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Existence, Uniqueness and Approximate Controllability of Impulsive Fractional Nonlinear Control System with Nonsingular Kernel

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Abstract
The main aim of this work is to investigate the existence and approximate controllability of mild solutions of impulsive fractional nonlinear control system with a nonsingular kernel in infinite dimensional space. Firstly, we set sufficient conditions to demonstrate the existence and uniqueness of the mild solution of the control system using the Banach fixed point theorem. Further, we prove the approximate controllability of the control system using the sequence method.
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Keywords: Approximate controllability, Fractional derivative, Control system, Nonsingular kernel. Banach fixed point theorem.

وجود وإمكانية تحكم تقريبية لنظام تحكم متسارع كسري غير خطي مع نواة غير منفردة

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الخلاصة
الهدف الرئيسي من هذا العمل هو التحقيق في وجود وإمكانية التحكم التقريبية للحل المعتدل لنظام التحكم التسارعي الكسري غير الخطي مع نواة غير منفردة في الفضاء ذي البعد غير المنتهي. في البداية، وضعنا شروطًا كافية لإثبات وجود ووحدانية الحل المعتدل لنظام التحكم باستخدام نظرية النقطة الصامدة لبناخ. أثبتنا المكانية التحكم التقريبية لنظام التحكم باستخدام طريقة المتسلسلات.

1. Introduction

Many researchers have focused on fractional calculus during the past three centuries. The importance of this topic lies in its ability to describe many scientific problems with high accuracy. Therefore, there are many applications of fractional calculus in various fields of science, such as economics, physics, medicine, engineering, and others. Several researchers have defined the fractional derivative in different types, for example, Caputo, Hilfer, Hadamard, Caputo and Fabrizio, Atangana and Baleanu, Hussain et al. and Hattaf. For more details on this topic, one can see [1–6].

Impulsive differential equations have attracted much research attention due to their significance in modelling processes exposed to short-time changing throughout their development. Many articles deal with impulsive differential equations and their solutions; see [7–10] .

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Controllability is one of the critical characteristics of applied dynamical systems. If a system is able to transform from any initial state to any final state using a control function, then it is said to be controllable [11]. Controllability is widely used in various fields, such as engineering practice, biological applications, etc. Two forms of controllability are most often considered in practical applications, namely exact controllability and approximate controllability. The system is exactly controllable if it reaches a required state at the given time using a specified admissible control. The system is approximately controllable if it reaches a state at the given time that lies in an ε-neighborhood of the required state using any admissible control. Several articles examine the exact controllability and approximate controllability of control systems, see [12–17].
Naji and Al-sharaa [13] introduced the mild solution and studied the controllability of the impulsive fractional nonlinear system

$$\begin{cases} {}^{C}D^{\rho,\omega}\left[y(t)-h\big(t,y(t)\big)\right] = \mathrm{A}y(t)+\mathrm{B}u(t)+f\big(t,y(t)\big) \quad t\in\mathcal{J}=[0,\Upsilon], t\neq t_\gamma, \\ \Delta y\big(t_\gamma\big)=q_\gamma\left(y\big(t_\gamma^-\big)\right), \gamma=1,2,\dots.p, \\ y(0)=y_{0,} \end{cases} \tag{1}$$

where $\rho,\omega\in(0,1)$, ${}^{C}D^{\rho,\omega}$ is the Hattaf-fractional derivative of order ρ,ω, $y(.)$ belong to Banach space X, A is a linear operator defined on $D(A)\subset$ X into X , $u\in L^p(\mathcal{J},U)$ is a control function with a Banach space U, the operator B: $L^p(\mathcal{J},U)\to$ X is bounded and linear, f and g are continuous functions where $f:\mathcal{J}\times\mathrm{X}\to\mathrm{X}$ and $h:\mathcal{J}\times\mathrm{X}\to\mathrm{D(A)}$, $0=t_0<t_1<t_2<\cdots<t_\gamma<t_{\gamma+1}=\Upsilon$, $y\big(t_\gamma^+\big)$ and $y\big(t_\gamma^-\big)$ indicate to the right and left limits of $y(t)$ at $t=t_\gamma$, respectively and $\Delta y\big(t_\gamma\big)=y\big(t_\gamma^+\big)-y\big(t_\gamma^-\big)$.
In this article, we set sufficient conditions to demonstrate the existence and uniqueness of the mild solution and prove the approximate controllability of the system (1).
The article is organized as follows. In section 2, we introduce some basic concepts and lemmas related to this work. In section 3, we investigate the existence and uniqueness of the mild solution of the system (1). The approximate controllability of system (1) discusses in section 4. In section 5, an example is given.

2. Preliminaries

In this part, we present some definitions and lemmas that we used throughout this work.
The Banach space of all piecewise continuous functions from $\mathcal{J}$ to X is denoted by $PC(\mathcal{J},X)$ with the norm $\|x\|_{PC}=sup_{t\in\mathcal{J}}\|x(t)\|$.
Definition (2.1) [5]**.** Let $\rho\in[0,1),\omega,\lambda>0$ and $f\in H^1(c,d)$. Then

$${}^{C}D_{\eta}^{\rho,\omega,\lambda}f(t)=\frac{N(\rho)}{1-\rho}\frac{1}{\eta(t)}\int_c^t E_\omega\left[-\frac{\rho}{1-\rho}(t-\zeta)^\lambda\right]\frac{d}{d\zeta}(\eta f)(\zeta)d\zeta, \tag{2}$$

is the Hattaf-fractional derivative of order ρ in sense of Caputo of the function f with respect to the weight function $\eta\in C^1(c,d)$, $\eta,\acute{\eta}>0$ on $[c,d]$. $N(\rho)$ is normalization function with $N(0)=N(1)=1$ and $E_\omega(t)=\sum_{\gamma=1}^{\infty}\frac{t^\gamma}{\Gamma(\omega\gamma+1)}$ is Mittag-Leffler function of one parameter ω.
When $\lambda=\omega$ and $N(\rho)=\eta(t)=1$, then the fractional derivative (2) will be in the form

$${}^{C}D^{\rho,\omega}f(t)=\frac{1}{1-\rho}\int_c^t E_\omega\left[-\frac{\rho}{1-\rho}(t-\zeta)^\omega\right]\frac{d}{d\zeta}f(\zeta)d\zeta. \tag{3}$$

In this work, we consider the fractional derivative (3) with $0<\omega<1$.
On a Banach space X, consider the linear operator $A: D(A)\subset X\to X$ is the generator of C_0-semigroup $\{\mathcal{G}(t), t\geq 0\}$, where $sup_{t\geq 0}\|\mathcal{G}(t)\|=\mathcal{S}, \mathcal{S}\geq 1$. We consider the bounded linear

operator $E \coloneqq \rho A_\rho A$ where $A_\rho = \left[(1-\rho)\left(\frac{1}{1-\rho}I - A\right)\right]^{-1}$ with $\|A_\rho\| \leq \eta$, $\eta > 0$. Clearly, E is the generator of uniformly continuous semigroup $\{T(t), t \geq 0\}$ and $sup_{t\geq 0}\|T(t)\| = \mathcal{S}$ [18].

Lemma (2.2) [13]. If $y \in PC(\mathcal{J}, X)$ is a solution of system (1), then it satisfies the following

$$y(t) = \begin{cases} A_\rho h(t, y(t)) + A_\rho \int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h(\zeta, y(\zeta)) d\zeta \\ +A_\rho T_\omega(t)(y_0 - h(0, y_0)) + (1-\rho)A_\rho[Bu(t) + f(t, y(t))] \\ +\rho {A_\rho}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[Bu(\zeta) + f(\zeta, y(\zeta))] d\zeta \quad t \in [0, t_1] \\ A_\rho h(t, y(t)) + A_\rho \int_0^t (t-\zeta)^{\omega-1} E L_\omega(t-\zeta) h(\zeta, y(\zeta)) d\zeta \\ +A_\rho T_\omega(t)(y_0 - h(0, y_0)) + (1-\rho)A_\rho[Bu(t) + f(t, y(t))] \\ +\rho {A_\rho}^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[Bu(\zeta) + f(\zeta, y(\zeta))] d\zeta \\ +A_\rho \sum_{\gamma=1}^{p} T_\omega(t-t_\gamma)\Delta y(t_\gamma) \qquad t \in (t_\gamma, t_{\gamma+1}] \end{cases}$$

where $L_\omega(t) = \omega \int_0^\infty \theta \varphi_\omega(\theta) T(\theta t^\omega) d\theta$, $T_\omega(t) = \int_0^\infty \varphi_\omega(\theta) T(\theta t^\omega) d\,\theta$,

$$\varphi_\omega(\delta) = \frac{1}{\omega}\delta^{-1-\frac{1}{\omega}}\psi_\omega\left(\delta^{-\frac{1}{\omega}}\right), 0 < \delta < \infty, 0 < \omega < 1$$

is probability density function, and

$$\psi_\omega(\delta) = \frac{1}{\pi}\sum_{i=1}^{\infty}(-1)^{i-1}\delta^{-\omega i-1}\frac{\Gamma(i\omega+1)}{i!}\sin(i\pi\omega)$$

is one-sided stable probability density.

Lemma (2.3) [19]. Assume that X is a Banach space. If $\Phi: X \to X$ is a contraction, then Φ has a unique fixed point.

3. The existence and uniqueness of the mild solution

To demonstrate the existence and uniqueness of the mild solution of system (1), the following conditions are assumed:

H1 : There exist constants $\mathcal{M}_h, \widehat{\mathcal{M}_h} > 0$ such that

$$\|Eh(t, y_1(t)) - Eh(t, y_2(t))\| \leq \mathcal{M}_h \|y_1(t) - y_2(t)\|$$

and

$$\widehat{\mathcal{M}_h} = sup_{t\in I}\|Eh(t, 0)\|.$$

H2 :The continuous function $f: \mathcal{J} \times X \to X$ satisfies Lipchitz condition i.e. there exists a constant $\mathcal{M}_f > 0$ such that

$$\|f(y_1) - f(y_2)\| \leq \mathcal{M}_f \|y_1(t) - y_2(t)\|$$

and

$$\widehat{\mathcal{M}_f} = sup_{t\in I}\|f(t, 0)\|.$$

where $\widehat{\mathcal{M}_f} > 0$.

H3 :The function $q_\gamma: X \to X, \gamma = 1,2, \dots, p$ is continuous and satisfies Lipchitz condition, i.e. there exists a constant $\mathcal{M}_\gamma > 0$ such that

$$\|q_\gamma(y_1) - q_\gamma(y_2)\| \leq \mathcal{M}_\gamma \|y_1(t) - y_2(t)\|$$

and

$$\sum_{\gamma=1}^{p} \mathcal{M}_\gamma = \mathcal{M},$$

where $\mathcal{M} > 0$.

Theorem (3.1). Suppose the hypotheses H1, H2 and H3 are satisfied, and

$$D = \eta \left[\|E^{-1}\| \mathcal{M}_h + \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_h + (1-\rho)\mathcal{M}_f + \eta\rho \frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)} \mathcal{M}_f + \mathcal{S}\mathcal{M} \right] < 1,$$

then the control system (1) has a unique mild solution on $PC(\mathcal{J}, \mathrm{X})$ for each $u \in L^P(\mathcal{J}, U)$.
Proof. Define the operator

$$(\widehat{\Phi} y)(t) = \begin{cases} \mathrm{A}_\rho h(t, y(t)) + \mathrm{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h(\zeta, y(\zeta)) d\zeta \\ +A_\rho T_\omega(t)(y_0 - h(0, y_0)) + (1-\rho)\mathrm{A}_\rho [Bu(t) + f(t, y(t))] \\ +\rho \mathrm{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[Bu(\zeta) + f(\zeta, y(\zeta))] d\zeta \quad t \in [0, t_1] \\ \mathrm{A}_\rho h(t, y(t)) + \mathrm{A}_\rho \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta) E h(\zeta, y(\zeta)) d\zeta \\ +A_\rho T_\omega(t)(y_0 - h(0, y_0)) + (1-\rho)\mathrm{A}_\rho [Bu(t) + f(t, y(t))] \\ +\rho \mathrm{A}_\rho^2 \int_0^t (t-\zeta)^{\omega-1} L_\omega(t-\zeta)[Bu(\zeta) + f(\zeta, y(\zeta))] d\zeta \\ +\mathrm{A}_\rho \sum_{\gamma=1}^{p} \Delta y(t_\gamma) T_\omega(t - t_\gamma), \qquad t \in (t_\gamma, t_{\gamma+1}]. \end{cases}$$

Step 1. We show the operator $\widehat{\Phi}$ maps $PC(\mathcal{J}, \mathrm{X})$ into itself.
For $0 \le s < s_1 \le t_1$,

$$\begin{aligned} &\|(\widehat{\Phi} y)(s) - (\widehat{\Phi} y)(s_1)\| \\ &= \Big\| \mathrm{A}_\rho h(s, y(s)) + \mathrm{A}_\rho \int_0^s (s-\zeta)^{\omega-1} L_\omega(s-\zeta) E h(\zeta, y(\zeta)) d\zeta \\ &+ \mathrm{A}_\rho T_\omega(s)(y_0 - h(0, y_0)) + (1-\rho)\mathrm{A}_\rho [Bu(s) + f(s, y(s))] \\ &+ \rho \mathrm{A}_\rho^2 \int_0^s (s-\zeta)^{\omega-1} L_\omega(s-\zeta)[Bu(\zeta) + f(\zeta, y(\zeta))] d\zeta - \mathrm{A}_\rho h(s_1, y(s_1)) \\ &- \mathrm{A}_\rho \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) E h(\zeta, y(\zeta)) d\zeta - \mathrm{A}_\rho T_\omega(s_1)(y_0 - h(0, y_0)) \\ &- (1-\rho)\mathrm{A}_\rho [Bu(s_1) + f(s_1, y(s_1))] \\ &- \rho \mathrm{A}_\rho^2 \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)[Bu(\zeta) + f(\zeta, y(\zeta))] d\zeta \Big\| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \mathrm{A}_\rho \left(h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big) \right) \right\| \\
&\quad + \left\| \mathrm{A}_\rho \left[\int_0^s (s-\zeta)^{\omega-1} L_\omega (s-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \right.\right. \\
&\quad \left.\left. - \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega (s_1-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \right] \right\| \\
&\quad + \left\| \mathrm{A}_\rho \big(y_0 - h(0, y_0)\big)\big(T_\omega(s) - T_\omega(s_1)\big) \right\| \\
&\quad + \left\| (1-\rho) A_\rho \left[B\big(u(s) - u(s_1)\big) + f\big(s, y(s)\big) - f\big(s_1, y(s_1)\big) \right] \right\| \\
&\quad + \left\| \rho \mathrm{A}_\rho^2 \left[\int_0^s (s-\zeta)^{\omega-1} L_\omega (s-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \right.\right. \\
&\quad \left.\left. - \int_0^{s_1} (s_1-\zeta)^{\omega-1} L_\omega (s_1-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \right] \right\| \\
&\leq \left\| \mathrm{A}_\rho \right\| \left[\left\| h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big) \right\| \right. \\
&\quad + \left\| \int_0^s (s-\zeta)^{\omega-1} L_\omega (s-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \right. \\
&\quad - \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega (s_1-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\
&\quad - \int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega (s_1-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\
&\quad + \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega (s-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \\
&\quad \left. - \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega (s-\zeta) E h\big(\zeta, y(\zeta)\big) d\zeta \right\| \\
&\quad + \left\| y_0 - h(0, y_0) \right\| \left\| T_\omega(s) - T_\omega(s_1) \right\| \\
&\quad + (1-\rho) \left[\|B\| \|u(s) - u(s_1)\| + \left\| f\big(s, y(s)\big) - f\big(s_1, y(s_1)\big) \right\| \right] \\
&\quad + \rho \left\| \mathrm{A}_\rho \right\| \left\| \int_0^s (s-\zeta)^{\omega-1} L_\omega (s-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \right. \\
&\quad - \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega (s_1-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\
&\quad - \int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega (s_1-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\
&\quad + \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega (s-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\
&\quad \left.\left. - \int_0^{s} (s_1-\zeta)^{\omega-1} L_\omega (s-\zeta) \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \right\| \right]
\end{aligned}
$$

$$\leq \|A_\rho\| \Big[\|h(s,y(s)) - h(s_1,y(s_1))\| + \left\| \int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) Eh(\zeta,y(\zeta)) d\zeta \right\|$$
$$+ \int_0^s \|(s_1-\zeta)^{\omega-1}\| L_\omega(s-\zeta) - L_\omega(s_1-\zeta) \| Eh(\zeta,y(\zeta)) \| d\zeta$$
$$+ \int_0^s \|(s-\zeta)^{\omega-1} - (s_1-\zeta)^{\omega-1}\| \|L_\omega(s-\zeta)\| \|Eh(\zeta,y(\zeta))\| d\zeta$$
$$+ \|y_0 - h(0,y_0)\| \|T_\omega(s) - T_\omega(s_1)\| + (1-\rho)\|B\| \|u(s) - u(s_1)\|$$
$$+ (1-\rho)\|f(s,y(s)) - f(s_1,y(s_1))\|$$
$$+ \rho\|A_\rho\| \left\| \int_s^{s_1} (s_1-\zeta)^{\omega-1} \|L_\omega(s_1-\zeta)\| \|Bu(\zeta) + f(\zeta,y(\zeta))\| d\zeta \right\|$$
$$+ \rho\|A_\rho\| \int_0^s (s_1-\zeta)^{\omega-1} \|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| \|Bu(\zeta) + f(\zeta,y(\zeta))\| d\zeta$$
$$+ \rho\|A_\rho\| \int_0^s \|(s-\zeta)^{\omega-1} - (s_1-\zeta)^{\omega-1}\| \|L_\omega(s-\zeta)\| \|Bu(\zeta)$$
$$+ f(\zeta,y(\zeta))\| d\zeta \Big]$$

$$\leq \eta \Big[\|h(s,y(s)) - h(s_1,y(s_1))\| + \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[s,s_1]} \|Eh(\xi,y(\xi))\|$$
$$+ sup_{\xi\in[0,s]} \|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]} \|Eh(\zeta,y(\zeta))\| \Big[\frac{s_1^\omega}{\omega}$$
$$- \frac{(s_1-s)^\omega}{\omega} \Big] + \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]} \|Eh(\zeta,y(\zeta))\|$$
$$+ \|y_0 - h(0,y_0)\| \|T_\omega(s) - T_\omega(s_1)\| + (1-\rho)\|B\| \|u(s) - u(s_1)\|$$
$$+ (1-\rho)\|f(s,y(s)) - f(s_1,y(s_1))\|$$
$$+ \rho\eta \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[s,s_1]} \|Bu(\zeta) + f(\zeta,y(\zeta))\|$$
$$+ \rho\eta sup_{\xi\in[0,s]} \|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]} \|Bu(\zeta)$$
$$+ f(\zeta,y(\zeta))\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega} \right]$$
$$+ \rho\eta \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]} \|Bu(\zeta) + f(\zeta,y(\zeta))\| \Big].$$

Let

$$O_1 = \eta\|h(s,y(s)) - h(s_1,y(s_1))\|$$
$$O_2 = \eta \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[s,s_1]} \|Eh(\xi,y(\xi))\|$$
$$O_3 = \eta sup_{\xi\in[0,s]} \|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]} \|Eh(\zeta,y(\zeta))\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega} \right]$$
$$O_4 = \eta \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]} \|Eh(\zeta,y(\zeta))\|$$
$$O_5 = \eta\|y_0 - h(0,y_0)\| \|T_\omega(s) - T_\omega(s_1)\|$$
$$O_6 = (1-\rho)\eta\|B\| \|u(s) - u(s_1)\|$$
$$O_7 = (1-\rho)\eta\|f(s,y(s)) - f(s_1,y(s_1))\|$$
$$O_8 = \rho\eta^2 \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[s,s_1]} \|Bu(\zeta) + f(\zeta,y(\zeta))\|$$

$$O_9 = \rho\eta^2 sup_{\xi\in[0,s]}\|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| sup_{\xi\in[0,s]}\left\|Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\right\| \left[\frac{s_1^\omega}{\omega} - \frac{(s_1-s)^\omega}{\omega}\right]$$

$$O_{10} = \rho\eta^2 \frac{\mathcal{S}(s_1-s)^\omega}{\Gamma(\omega+1)} sup_{\xi\in[0,s]}\left\|Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\right\|.$$

Since f, h are continuous functions on $\mathcal{J}$, then O_1 and O_7 tend to zero as $s \to s_1$.
Since L_ω, T_ω are continuous in the uniform operator topology, then
O_3, O_5 and O_9 tend to zero as $s \to s_1$.
Since u is measurable, then $u(s) \to u(s_1)$ a.e. $s \to s_1$, then O_6 tends to zero.
Clearly, O_2, O_4, O_8 and O_{10} tend to zero as $s \to s_1$. Therefore,
$\left\|\big(\widehat{\Phi}y\big)(s) - \big(\widehat{\Phi}y\big)(s_1)\right\| \to 0$ as $s \to s_1$. Thus $\big(\widehat{\Phi}y\big)(t) \in C[0, t_1]$.
Now, for $t_k < s < s_1 \le t_{k+1}$, we have

$$\begin{aligned}
&\left\|\big(\widehat{\Phi}y\big)(s) - \big(\widehat{\Phi}y\big)(s_1)\right\| \\
&\le \|A_\rho\| \Bigg[\left\|h\big(s, y(s)\big) - h\big(s_1, y(s_1)\big)\right\| \\
&+ \left\|\int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta) Eh\big(\zeta, y(\zeta)\big) d\zeta\right\| \\
&+ \int_0^s \|(s_1-\zeta)^{\omega-1}\| L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\left\|Eh\big(\zeta, y(\zeta)\big)\right\| d\zeta \\
&+ \int_0^s \|(s-\zeta)^{\omega-1} - (s_1-\zeta)^{\omega-1}\| \|L_\omega(s-\zeta)\| \left\|Eh\big(\zeta, y(\zeta)\big)\right\| d\zeta \\
&+ \|y_0 - h(0, y_0)\| \|T_\omega(s) - T_\omega(s_1)\| + (1-\rho)\|B\| \|u(s) - u(s_1)\| \\
&+ (1-\rho)\left\|f\big(s, y(s)\big) - f\big(s_1, y(s_1)\big)\right\| \\
&+ \rho\|A_\rho\| \left\|\int_s^{s_1} (s_1-\zeta)^{\omega-1} L_\omega(s_1-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta\right\| \\
&+ \rho\|A_\rho\| \int_0^s (s_1-\zeta)^{\omega-1} \|L_\omega(s-\zeta) - L_\omega(s_1-\zeta)\| \big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\
&+ \rho\|A_\rho\| \int_0^s \|(s-\zeta)^{\omega-1} - (s_1-\zeta)^{\omega-1}\| L_\omega(s-\zeta)\big[Bu(\zeta) + f\big(\zeta, y(\zeta)\big)\big] d\zeta \\
&+ \Sigma\Bigg],
\end{aligned}$$

where

$$\Sigma = \sum_{\gamma=1}^{p} \left\|\Delta y\big(t_\gamma\big)\right\| \left\|T_\omega\big(s_1 - t_\gamma\big) - T_\omega\big(s - t_\gamma\big)\right\|,$$

Since T_ω is continuous in the uniform operator topology, then Σ tends to zero as $s \to s_1$. From above, we have $\left\|\big(\widehat{\Phi}y\big)(s) - \big(\widehat{\Phi}y\big)(s_1)\right\|$ tends to zero as $s \to s_1$. Therefore, $\widehat{\Phi}y \in PC[0, \Upsilon]$.
Step 2. We show the operator $\widehat{\Phi}$ is contraction on $PC(\mathcal{J}, X)$.
For $y_1, y_2 \in PC(\mathcal{J}, X)$, and for each $t \in [0, t_1]$,

$$\begin{aligned}
&\left\|\left(\widehat{\Phi}y_1\right)(t)-\left(\widehat{\Phi}y_2\right)(t)\right\| \\
&\qquad \leq \left\|A_\rho\left[h\left(t,y_1(t)\right)-h\left(t,y_2(t)\right)\right]\right\| \\
&\qquad + \left\|A_\rho\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)E\left[h\left(\zeta,y_1(\zeta)\right)-h\left(\zeta,y_2(\zeta)\right)\right]d\zeta\right\| \\
&\qquad + \left\|(1-\rho)A_\rho\left[f\left(t,y_1(t)\right)-f\left(t,y_2(t)\right)\right]\right\| \\
&\qquad + \left\|\rho A_\rho^2\int_0^t (t-\zeta)^{\omega-1}L_\omega(t-\zeta)\left[f\left(\zeta,y_1(\zeta)\right)-f\left(\zeta,y_2(\zeta)\right)\right]d\zeta\right\| \\
&\leq \eta\left\|h\left(t,y_1(t)\right)-h\left(t,y_2(t)\right)\right\|+\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h\|y_1-y_2\| \\
&\qquad + (1-\rho)\eta\left\|f\left(t,y_1(t)\right)-f\left(t,y_2(t)\right)\right\|+\rho\eta^2\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f\|y_1-y_2\| \\
&\leq \eta\mathcal{M}_hE^{-1}\|y_1-y_2\|+\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h\|y_1-y_2\|+(1-\rho)\eta\mathcal{M}_f\|y_1-y_2\| \\
&\qquad + \rho\eta^2\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f\|y_1-y_2\| \\
&= \eta\left[\mathcal{M}_hE^{-1}+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f+\rho\left\|A_\rho\right\|\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f\right]\|y_1-y_2\|.
\end{aligned}$$

Now, for $t\in(t_\gamma,t_{\gamma+1}]$ using our assumption, we have

$$\begin{aligned}
&\left\|\left(\widehat{\Phi}y_1\right)(t)-\left(\widehat{\Phi}y_2\right)(t)\right\| \leq \\
&\qquad = \eta\left[\mathcal{M}_hE^{-1}+\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_h+(1-\rho)\mathcal{M}_f+\rho\left\|A_\rho\right\|\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\mathcal{M}_f\right]\|y_1 \\
&\qquad - y_2\|+\eta\mathcal{M}\mathcal{S}\|y_1-y_2\| \\
&\qquad = D\|y_1-y_2\|
\end{aligned}$$

and by our assumption, then $\widehat{\Phi}$ is contraction. According to Banach fixed point theorem, the operator $\widehat{\Phi}$ has a unique fixed point y such that $\widehat{\Phi}y=y$. Therefore, the proof is complete. ■

4. Approximate controllability results

In this part, we study the approximate controllability of the system (1). Define the bounded linear operator $\Lambda: X\to X$ as

$$\Lambda(\mathcal{y})=(1-\rho)A_\rho\mathcal{y}(\Upsilon)+\rho A_\rho^2\int_0^\Upsilon(\Upsilon-s)^{\omega-1}L_\omega(\Upsilon-s)\mathcal{y}(s)ds.$$

The following condition is important to prove the approximate controllability of the system (1),

H4 $\quad\forall\epsilon>0,\forall\,\mathcal{y}\in X,\exists\,\mathcal{u}\in L^2(\mathcal{J},U)$ such that

$$\|\Lambda(\mathcal{y})-\Lambda(B\mathcal{u})\|<\varepsilon$$

and

$$\|B\mathcal{u}(\cdot)\|<a\|\mathcal{y}(\cdot)\|$$

where $a>0$.

Definition (4.1). The system (1) is approximately controllable on $\mathcal{J}$ if $\overline{\mathcal{K}_\Upsilon(f)}=X$, where $\mathcal{K}_\Upsilon(f)=\{y(\Upsilon;\mathcal{u}):\mathcal{u}(\cdot)\in U\}$ is a reachable set of the system (1).

Lemma (4.2). Assume the conditions H1, H2, H3 and H4 are hold, then

i. $\quad\|y(t)\|_X\leq D\|y\|_{PC}+\widehat{D}+\left[(1-\rho)+\rho\eta\frac{\mathcal{S}\Upsilon^\omega}{\Gamma(\omega+1)}\right]\eta\|B\|\|\mathcal{u}(t)\|.$

where $\widehat{D} = \eta\left[\|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big) + (1-\rho)\widehat{\mathcal{M}_f} + \eta\rho\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\right]$.

ii. For $y_1, y_2 \in X$, then

$$\|y_2(t) - y_1(t)\|_X \le \frac{\eta}{1-D}\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|.$$

Proof.

i. $\|y(t)\|_X \le \|A_\rho\|\Big[\|h(t,y(t))\| + \frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Eh(\zeta,y(\zeta)\|d\zeta + \mathcal{S}\big\|\big(y_0 - h(0,y_0)\big)\big\| + (1-\rho)\big[\|Bu(t)\| + \big\|f\big(t,y(t)\big)\big\|\big] + \rho\|A_\rho\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\|Bu(\zeta)\|d\zeta + \rho\|A_\rho\|\frac{\mathcal{S}}{\Gamma(\omega)}\int_0^t (t-\zeta)^{\omega-1}\big\|f\big(\zeta,y(\zeta)\big)\big\|d\zeta + \mathcal{S}\sum_{\gamma=1}^{p}\big\|\Delta y\big(t_\gamma\big)\big\|\Big]$

$$\le \eta\Bigg[\|E^{-1}\|\mathcal{M}_h\|y\| + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\big(\mathcal{M}_h\|y\| + \widehat{\mathcal{M}_h}\big) + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big) + (1-\rho)\|B\|\|u(t)\| + (1-\rho)\big(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\big) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\|B\|\|u(t)\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\omega\Gamma(\omega)}\big(\mathcal{M}_f\|y\| + \widehat{\mathcal{M}_f}\big) + \mathcal{S}\mathcal{M}\|y\| + \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\|\Bigg]$$

$$= \eta\Bigg[\|y\|\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right] + \|E^{-1}\|\widehat{\mathcal{M}_h} + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_h} + \mathcal{S}\big(\|y_0\| + \|E^{-1}\|\widehat{\mathcal{M}_h}\big) + (1-\rho)\widehat{\mathcal{M}_f} + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\widehat{\mathcal{M}_f} + \mathcal{S}\sum_{\gamma=1}^{p}\|q_\gamma(0)\| + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\|B\|\|u(t)\|\Bigg]$$

$$= D\|y\| + \widehat{D} + \left[(1-\rho) + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\right]\eta\|B\|\|u(t)\|$$

ii. $\|y_2(t) - y_1(t)\| \le \|A_\rho\|\Big[\big\|h\big(t,y_2(t)\big) - h(t,y_1(t))\big\| + \int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\|Eh(\zeta,y_2(\zeta) - Eh(\zeta,y_1(\zeta)\|d\zeta + (1-\rho)\|B\|\|u_2(t) - u_1(t)\| + (1-\rho)\big\|f\big(t,y_2(t)\big) - f\big(t,y_1(t)\big)\big\| + \rho\|A_\rho\|\int_0^t (t-\zeta)^{\omega-1}\|L_\omega(t-\zeta)\|\,\|B\|\|u_2(\xi) - u_1(\xi)\|d\zeta + \sum_{\gamma=1}^{p}\|T_\omega\|\big\|q_\gamma y_2\big(t_\gamma\big) - q_\gamma y_1\big(t_\gamma\big)\big\|\Big]$

$$\le \eta\left[\|E^{-1}\|\mathcal{M}_h\|y_2 - y_1\| + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h\|y_2 - y_1\| + (1-\rho)\|B\|\|u_2(t) - u_1(t)\| + (1-\rho)\mathcal{M}_f\|y_2 - y_1\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\left[\|B\|\|u_2(t) - u_1(t)\| + \mathcal{M}_f\|y_2 - y_1\|\right] + \mathcal{S}\mathcal{M}\|y_2 - y_1\|\right]$$

$$= \eta\left[\|E^{-1}\|\mathcal{M}_h + \frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_h + (1-\rho)\mathcal{M}_f + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\mathcal{M}_f + \mathcal{S}\mathcal{M}\right]\|y_2 - y_1\| + \eta\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|$$

$$= D\|y_2 - y_1\| + \eta\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|.$$

It follows,

$$\|y_2 - y_1\| - D\|y_2 - y_1\| \le \eta\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|.$$

Thus,

$$\|y_2(t) - y_1(t)\| \le \frac{\eta}{1-D}\left[(1-\rho)\|B\| + \rho\eta\frac{\mathcal{S}\Upsilon^{\omega}}{\Gamma(\omega+1)}\|B\|\right]\|u_2(t) - u_1(t)\|.$$

Theorem 4.3. Suppose the hypotheses H1, H2, H3 and H4 hold, then the system (1) is approximate controllability provided

$$\left(\mathcal{M}_f + \frac{\|\mathrm{A}_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda\frac{\|\mathrm{A}_\rho\|}{1-D}[(1-\rho)+1]\|B\| < 1. \qquad (4)$$

Proof. Since the domain $D(\mathrm{A})$ of operator A is dense in X [18], i.e. $\overline{D(\mathrm{A})} = X$. It is sufficient to prove $D(\mathrm{A}) \subset \mathcal{K}_\Upsilon(f)$, that is mean we have to show for any $\epsilon > 0$ and $x \in D(\mathrm{A})$, there exists $u \in L^p(\mathcal{J}, U)$
such that

$$\left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - \mathrm{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) + \frac{1-\rho}{\rho}Eh\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathrm{A}_\rho^{-1}}{\rho}\Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big)\right\|$$

$$= \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) + \left(\frac{1-\rho}{\rho}\rho\mathrm{A}\mathrm{A}_\rho - \mathrm{A}_\rho\right)h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathrm{A}_\rho^{-1}}{\rho}\Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big)\right\|$$

$$= \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) + \big((1-\rho)\mathrm{A} - I\big)\mathrm{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathrm{A}_\rho^{-1}}{\rho}\Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big)\right\|$$

$$= \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - \mathrm{A}_\rho^{-1}\mathrm{A}_\rho h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathrm{A}_\rho^{-1}}{\rho}\Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big)\right\|$$

$$= \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - \Sigma_\Upsilon - \Lambda(Bu) - \Lambda(f) - h\big(\Upsilon, y(\Upsilon)\big) - \frac{\mathrm{A}_\rho^{-1}}{\rho}\Lambda\Big(Eh\big(\Upsilon, y(\Upsilon)\big)\Big)\right\| < \epsilon,$$

where $\Sigma_\Upsilon = \sum_{\gamma=1}^{p}\Delta y(t_\gamma)\sigma_\gamma(\Upsilon)T_\omega(\Upsilon - t_\gamma)$.

For any initial $y_0 \in X$, since $T(t)$ is differentiability semigroup for each $t > 0$ then

$$\left[\mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) + h\big(\Upsilon, y(\Upsilon)\big) + \Sigma_\Upsilon\right] \in D(\mathrm{A})$$

and we can see there exists a function $\mathcal{Q} \in L^p(\mathcal{J}, X)$ such that

$$\Lambda\big(\mathcal{Q}(\cdot)\big) = x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon.$$

For example,

$$\mathcal{Q}(t) = \begin{cases} 0 & t = \Upsilon \\ \dfrac{(\Upsilon - t)^{1-\omega}}{\Upsilon\rho}\left(\Gamma(\omega)\right)^2 A_\rho^{-2}\left(x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon\right) \\ \quad \times\left(L_\omega(\Upsilon - t) + 2t\dfrac{d}{dt}L_\omega(\Upsilon - t)\right) & t \in [0,\Upsilon) \end{cases}$$

then,

$$\Lambda\left(\mathcal{Q}(t)\right) = \rho A_\rho^2 \int_0^\Upsilon (\Upsilon - s)^{\omega-1} L_\omega(\Upsilon - s)\frac{(\Upsilon - s)^{1-\omega}}{\Upsilon\rho}\left(\Gamma(\omega)\right)^2 A_\rho^{-2}\Big(x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon\Big)\left(L_\omega(\Upsilon - s) + 2s\frac{d}{ds}L_\omega(\Upsilon - s)\right)ds$$

$$= \frac{\left(\Gamma(\omega)\right)^2}{\Upsilon}\Big(x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon\Big)\int_0^\Upsilon\left(\left(L_\omega(\Upsilon - s)\right)^2 + 2sL_\omega(\Upsilon - s)\frac{d}{ds}L_\omega(\Upsilon - s)\right)ds$$

$$= \frac{\left(\Gamma(\omega)\right)^2}{\Upsilon}\left(x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon\right)\left[\int_0^\Upsilon\left(L_\omega(\Upsilon - s)\right)^2 ds + \int_0^\Upsilon s\frac{d}{ds}\left(L_\omega(\Upsilon - s)\right)^2 ds\right].$$

Using integral by parts, we have

$$\Lambda\left(\mathcal{Q}(t)\right) = \frac{\left(\Gamma(\omega)\right)^2}{\Upsilon}\left(x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon\right)\left[\int_0^\Upsilon\left(L_\omega(\Upsilon - s)\right)^2 ds + \frac{\Upsilon}{\left(\Gamma(\omega)\right)^2} - \int_0^\Upsilon\left(L_\omega(\Upsilon - s)\right)^2 ds\right]$$

$$= \left(x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon\right).$$

Now, for any given $\epsilon > 0$ and by H4, there exists a control u such that

$$\left\|x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon - \Lambda\left(Bu(t)\right)\right\| < \frac{\epsilon}{2^3}. \quad (5)$$

Next, we show there is a control $u \in L^p(\mathcal{J}, U)$ such that the inequality (5) holds.
Let $u_1 \in L^P(\mathcal{J}, U)$, then by H4, there exists $u_2 \in L^P(\mathcal{J}, U)$ such that

$$\left\|\Lambda\left[Bu(t) - f\left(t, y_1(t)\right) - \frac{A_\rho^{-1}}{\rho}Eh\left(t, y_1(t)\right)\right] - \Lambda\left(Bu_2(t)\right)\right\| < \frac{\epsilon}{2^3}, \quad (6)$$

where $y_1 = y(\mathrm{t}; u_1), t \in \mathcal{J}$.
From (5) and (6), we have

$$\left\|x - \mathrm{A}_\rho T_\omega(\Upsilon)\left(y_0 - h(0,y_0)\right) - h\left(\Upsilon, y(\Upsilon)\right) - \Sigma_\Upsilon - \Lambda\left[f\left(t, y_1(t)\right) + \frac{A_\rho^{-1}}{\rho}Eh\left(t, y_1(t)\right)\right] - \Lambda\left(Bu_2(t)\right)\right\|$$

$$= \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(B u(t)\big) + \Lambda\big(B u(t)\big) - \Lambda\left[f\big(t,y_1(t)\big) + \frac{A_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\big(B u_2(t)\big) \right\|$$

$$\leq \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big(B u(t)\big) \right\| + \left\| \Lambda\big(B u(t)\big) - \Lambda\left[f\big(t,y_1(t)\big) + \frac{A_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\big(B u_2(t)\big) \right\| \leq \frac{\epsilon}{2^2}.$$

Denote $y_2 = y(\mathrm{t}; u_2), t \in \mathcal{J}$, then by H4, there exists $w_2 \in L^p(\mathcal{J}, U)$ such that

$$\left\| \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_2(t)\big)\right] - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\big(B w_2(t)\big) \right\| \leq \frac{\epsilon}{2^3},$$

and

$$\|B w_2(t)\| \leq \lambda \left\| f\big(t,y_2(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_2(t)\big) - f\big(t,y_1(t)\big) - \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big) \right\|$$

$$\leq \lambda \left\| f\big(t,y_2(t)\big) - f\big(t,y_1(t)\big) \right\| + \lambda \frac{\|\mathrm{A}_\rho^{-1}\|}{\rho} \left\| Eh\big(t,y_2(t)\big) - Eh\big(t,y_1(t)\big) \right\|$$

$$\leq \left(\mathcal{M}_f + \frac{\|\mathrm{A}_\rho^{-1}\|}{\rho} \mathcal{M}_h\right) \lambda \|y_2 - y_1\|$$

$$\leq \left(\mathcal{M}_f + \frac{\|\mathrm{A}_\rho^{-1}\|}{\rho} \mathcal{M}_h\right) \lambda \frac{\|\mathrm{A}_\rho\|}{1-D} [(1-\rho)+1] \|B\| \|u_2(t) - u_1(t)\|.$$

Let $u_3(t) = u_2(t) - w_2(t)$, $u_3(\cdot) \in L^p(\mathcal{J}, U)$. It follows

$$\left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_2(t)\big)\right] - \Lambda\big(B u_3(t)\big) \right\|$$

$$= \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon + \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_2(t)\big)\right] - \Lambda\big(B u_3(t)\big) \right\|$$

$$\leq \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0,y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\big(B u_2(t)\big) \right\| + \left\| \Lambda\left[f\big(t,y_1(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_1(t)\big)\right] - \Lambda\left[f\big(t,y_2(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t,y_2(t)\big)\right] + \Lambda\big(B w_2(t)\big) \right\| \leq \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3}.$$

By mathematical induction we can see that the sequence $\{u_n, n = 0,1,2, \dots\} \subset L^p(J, U)$, consequently,

$$\left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_n(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t, y_n(t)\big)\right] - \Lambda\big(Bu_{n+1}(t)\big) \right\| \leq \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots + \frac{\epsilon}{2^{n+1}}$$

where $y_n = y(\mathrm{t}; u_n)$ and

$$\|Bu_{n+1} - Bu_n\| \leq \left(\mathcal{M}_f + \frac{\|A_\rho^{-1}\|}{\rho}\mathcal{M}_h\right)\lambda \frac{\|A_\rho\|}{1 - D}\big((1 - \rho) + 1\big)\|Bu_n(t) - Bu_{n-1}(t)\|$$

and from our assumption, we get the sequence $\{Bu_n(t), n = 1,2,3, \dots\}$ is a Cauchy sequence on X. Since X is a Banach space, then there exists a point $\delta(t) \in X$ such that $Bu_n \to \delta(t)$ as $n \to \infty$. Then for any $\epsilon > 0$, there exists a positive integer k such that

$$\left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\left[f\big(t, y_k(t)\big) + \frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t, y_k(t)\big)\right] - \Lambda\big(Bu_k(t)\big) \right\|$$

$$\leq \left\| x - \mathrm{A}_\rho T_\omega(\Upsilon)\big(y_0 - h(0, y_0)\big) - h\big(\Upsilon, y(\Upsilon)\big) - \Sigma_\Upsilon - \Lambda\big[f\big(t, y_k(t)\big)\big] - \Lambda\left[\frac{\mathrm{A}_\rho^{-1}}{\rho} Eh\big(t, y_k(t)\big)\right] - \Lambda\big(Bu_{k+1}(t)\big) \right\| + \big\|\Lambda\big(Bu_{k+1}(t)\big) - \Lambda\big(Bu_k(t)\big)\big\|$$

$$\leq \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \cdots + \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2} < \epsilon.$$

Therefore, we get a sequence $\{y_k, k = 1,2, \dots\} \subset \mathcal{K}_\Upsilon(f)$ converge to $x \in D(\mathrm{A})$, thus $x \in \overline{\mathcal{K}_\Upsilon(f)}$, which mean $\overline{\mathcal{K}_\Upsilon(f)} = X$. ■

5. Example

Consider the following nonlinear fractional control system with nonsingular kernel

$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}}[y(t,\gamma) - h(t, y(t,\gamma))] = Ay(t,y) + Bu(t) + f\big(t, y(t,\gamma)\big), \\ \qquad \gamma \in [0,\pi], t \in [0, t_1) \cup (t_1, 1], \\ y(t,0) = y(t,\pi) = 0, t \in [0,1], \\ \qquad \Delta y(t_1^+) = q_1\big(y(t_1^-)\big), \qquad t_1 = \dfrac{1}{2}, \end{cases} \tag{7}$$

Setting $X = L^2([0.\pi], R) = U$, and define the operator $A: D(A) \subset X \to X$ by

$$Ay(t,\gamma) = \frac{\partial^2 y}{\partial \gamma^2}(t,\gamma).$$

where

$$D(A) = \left\{y \in X: \frac{\partial y}{\partial \gamma}, \frac{\partial^2 y}{\partial \gamma^2} \in X \text{ and } \gamma(0) = \gamma(\pi) = 0\right\}.$$

For $y \in D(A)$ then A can be written as the following

$$Ay = \sum_{s=1}^{\infty} -s^2 \langle y, y_s \rangle y_s,$$

where $y_s(\gamma) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(s\gamma)$, $s = 1,2,3 \ldots$. Then $\{y_s(\gamma)\}$ is an orthonormal basis for X and y_s is an eigenfunction corresponding to the eigenvalue $\lambda_s = -s^2$ of the operator A, $s = 1,2,3 \ldots$.

Therefore, A is the generator of C_0 −semigroup $\{T(t), t \geq 0\}$ in $L^2[0,\pi]$ such that $T(t)y = \sum_{s=1}^{\infty} e^{-s^2 t} \langle y, y_s \rangle y_s$, $y \in D(A)$, and $\|T(t)\| < 1 = \mathcal{S}$. The functions h, f and q_1 are defined as follows:

- h: $[0,1] \times X \to D(A)$ such that
$$h\big(t, y(t,\gamma)\big) = \int_0^{\gamma} \sin y(t,\zeta)\, d\zeta\,, t \in [0,1], \gamma \in [0,\pi], y \in X.$$

- f: $[0,1] \times X \to X$ such that
$$f(t, y(t,\gamma)) = \frac{t^2 e^{-t} |y(t,\gamma)|}{b}, \qquad t \in [0,1], \gamma \in [0,\pi], y \in X, b > 0.$$
- q_1: $X \to X$ such that
$$q_1(y(t_1,\gamma)) = \frac{\left|y\left(\frac{1}{2}^{-}, \gamma\right)\right|}{2\left(1 + \left|y(\frac{1}{2}^{-}, \gamma)\right|\right)}, t \in [0,1], \gamma \in [0,\pi], y \in X.$$

According to Hille-Yosida Theorem $\|\mathcal{G}(t)\| = \|T(t)\| = \mathcal{S}$.
Naji and Al-sharaa [13] show that the system (7) satisfies the conditions H1, H2 and H3.
For every $u(\cdot) \in L^2(J,U)$ of the form $u(t) = \sum_{s=1}^{\infty} u_s(t) y_s$, we define
$$Bu(t) = \sum_{s=1}^{\infty} \hat{u}_s(t) y_s$$
where
$$\hat{u}_s(t) = \begin{cases} 0 & 0 \leq t < 1 - \frac{1}{s^2} \\ u_s(t) & 1 - \frac{1}{s^2} \leq t \leq 1 \end{cases}.$$
It is clear that $\|Bu\| \leq \|u(\cdot)\|$. Therefore, B is a bounded linear operator from $L^2(J,U)$ into X.
Now, we shall prove condition H4. Consider the corresponding linear system of the system (7) as follows:
$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}} y_s(t) + s^2 y_s(t) = \hat{u}_s(t), \qquad 1 - \frac{1}{s^2} \leq t \leq 1 \\ \Delta y(t_1^+) = q_1\big(y(t_1^-)\big), \end{cases}$$
Let $\wp(\cdot)$ be an arbitrary element in $L^2(J,X)$ and $k \in X$ which is defined as
$$k = (1-\rho)A_\rho \wp(1) + \rho A_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \wp(\xi) d\xi.$$
Assume that $\wp(t) = \sum_{s=1}^{\infty} \wp_s(t) y_s$ and $K = \sum_{s=1}^{\infty} k_s y_s$. we can choose the control function
$$\tilde{u}_s(t) = \frac{2s^2}{1 - e^{-2}} k_s e^{-s^2(1-t)}, 1 - \frac{1}{s^2} \leq t \leq 1,$$
then
$$(1-\rho)A_\rho Bu(1) + \rho A_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) Bu(\xi) d\xi$$
$$= (1-\rho)A_\rho Bu(1) + \rho A_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \sum_{s=1}^{\infty} \tilde{u}_s(\xi) y_s\, d\xi$$

$$= (1-\rho)A_\rho Bu(1) + \rho A_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \sum_{s=1}^{\infty} \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-\xi)} y_s \, d\xi$$

$$= k = (1-\rho)A_\rho \wp(1) + \rho A_\rho^2 \int_0^1 (1-\xi)^{\omega-1} L_\omega(1-\xi) \wp(\xi) d\xi.$$

Therefore, the first part of condition H1 holds.
Now,

$$\|Bu(t)\|^2 = \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} |\tilde{u}_s(t)|^2 dt$$

$$= \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} \left| \frac{2s^2}{1-e^{-2}} k_s e^{-s^2(1-t)} \right|^2 dt$$

$$= \sum_{s=1}^{\infty} \int_{1-\frac{1}{s^2}}^{1} \frac{4s^4}{(1-e^{-2})^2} k_s^2 e^{-2s^2(1-t)} dt$$

$$= \sum_{s=1}^{\infty} \frac{2s^2}{1-e^{-2}} k_s^2$$

$$= \frac{1}{1-e^{-2}} \sum_{s=1}^{\infty} \left(1-e^{-2s^2}\right) \int_0^1 |\hat{\wp}_s(t)|^2 dt$$

$$\leq \frac{1}{1-e^{-2}} \|\wp(\cdot)\|^2.$$

Therefore, the condition H4 holds.
If

$$D = \eta \left[\left(\|E^{-1}\| + \frac{1}{\Gamma\left(\frac{4}{3}\right)} \right) \mathcal{M}_h + \frac{1}{2} \left(1 + \eta \frac{1}{\Gamma\left(\frac{4}{3}\right)} \right) \mathcal{M}_f + \mathcal{M} \right] < 1,$$

and

$$\left(\mathcal{M}_f + 2\left\|A_\rho^{-1}\right\|\mathcal{M}_h\right) \frac{1}{2(1-e^{-2})} \frac{\eta}{1-D} \|B\| < 1$$

then the system (7) is approximately controllable.

Conclusion

In this work, the existence and uniqueness of the mild solution to the system (1) have been proved in a Banach space using Banach Fixed Point Theorem.
The approximate controllability for system (1) was discussed using the approximate sequence method. The efficacy of our result has been shown using an example.

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