\vec{\bar{u}} - \vec{u}) dxdt$$

Proof: At first let, the weak form of the adjoint quaternary linear hyperbolic boundary value problem is given for all $v_i \in V$ a.e. on I by

$$(Z_{1tt}, v_1) + (\nabla Z_1, \nabla v_1) + (Z_1, v_1) + (Z_2, v_1) - (Z_3, v_1) - (Z_4, v_1) = (y_1 - y_{1d}, v_1) \quad (71a)$$
$$(Z_1(T), v_1) = (Z_{1t}(T), v_1) = 0 \quad (71b)$$
$$(Z_{2tt}, v_2) + (\nabla Z_2, \nabla v_2) - (Z_1, v_2) + (Z_2, v_2) + (Z_3, v_2) + (Z_4, v_2) = (y_2 - y_{2d}, v_2) \quad (72a)$$
$$(Z_2(T), v_2) = (Z_{2t}(T), v_2) = 0 \quad (72b)$$
$$(Z_{3tt}, v_3) + (\nabla Z_3, \nabla v_3) + (Z_1, v_3) - (Z_2, v_3) + (Z_3, v_3) - (Z_4, v_3) = (y_3 - y_{3d}, v_3) \quad (73a)$$
$$(Z_3(T), v_3) = (Z_{3t}(T), v_3) = 0 \quad (73b)$$
$$(Z_{4tt}, v_4) + (\nabla Z_4, \nabla v_4) + (Z_1, v_4) - (Z_2, v_4) + (Z_3, v_4) + (Z_4, v_4) = (y_4 - y_{4d}, v_4) \quad (74a)$$
$$(Z_4(x,T), v_4) = (Z_{4t}(T), v_4) = 0 \quad (74b)$$

One can easily show that the weak form ((71)-(74)) has a unique solution $\vec{Z} = (Z_1, Z_2, Z_3, Z_4) \in L^2(Q)$ by using the same way that is employed in the proof of Theorem 3.1. Now, substituting $v_i = \delta y_{i\varepsilon}$ for $i = 1,2,3,4$ in (71a), (72a), (73a) and (74a), respectively.

$$\int_0^T (\delta y_{1\varepsilon}, Z_{1tt}) dt + \int_0^T [(\nabla Z_1, \nabla \delta y_{1\varepsilon}) + (Z_1, \delta y_{1\varepsilon}) + (Z_2, \delta y_{1\varepsilon}) - (Z_3, \delta y_{1\varepsilon}) - (Z_4, \delta y_{1\varepsilon})] dt$$
$$= \int_0^T (y_1 - y_{1d}, \delta y_{1\varepsilon}) dt \quad (75)$$

$$\int_0^T (\delta y_{2\varepsilon}, Z_{2tt}) dt + \int_0^T [(\nabla Z_2, \nabla \delta y_{2\varepsilon}) - (Z_1, \delta y_{2\varepsilon}) + (Z_2, \delta y_{2\varepsilon}) + (Z_3, \delta y_{2\varepsilon}) + (Z_4, \delta y_{2\varepsilon})] dt$$
$$= \int_0^T (y_2 - y_{2d}, \delta y_{2\varepsilon}) dt \quad (76)$$

$$\int_0^T (\delta y_{3\varepsilon}, Z_{3tt}) dt + \int_0^T [(\nabla Z_3, \nabla \delta y_{3\varepsilon}) + (Z_1, \delta y_{3\varepsilon}) - (Z_2, \delta y_{3\varepsilon}) + (Z_3, \delta y_{3\varepsilon}) - (Z_4, \delta y_{3\varepsilon})] dt$$
$$= \int_0^T (y_3 - y_{3d}, \delta y_{3\varepsilon}) dt \quad (77)$$

$$\int_0^T (\delta y_{4\varepsilon}, Z_{4tt}) dt + \int_0^T [(\nabla Z_4, \nabla \delta y_{4\varepsilon}) + (Z_1, \delta y_{4\varepsilon}) - (Z_2, \delta y_{4\varepsilon}) + (Z_3, \delta y_{4\varepsilon}) + (Z_4, \delta y_{4\varepsilon})] dt$$
$$= \int_0^T (y_4 - y_{4d}, \delta y_{4\varepsilon}) dt \quad (78)$$

Now, let $\vec{\vec{u}}, \vec{u} \in L^2(Q)$, $\delta\vec{u} = \vec{\vec{u}} - \vec{u}$, for $\varepsilon > 0$, $\vec{u}_\varepsilon = \vec{u} + \varepsilon\delta\vec{u} \in L^2(Q)$, then from Theorem 3.1, $\vec{y} = \vec{y}_{\vec{u}}$, $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon}$ are their corresponding state quaternary vector solution, we assume $\delta\vec{y}_\varepsilon = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}, \delta y_{3\varepsilon}, \delta y_{4\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, setting $v_i = Z_i$ for $i = 1,2,3,4$ in (61a), (62a), (63a) and (64a), respectively. We integrate both sides on $[0, T]$, then integrating by parts twice the first in the LHS of each equation, we get

$$\int_0^T (\delta y_{1\varepsilon}, Z_{1tt})dt + \int_0^T [(\nabla\delta y_{1\varepsilon}, \nabla Z_1) + (\delta y_{1\varepsilon}, Z_1) - (\delta y_{2\varepsilon}, Z_1) + (\delta y_{3\varepsilon}, Z_1) + (\delta y_{4\varepsilon}, Z_1)]dt = \int_0^T (\varepsilon\delta u_1, Z_1)dt \quad (79)$$

$$\int_0^T (\delta y_{2\varepsilon}, Z_{2tt})dt + \int_0^T [(\nabla\delta y_{2\varepsilon}, \nabla Z_2) + (\delta y_{1\varepsilon}, Z_2) + (\delta y_{2\varepsilon}, Z_2) - (\delta y_{3\varepsilon}, Z_2) - (\delta y_{4\varepsilon}, Z_2)]dt = \int_0^T (\varepsilon\delta u_2, Z_2)dt \quad (80)$$

$$\int_0^T (\delta y_{3\varepsilon}, Z_{3tt})dt + \int_0^T [(\nabla\delta y_{3\varepsilon}, \nabla Z_3) - (\delta y_{1\varepsilon}, Z_3) + (\delta y_{2\varepsilon}, Z_3) + (\delta y_{3\varepsilon}, Z_3) + (\delta y_{4\varepsilon}, Z_3)]dt = \int_0^T (\varepsilon\delta u_3, Z_3)dt \quad (81)$$

$$\int_0^T (\delta y_{4\varepsilon}, Z_{4tt})dt + \int_0^T [(\nabla\delta y_{4\varepsilon}, \nabla Z_4) - (\delta y_{1\varepsilon}, Z_4) + (\delta y_{2\varepsilon}, Z_4) - (\delta y_{3\varepsilon}, Z_4) + (\delta y_{4\varepsilon}, Z_4)]dt = \int_0^T (\varepsilon\delta u_4, Z_4)dt \quad (82)$$

By subtracting (75) from (79), (76) from (80), (77) from (81)and (78) from (82), the collecting of all the above obtained equations yields to

$$\varepsilon\int_0^T [(\delta u_1, Z_1) + (\delta u_2, Z_2) + (\delta u_3, Z_3) + (\delta u_4, Z_4)]dt = \int_0^T [(y_1 - y_{1d}, \delta y_{1\varepsilon}) + (y_2 - y_{2d}, \delta y_{2\varepsilon}) + (y_3 - y_{3d}, \delta y_{3\varepsilon}) + (y_4 - y_{4d}, \delta y_{4\varepsilon})]dt \quad (83)$$

On the other hand, we have

$$G(\vec{u}_\varepsilon) - G(\vec{u}) = \int_Q ((y_1 - y_{1d})\delta y_{1\varepsilon} + \varepsilon\beta u_1\delta u_1)dxdt + \int_Q ((y_2 - y_{2d})\delta y_{2\varepsilon} + \varepsilon\beta u_2\delta u_2)dxdt + \int_Q ((y_3 - y_{3d})\delta y_{3\varepsilon} + \varepsilon\beta u_3\delta u_3)dxdt + \int_Q ((y_4 - y_{4d})\delta y_{4\varepsilon} + \varepsilon\beta u_4\delta u_4)dxdt + O_1(\varepsilon) \quad (84)$$

Where $O_1(\varepsilon) = \frac{1}{2} \| \delta\vec{y}_\varepsilon \|^2_{L^2(Q)} + \frac{\beta}{2}\varepsilon^2 \| \delta\vec{u} \|^2_{L^2(Q)}$, with $O_1(\varepsilon) \to 0$ as $\varepsilon \to 0$

Now, by using (83) in (84), one has that

$$G(\vec{u}_\varepsilon) - G(\vec{u}) = \varepsilon\int_Q (\vec{Z} + \beta\vec{u})\delta\vec{u}\, dxdt + O_1(\varepsilon)$$

Finally, we divide both sides by ε, and take the limit $\varepsilon \to 0$, we get

$$DG(\vec{u}, \vec{\vec{u}} - \vec{u}) = \int_Q H_{\vec{u}}(x, t, \vec{y}, \vec{u}, \vec{Z})(\vec{\vec{u}} - \vec{u})dxdt = \int_Q (\vec{Z} + \beta\vec{u})\delta\vec{u}\, dxdt$$

Where $H_{\vec{u}}(x, t, \vec{y}, \vec{u}, \vec{Z}) = (Z_1 + \beta u_1 \quad Z_2 + \beta u_2 \quad Z_3 + \beta u_3 \quad Z_4 + \beta u_4)^T$

Theorem 5.2: The necessary optimality condition for the optimal classical continuous control quaternary vector of problem (14-17), and (71-74) with the cost functional (13) is
$DG(\vec{u},\vec{\vec{u}}-\vec{u}) = \vec{Z} + \beta\vec{u} = 0$ with $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{Z} = \vec{Z}_{\vec{u}}$

Proof: If $\vec{u}$ is a optimal classical continuous control quaternary vector of the problem, then $G_0(\vec{\vec{u}}) = \min_{\vec{u}\in\vec{W}_A} G_0(\vec{u}), \forall\vec{u} \in L^2(Q)$ that means
$DG(\vec{u},\vec{\vec{u}}-\vec{u}) = 0$. This gives $\vec{Z} + \beta\vec{\vec{u}} = 0, \delta\vec{u} = \vec{w} - \vec{\vec{u}}.$
Hence, the necessary condition is $(\vec{Z} + \beta\vec{\vec{u}}, \delta\vec{u}\,) \geq 0$. This implies that $(\vec{Z} + \beta\vec{\vec{u}}, \vec{w}\,) \geq (\vec{Z} + \beta\vec{\vec{u}}, \vec{\vec{u}}\,)$ for all $\vec{w} \in L^2(Q)$.

6. Conclusions

The method of Galerkin is successfully employed to prove the existence theorem for the unique quaternary state vector solution of the quaternary linear hyperbolic boundary value problem where the classical continuous control quaternary vector is known.
Also, the existence theorem of an optimal classical continuous control quaternary vector related to the quaternary linear hyperbolic boundary value problem is demonstrated. The existence of a unique solution to the adjoint quaternary linear hyperbolic boundary value problem associated with the quaternary linear hyperbolic boundary value problem is formulated and studied. The directional derivative for the cost functional is derived. Finally, the necessary optimality theorem for the optimal classical continuous control quaternary vector is stated and proved.

7. References

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