DOI: 10.24996/ijs.2017.58.2C.10

ISSN: 0067-2904

Strongly Coretractable Modules

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Abstract

Let R be a ring with identity and M be a right unitary R-module. In this paper we introduce the notion of strongly coretractable modules. Some basic properties of this class of modules are investigated and some relationships between these modules and other related concepts are introduced.

Keywords: Coretractable Modules , Strongly Coretractable Modules , (Cocompressible) Epi-Coretractable Modules And Coquasi-Dedekind Module .

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الخلاصة

لتكن R حلقة ذات محايد و M مقاساً احاديا ايمن على الحلقة R. في هذا البحث قدمنا مفهوم المقاسات المنكمشة المضادة بقوة . بعض الخواص الاساسية حول هذا الصنف من المقاسات فد اعطيت وكذلك تم تقديم بعض العلاقات التي تربط هذه المقاسات بالمفاهيم ذات العلاقة .

Introduction

Throughout this paper all rings have identities and all modules are unital right R-modules. A module M is called coretractable if for each a proper submodule N of M , there exists a nonzero R-homomorphism $f:M/N\rightarrow M$. Equivalently , M is coretractable if for each proper submodule N of M , there exists $g\in End_R(M)$, $g\neq 0$ such that $g(N)=0$ [1] . In this paper , our main aim to introduce and study strongly coretractable modules where an R-module M is called strongly coretractable module if for each proper submodule N of M , there exists a nonzero R-homomorphism $f:M/N\rightarrow M$ such that $\operatorname{Im} f+N=M$ [2] . It is clear every strongly coretractable module is coretractable but it is not conversely . This work consists of two sections , in section one we supply some basic properties of strongly coretractable modules . A characterization of strongly coretractable modules is given . We prove that a direct sum of two strongly coretractable modules is also strongly coretractable module (Theorem1.16) also we prove that the isomorphic image and quotient of strongly coretractable modules is again strongly coretractable modules (see Proposition1.4 and Theorem1.5), but a submodule of strongly coretractable module may be not strongly coretractable module. In section two, many relationships between strongly coretractable modules and other concepts are presented

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1. Basic Properties of Strongly Coretractable Modules

In this section, we study the concept of strong coretractability for modules . Many characterizations of strongly coretractable modules under certain classes of module are presented. Beside these several interested properties of strongly coretractable modules are given.

Definition(1.1) [2]**:** An R-module M is called strongly coretractable module if for each proper submodule N of M , there exists a nonzero R-homomorphism f:M/N→M such that Imf+N=M . A ring R is called strongly coretractable if R is strongly coretractable R-module . Equivalently , M is a strongly coretractable R-module if and only if for each proper submodule N of M , there exists $g\in End_R(M)$, g≠0 such that Img+N=M and g(N)=0 .

Examples and Remarks(1.2) :

(1) It is clear that every strongly coretractable module is coretractable module but not conversely as the following example shows :

Consider the Z-module Z_4 .The only proper nonzero submodule of M is $<\bar{2}>$, but there exists nonzero Z-homomorphism from $Z_4/<\bar{2}>$ into Z_4 which is defined by $f(\bar{0}+<\bar{2}>)=\bar{0}$ and $f(\bar{1}+<\bar{2}>)=\bar{2}$. But $Imf+<\bar{2}> = f(Z_4/<\bar{2}>) + <\bar{2}> \neq Z_4$. Thus Z_4 is not strongly coretractable module. However Z_4 is coretractable .

Similarly Z_{12} as Z-module is not strongly coretractable but Z_{12} is coretractable module.

(2) Each of the Z-module Z and Q are not strongly coretractable modules, since each of them is not coretractable .

(3) Every semisimple (simple) R-module is strongly coretractable R-module .

(4) It is clear that every epi-coretractable R-module is strongly coretractable . Where an R-module is called epi-coretractable module if for each proper submodule K of M , there exists a homomorphism f∈Hom(M/K,M) such that f is epimomorphism , some authors called it cocompressible (see [6]). However Z_6 as Z- module is strongly coretractable module by part(3) , but it is not epi-coretractable module . However , if $M=Z_{p^\infty}$ as Z-module . Then for each proper submodule N of M , M/N≅M and so there exists an isomorphism $f\in Hom_R(M/N,M)$ which implies that M is an epi-coretractable module and hence M is a strongly coretractable module.

(5) If M is a prime R-module with soc(M) ≠ 0, then M is strongly coretractable module.

Proof : Since M is a prime module with soc(M) ≠ 0, then M is semisimple by [4,Proposition(2.1.8)]. Thus M is strongly coretractable module by part(3).

(6) Let N be a proper submodule of M such that N and M/N are strongly coretractable modules , then it is not necessary that M is strongly coretractable module . For example , the Z-module $M=Z_4$ is not strongly coretractable Z-module . But $N=<\bar{2}>$ is proper submodule of M , and $M/N \cong <\bar{2}>$ is simple , so semisimple and so N , M/N are strongly coretractable .

(7) Let M be a module over commutative ring R such that annM⊂ [N:M] ⊆ annN for any nonzero proper submodule N of M . Then M is a coretractable module but it is not strongly coretractable module .

Proof : Let N be a proper non-trivial submodule of M. By hypothesis annM⊂ [N: M] ⊆annN , so there exists t∈[N:M] , t≠0 , such that Mt ⊆N , Nt=0 and Mt≠0 . Define f:M→ M by f(m)=mt for all m∈M , f is an R-homomorphism , since R is a commutative ring and f(M) = Mt ≠ 0, f(N) = Nt = 0 . Thus M is a coretractable module, while f(M) = Mt ⊆ N implies f(M) + N ≠ M. Thus M is not strongly coretractable module.

(8) Let M be a strongly coretractable nonsimple R-module . Then there exists a nonzero $\varphi \in End_R(M)$ such that $\varphi(M) = \varphi^2(M)$ and $\varphi\neq I_M$ (identity on M).

Proof: Since M is not simple module , there exists a non-trivial submodule N of M . As M is strongly coretractable , there exists $0_M\neq\varphi\in End_R(M)$ with Imφ +N=M and φ(N)=0 . Hence $\varphi\neq I_M$. It follows that φ(M)= φ(φ(M) + φ(N)). Thus $\varphi(M)=\varphi^2(M)$.

Recall that a module M over a commutative ring R is called scalar module if for all nonzero $f\in End_R(M)$, there exists 0≠ r ∈ R such that f(m)=mr for all m∈M [5].

Proposition(1.3): Let M be a (nonsimple) strongly coretractable and scalar R-module . Then there exists r∈R, r≠0 such that $Mr = M r^2$, where R is a commutative ring .

Proof : By part(8) there exists $0_M\neq \varphi \in End_R(M)$ such that φ≠Identity and $\varphi(M)=\varphi^2(M)$. As M is a scalar R-module , so there exists r∈ R , r≠ 0,r≠ 1 such that φ (m) = mr for each m∈M. Thus $Mr = Mr^2$.

Proposition(1.4): Let M≅M' where M is a strongly coretractable R-module. Then M' is a strongly coretractable module.
Proof : It is clear.
Theorem(1.5): Let M be a strongly coretractable R-module and N be a proper submodule of M . Then M/N is a strongly coretractable module.
Proof : Let W/N be a proper submodule of M/N , so W≠M . Since M is a strongly coretractable module . Then there exists a nonzero R-homomorphism g:M/W⟶M such that Img+W =M. But (M/N)/(W/N) ≅M/W . Set f= π∘g where π is the natural epimorphism from M into M/W. Then

$$f(\frac{M}{W}) + \frac{W}{N} = \pi(g(\frac{M}{W})) + \frac{W}{N} = \frac{g\left(\frac{M}{W}\right)+N}{N} + \frac{W}{N} = \frac{g\left(\frac{M}{W}\right)+N+W}{N} = \frac{M+N}{N} = \frac{M}{N}.$$

Thus M/N is astrongly coretractable module .
Corollary(1.6): Let f :M ⟶ M' be an R-epimorphism module. If M is a strongly coretractable R-module , then M' is a strongly coretractable module.
Proof: It follows by Theorem(1.5) and by Proposition(1.4).
Corollary(1.7): Let N be a nonzero direct summand of strongly coretractable R-module M. Then N is strongly coretractable module.
Proof: It follows by Theorem(1.5) and by Proposition(1.4) .
Remark(1.8): A proper submodule of strongly coretractable R-module is not necessarily strongly coretractable module . For example :

Consider the Z-module $M=Z_{2^\infty}$. Let $N=<1/2^2+Z>$ be a proper submodule of M, then $N\cong Z_{2^2}=Z_4$, but Z_4 is not a strongly coretractable . Thus N is not a strongly coretractable module.

2. Strongly Coretractable Modules and Related Concepts

In this section we investigate the relationships between strongly coretractable modules and other modules such as nonsingulsr , semisimple , prime, duo , hollow and finitely generated modules . First, we need recall that A ring R is called a Von Neumann regular ring if for any a ∈R , there exists b∈R such that a = aba .

Under the class of modules whose endomorphism rings are von Neumann regular, the two concepts coretractable and strongly coretractable module are coincide.

Proposition(2.1)[2]: Let M be an R-module and $S=End_R(M)$ be a von Neumann regular ring .Then the following statements are equivalent:
(1) M is a coretractable module ;
(2) M is a strongly coretractable module ;
(3) M is a semisimple module .

Proof: (1)⟹(3) Since M is coretractable module and S is regular ring , then M is a semisimple module by[1, Proposition(4.4)] .
(3)⟹(2) and (2)⟹(1) follow by Examples and Remarks(1.2(3,1)) .

Consider the Z-module $M=Q\oplus Z_2$. $End_Z(M)$ is von Neumann regular ring by [6, Example (2.4.32),P.63] . But M is not semisimple module . Hence M is not coretractable module and so it is not strongly coretractable module .

Recall that an R-module M has C_2 condition if for any submodule N of M which is isomorphic to a direct summand of M is a direct summand [6] .

Also $M=Q\oplus(\oplus_{n=1}^{\infty}Z_2)$ as right Z-module is Rickart with C_2 condition [6, Example(2.4.32),P.63] where M is called Rickart if for each f∈End(M) , kerf is a direct summand of M [6] . Then $End_R(M)$ is von Neumann regular ring[6]. But M is not semisimple module . Therefore M is not coretractable module and so it is not strongly coretractable module , by Proposition(2.1) .

Corollary(2.2): Let M be a strongly coretractable R-module and $End_R(M)$ is regular ring . Then every submodule of M is strongly coretractable module .
Proof : It follows by Proposition(2.1) and by Corollary(1.7) .
Corollary(2.3): Let M be a scalar faithful R-module over a regular commutative ring R . Then the following statements are equivalent:
(1) M is a coretractable module ;
(2) M is a semisimple module ;
(3) M is a strongly coretractable module .

Proof : As M is a scalar faithful R-module , hence by [7, Lemma(6.1)] , $S=End_R(M)\cong R$, and hence $End_R(M)$ is regular ring . Therefore 1⟺2⟺3 by Proposition(2.1) .

Proposition(2.4): Let M be a prime R-module . If $\overline{M}$ (The quasi-injective hull of M) is strongly coretractable . Then M is also strongly coretractable module .

Proof : Since M is a prime module , hence $End_R(\overline{M})$ is regular [8,Proposition (3.7) , P.36] . But $\overline{M}$ is strongly coretractable , hence $\overline{M}$ is semisimple by Proposition(2.1) , and so M is semisimple . Therefore M is strongly coretractable .

Remark(2.5): The condition M is prime in Proposition(2.4) cannot be dropped , for example , the Z-module Z_4 is not prime and it is not strongly coretractable, but $\overline{M}=Z_{2^\infty}$ is a strongly coretractable module .

Proposition(2.6): Let R be a semisimple ring . Then every R-module is a strongly coretractable module .

Proof : Since every module over semisimple ring R is semisimple module , hence every R-module is strongly coretractable module .

Recall that an R-module M is called a duo module if every submodule N of M is fully invariant [9].

Theorem(2.7): Let $M=M_1 \oplus M_2$ where M_1 and M_2 be R-modules and $annM_1+annM_2 = R$ (or M is duo R-module or distributive module) . Then M is strongly coretractable module if and only if M_1 and M_2 are strongly coretractable modules .

Proof : **(⟹)** The natural projections ρ_1(ρ_2): $M_1\oplus M_2 \longrightarrow M_1$ (respectively, M_2) are epimorphisims . Therefore M_1 and M_2 are strongly coretractable modules by Corollary(1.6) .

(⟸) Let K be a proper nonzero submodule of M . Since $annM_1+annM_2= R$ (or M is duo R-module or distributive module) , then $K=K_1 \oplus K_2$ for some $K_1 \leq M_1$ and $K_2\leq M_2$.

Case(1): K_1 is a proper submodule of M_1 and K_2 is a proper submodule of M_2 . Since M_1 and M_2 are strongly coretractable module , then there exists $f : M_1/K_1 \longrightarrow M_1$ such that $Imf+K_1=M_1$, and there exists $g:M_2/K_2\longrightarrow M_2$ such that $Img+K_2=M_2$.

Now, define a nonzero R-homomorphism $h:M/K\rightarrow M$;

that is $h:(M_1\oplus M_2)/(K_1\oplus K_2)\longrightarrow M_1\oplus M_2$ by $h[(m_1,m_2)+K_1\oplus K_2]=[f(m_1\oplus K_1) , g(m_2\oplus K_2)]$. Then h is well-defined . Now , $Imh+K=Imh +(K_1\oplus K_2)=(Imf\oplus Img)+ (K_1\oplus K_2) =(Imf+K_1)\oplus (Img+K_2) =M_1\oplus M_2=M$.

Case(2): $K_1=M_1$ and K_2 is a nonzero proper submodule of M_2 .

Consider $(M_1\oplus M_2)/(K_1\oplus K_2)=(M_1\oplus M_2)/(M_1\oplus K_2)\cong M_2/K_2$.

Since M_2 is strongly coretractable module , so there exists $f:M_2/K_2\longrightarrow M_2$ such that $Imf + K_2 = M_2$. Define $g : M_2/ K_2 \longrightarrow M_1 \oplus M_2$ by $g = i\circ f$ where i is the inclusion mapping from M_2 into $M_1\oplus M_2$. Therefore $Img+K= Img+ (K_1\oplus K_2) = Img+(M_1\oplus K_2) = M_1\oplus(Img+K_2) = M_1\oplus((0)\oplus Imf)+K_2) = M_1\oplus((0)+(Imf+K_2) = M_1\oplus((0)+M_2) =M_1\oplus M_2 = M$.

Case(3): K_1 is a nonzero proper submodule of M_1 and $K_2 = M_2$, then by a similar proof case(2) , we can get the result

Case(4): $K_1=0$ and $K_2 = M_2$.

Consider $M/K\cong (M_1\oplus M_2)/(0\oplus K_2) \cong M_1$. Let i is the inclusion mapping from M_1 into $M_1\oplus M_2$, hence $i(M_1)+K=(M_1 \oplus 0) \oplus(0\oplus M_2)=M$

Case(5): $K_1=M_1$ and $K_2=0$, then the proof is similar to Case(4) .

Recall that a submodule N of M is called coquasi-ivertable submodule of M if $Hom_R(M,N) = 0$ [10,P.8] and A nonzero R-module M is called coquasi-Dedekind module if every proper submodule of M is coquasi-ivertable module of M [10,P.32].

Equivalently , M is coquasi-Dedekind module if for each $f\in End_R(M)$, $f\neq 0$, f is an epimomorphism. [10 ,Theorem(2.1.4) ,P.33] .

Proposition(2.8): Let M be a coretractable R-module . If M is a coquasi-Dedekind module , then M is cocompressible module and hence M is strongly coretractable module .

Proof : Since M is a coretractable module . Then for each proper submodule K of M , there exists a nonzero R-homomorphism $f\in Hom(M/K,M)$. So $f^{\circ}\pi \in End_R(M)$ where π is the natural R-epimomorphisim from M into M/K . Now, since $f\neq 0$, so there exists $m+K\in M/K$ such that $m+K\neq 0$ and $f(m+K) \neq 0$ and so $f^{\circ}\pi(m)\neq 0$. But M is coquasi-Dedekind module , then $f^{\circ}\pi$ is an epimomorphisim . Therefore f is an epimomorphisim and hence M is a cocompressible module . Thus M is strongly coretractable module .

Proposition(2.9):Any maximal submodule of a strongly coretractable R-module M is a direct summand of M .

Proof : Let K be a maximal submodule of M . If K=0, then K is a direct summand of M . If K≠0 . Since K is a proper submodule of M and M is a strongly coretractable module , then there exists a nonzero R-homomorphism f:M/K→M with Imf+K=M .

Now, if Imf =M . As M/K is a simple module , so Imf is a simple module and hence M is simple and so that K=0 which is a contradiction . Thus Imf ≠ M , but as M/K is simple , then kerf=(0) .

On the other hand , (M/K)/kerf ≅ Imf by 1st isomorphism Theorem . Hence M/K=Imf , which implies that Imf is simple and hence Imf∩K=0 . But Imf+K=M . Therefore Imf ⊕ K=M and hence K is a direct summand of M .

Corollary(2.10): Let R be a commutative ring with identity and I be a maximal ideal in R . If R is a strongly coretractable ring . Then I=eR for some an idempotent element e in R .

Corollary(2.11): Every maximal submodule K of a strongly coretractable R-module M is strongly coretractable module .

Proof : It follows by Proposition(2.9) and Corollary(1.7) .

Recall that an R-module M is called multiplication module if for each submodule N of M, there exists a right ideal in R such that MI=N [11] .

Corollary(2.12): Let M be a finitely generated R-module (or multiplication module) . If M is a strongly coretractable module . Then M has a nonzero simple direct summand.

Proof : Since M is a finitely generated module (or multiplication module) , then M has a maximal submodule , say N . Hence by Proposition(2.9) , N is a direct summand of M ; that is M=N⊕W for some a submodule W of M . It follows that M/N≅W. But M/N is simple R-module, since N is maximal submodule. Hence W is a simple submodule of M and it is a direct summand. Therefore M has a nonzero simple direct summand.

The condition M is a finitely generated module (or M is a multiplication module) is necessary in proposition (2.12). For example, consider the Z-module $M=Z_{p^\infty}$ is strongly coretractable module , but it is not finitely generated and it is not multiplication module . However M has no simple direct summand.

Corollary (2.13): Let M be a strongly coretractable R-module . If M is a finitely generated module (or multiplication module). Then every proper submodule of M is contained in a direct summand.

Proof: It follows by Proposition (2.9).

Corollary (2.14): Let M be a finitely generated R-module or (a multiplication module). Then M is a strongly coretractable R-module if and only if M has no proper essential submodule (that is M is a semisimple).

Proof: It is clear.

As application of Corollary(2.14) , each of Z-module Z , Z_4 , Z_8 , Z_9 , Z_{12} , … is not strongly coretractable module .

Corollary (2.15): Let R be a ring with identity. Then R is strongly coretractable ring if and only if R is semisimple.

Proof: It is clear by Corollary (2.14).

Corollary (2.16): Let R be a commutative ring with identity. If for each proper nonzero ideal J of R, there exists r ∈ R with r ∈ annJ, 1-r ∈ J then R is a strongly coretractable ring .

Proof: Let (0) ≠J < R . Since there exists r∈R , r∈ annJ , 1- r∈ J , R=J + annJ . Now, let x∈J∩annJ. Then x=x.1= x (1-r)+xr=0 and so R=J⊕annJ. Thus R is a semisimple and hence R is a strongly coretractable ring

Corollary (2.17): Let R be a commutative ring with identity . If R is strongly coretractable ring , then R is regular ring and hence L(R)=0 .

Proof: By Corollary (2.16) , R is a semisimple , so it is regular and hence L(R)=0 .

Corollary (2.18): Let R be a commutative ring with identity. If R is strongly coretractable ring. Then R has no nonzero nilpotent ideal.

Proof: By Proposition (2.17), R is a regular ring . Suppose R has a nilpotent ideal I of R, then $I^n=0$ for some positive integer n. But R is regular ring, so $I^n=\underset{n-time}{I\cap I\ldots\ldots\cap I}=I$. Thus I= 0 .

Proposition(2.19): Let R be a ring with identity . Then the following conditions are equivalent:

1. R is a semisimple ring.

2. Every R-module is strongly coretractable.
3. Every finitely generated R-module is strongly coretractable.
4. Every free R-module is strongly coretractable.

Proof: It is clear.

By Examples and Remarks(1.2(4)) , we show that every epi-coretractable module is strongly coretractable , but not conversely . However the converse is true under the class of hollow modules.

Recall that a proper submodule N of an R-module M is small (denoted by N<<M) whenever N+W=M, where $W \leq M$ then W=M [8]. And an R-module is called hollow if every proper submodule of M is small submodule [12].

Proposition (2.20): Let M be a hollow R-module. If M is a strongly coretractable R-module. Then M is an epi-coretractable R-module.

Proof: Let N be a proper submodule of M . Since M is a strongly coretractable module , then there exists $f \neq 0$ such that $f : M/N \to M$ and Imf+N=M . But M is hollow module , so N<<M which implies that Imf=M . Thus M is epi-coretractable module .

Recall that a submodule N of M is called essential small (briefly e-small and denoted by $N \ll_e M$) if whenever N+W=M and W is an essential submodule of M , then W=M [13] . It is clear that every small submodule is e-small but not conversely , for example in the Z-module Z_{12} , $N=<\bar{2}>$ is e-small submodule , but it is not small . An R-module M is called e-hollow if every proper submodule is e-small [12] . It is clear that every semisimple module is e-hollow , but not conversely, for example the Z-module Z_4 is e-hollow and it is not semisimple .

Hence we have the following corollary :

Corollary(2.21): Let M be a finitely generated (or a multiplication R-module) . If M is a strongly coretractable R-module , then M is e-hollow module .

Proof : By Corollary(2.12) , M has no proper essential submodule . Thus M is e-hollow module .

Corollary(2.22): Let M be a strongly coretractable e-hollow R-module .Then M is semisimple .

Proof : Suppose N is a proper essential submodule of M . As M is strongly coretractable , there exists $f \neq 0$ such that $f : M/N \to M$ and Imf+N=M , but this contradict the hypothesis , M is e-hollow . Thus M has no proper essential , that is M is semisimple .

T.A.Kalati in[14] defined the notion of annihilator small submodule where a submodule N of an R-module M is called an annihilator small (denoted by $N \ll_a M$) if N+W=M for some submodule W of M implies that $l\text{-ann}_S(W)=0$ where $S= End_R(M)$, and $l\text{-ann}_S(W)=\{f \in S : f(W)=0\}$. It is clear that every small is an annihilator small and not conversely (see[14]).

Corollary (2.23): Let M be a strongly coretractable R-module .Then $K \ll_a M$ and $N/K \ll_a M/K$ if and only if $N \ll_a M$.

Proof : **(⟹)** Let $K \ll_a M$. Since M is strongly coretractable module , then M is coretractable module and hence K << M by [13,Proposition(2.2)] . But M/K is strongly coretractable module by Theorem(1.5) , hence M/K is coretractable module but by hypothesis $N/K \ll_a M/K$, So by [14,Proposition(2.2)] , N/K<<M/K , hence N<< M which implies that $N \ll_a M$.

(⟸) It is clear .

Remark (2.24): Let M be an R-module . If E(M) (The injective hull of M) is a strongly coretractable module . It may be that M is not coretractable module and hence it is not strongly coretractable module .

For example , consider the Z-module Z_4 is not strongly coretractable module see Examples and Remarks(1.3(1)) . In spite of $E(Z_4) = Z_{2^\infty}$ is strongly coretractable module .

Remark(2.25): Let M be an R-module . If E(M) is a finitely generated and strongly coretractable R-module . Then M=E(M) .

Proof: Since E(M) is finitely generated and strongly coretractable module . Then E(M) has no proper essential submodule by Corollary(1.12). But $M \leq_e E(M)$, so M=E(M) .

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