DOI: 10.24996/ijs.2023.64.9.22

ISSN: 0067-2904

Generalized Pre α-Regular and Generalized Pre α-Normal Spaces in Topological Spaces

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Received: 16/12/2021 Accepted: 16/10/2022 Published: 30/9/2023

Abstract

The concept of separation axioms constitutes a key role in general topology and all generalized forms of topologies. The present authors continued the study of gpα-closed sets by utilizing this concept, new separation axioms, namely gpα-regular and gpα-normal spaces are studied and established their characterizations. Also, new spaces namely gpα-T_k for k = 0, 1, 2 are studied.

Keywords: gpα-closed set, gpα-open set, gpα-T_k-spaces, gpα-continuous function, gpα-regular spaces, gpα-normal spaces.

1. Introduction

O.Njastad [1] introduced and defined α-open sets. Following the work on α-open sets, many topologists focused on generalization of topological concepts using semi-open and α-open sets. These sets play an important role in the generalization of continuity in topological spaces. Mashhour et al . [2, 3] proposed pre-open sets. Since then, many topologists have applied these ideas to investigate the weak separation axioms, weak regularity and weak normality.

Munshi [4] proposed separation axioms. The literature survey on separation axiom revealed significant work on T_0, T_1, and T_2 space, regular and normal spaces. Maheshwari and Prasad [5] introduced s-normal spaces using semi-open sets. Nori and Popa [6], Dorsett [7] and Arya [8] studied g-regular and g-normal spaces using g-closed sets in topological spaces. Further many topologists studied the separation axioms [9, 10, 11, 12, 13 and 14].

Recently, Patil et al. [15, 16] developed and researched the idea of gpα-closed sets and gpα-continuous functions.

In this paper our aim is to establish and study the weak separation axioms such as gpα-T_0, gpα-T_1 and gpα-T_2 spaces and new classes of spaces, gpα-regular and gpα-normal spaces, using gpα-closed sets. We have also defined the fundamental properties that relate to them.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) symbolise nonempty topological spaces on which no separation axioms are assumed unless explicitly stated and they are simply written X, Y and Z respectively.

Definition 2.1: [15] A subset A of a topological space X is said to be gpα-closed if pcl(A) $\subseteq$U whenever A $\subseteq$ U and U is α-open set in X.

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Definition 2.2: A subset A of X is called
(i) gpα - closure of A and is denoted by gpα-cl(A).[15] and defined as gpα-cl(A)=∩{G: A ⊆ G, G is gpα-closed in X}.
(ii) gpα - interior of A and denoted by gpα- int(A) [15] and defined as gpα-int(A) =∪{G:G⊆ A, G is gpα-open in X}.

Definition 2.3: A function f: X → Y is called
(i) gpα - open [15] if f(V) is gpα-open in X, for every open set V in X.
(ii) gpα -continuous [16] (briefly gpα-c) if $f^{-1}(U)$ is gpα-closed in X, for every closed set U in Y.
(iii) gpα -irresolute [16] (briefly gpα-I) if $f^{-1}(V)$ is gpα-closed in X for every gpα-closed set V in Y.
(iv) Pre-gpα - open [17] if f(V) is gpα- open in X, for every gpα-open set V in X.
(v) Pre-continuous [18] if $f^{-1}(V)$ is pre-open in X, for every open set V in Y.

3. Generalized Pre α-Separation Axioms

The weaker forms of separation axioms are found in this section, such as gpα-T_0, gpα-T_1 and gpα-T_2 spaces and their related concepts.

Definition 3.1: Let (X, τ) be a topological space. Then X is said to be gpα-T_0 if for each pair of distinct points x, y in X, there exists a gpα-open set containing one point but not the other.

Example 3.2: Let X={a_1, a_2, a_3} and τ = {X, φ, {a_3}}. Then the space (X, τ) is gpα-T_0 space.

Theorem 3.3: A space (X, τ) is gpα-T_0 if and only if gpα-closures of distinct points are distinct.

Proof: Let (X, τ) be a gpα-T_0 and x, y ∈ X with x ≠ y. Then there exists gpα-open set U such that x ∈ U, y ∉ U. Also, x ∉ X − U and y ∈ X − U, such that X − U is gpα-closed in X. But gpα−cl({y}) is intersection of the gpα-closed sets, which contains y and hence y ∈ gpα− cl({y}). Since, x ∉ X − U, so x ∉ gpα−cl({n}). Therefore gpα−cl({x})≠ gpα−cl({y}).
Conversely, assume that x, y ∈ X for any pair of different points gpα−cl({x})≠gpα− cl({y}). Then there exists at least one point q ∈ X such that q ∈ gpα−cl({x}) but q ∉ gpα− cl({y}). We have to show x ∉ gpα−cl ({y}). However, assume that x ∈gpα−cl({n}). Then gpα−cl ({x})⊆gpα−cl ({y}). Hence, q ∈gpα−cl({y}) which is a contradiction. Thus x∉ gpα−cl ({y}) implies x ∈ M −gpα-cl ({y}) such that X-gpα-cl({y}) is a gpα-open set in M containing m but not n. So M is gpα-T_0 space.

Theorem 3.4: gpα-T_0 space is hereditary property.
Proof: Let y_1, y_2 be any two distinct points in Y. As Y ⊆ X, then y_1, y_2 are distinct points in X. Since X is gpα-T_0, then there exists gpα-open set U such that $y_1 \in U$, $y_2 \notin U$. Then Y ∩ U is gpα-open set in Y containing y_1 but not y_2. Therefore Y is gpα-T_0 space.

Definition 3.5: A function f: X → Y is pre-gpα-open, if f(V) is gpα-open in X, for every gpα-open set V in X.

Theorem 3.6: Let f: X → Y be bijective and pre gpα-open. If X is gpα-T_0, then Y is also gpα-T_0 space.

Proof: Let f: X → Y be bijective and pre-gpα open and X is gpα-T_0 space. Let y_1, y_2 ∈ Y with $y_1 \neq y_2$. As f is bijective, and let x_1, x_2 ∈ X such that $p(x_1) = y_1$, $p(x_2) = y_2$. Then there exists gpα-open set U in X as a result $x_1 \in U$, $x_2 \notin U$ as X is gpα-T_0. So, f(U) is a gpα-open set containing $f(x_1)$ but not $f(x_2)$. So, there exists gpα-open set f(U) ∈ N such that $y_1 \in f(U)$, $n_2 \notin f(U)$. Thus Y is gpα-T_0 space.

Theorem 3.7: For a topological space X, let gpα-O(X) is open under arbitrary union. The attributes listed below are equivalent:
(i) X is gpα-T_0,
(ii) Each singleton set is gpα-closed,
(iii) Each subset of X is the intersection of all gpα-open set containing it,
(iv) The set {y} is the intersection of all gpα-open set containing the point x ∈ X.

Proof: (i) → (ii) Let x ∈ X where X is gpα-T_0. Then for each y ∈ X such that y ≠ x, there exists gpα-open set U_y containing y but not x. Therefore, $y \in U_y \subseteq \{x\}^c$. Then $\{x\}^c = \cup\{ U_y : y \in \{x\}^c \}$, that is $\{x\}^c$ is the union of gpα-open sets. Hence {x} is gpα-closed.
(ii) → (iii) Assume that (ii) holds. Let A ⊆ X. Then for each point y ∉ A, there exists $\{y\}^c$ such that $A \subseteq \{y\}^c$ where $\{y\}^c$ is gpα-open in X. Hence $A = \cap\{\{y\}^c : y \in A^c\}$ and as a result, set A is the intersection of all gpα-open sets containing A.
(iii) → (iv) Obvious.
(iv) → (i) Assume that (4) holds. Let x, y ∈ X with x ≠ y. According to the hypothesis, there is a gpα-open set U_x, such that $x \in U_x$ and $y \notin U_x$, so the gpα-T_0 space condition is satisfied. As a result, X is gpα-T_0 space.

Definition 3.8: A topological space (X, τ) is said to be a gpα-T_1 space if for each pair of distinct points x, y in X, there are gpα-open sets U and V such that x ∈ U, y ∉ U and y ∈ V, x ∉ V.

Example 3.9: Let $X = \{a_1, a_2, a_3\}$ and $\tau = \{X, \varphi, \{a_1\}, \{a_2, a_3\}\}$. Then the space (X, τ) is gpα-T_1 space.

Remark 3.10: Every T_1- space is gpα-T_1, but the reverse implication is not true.

Example 3.11: Let $X = \{a_1, a_2, a_3\}$ and $\tau = \{X, \varphi, \{a_1\}, \{a_2, a_3\}\}$. Then, the space X is gpα-T_1 space but not T_1- space.

Theorem 3.12: Every gpα-T_1 space is gpα-T_0 space.
Proof: Consider X is gpα-T_1 space. Let c and d be two distinct points in X. Since X is gpα-T_1 space there exist gpα-open sets U and V such that c ∈ U, d ∉ U and c ∉ V, d ∈ V. We have c ∈ U and d ∉ U. Thus, X is gpα-T_0 space.

Remark 3.13: Following example shows the converse of the preceding theorem is not true in general.

Example 3.14: Let $X = \{a_1, a_2, a_3\}$ and $\tau = \{ X, \varphi, \{a_1\}\}$. Then the space X is gpα-T_0 space but not gpα-T_1 space. As the set $\{a_1\}$ is gpα-open for two different points $a_1, a_2 \in X$, such that $a_1 \in \{a_1\}$ and $a_2 \notin \{ a_1\}$, but there is no gpα-open set G such that $a_1 \in \{a_1\}$ and $a_2 \notin \{a_1\}$, for $a_1 \neq a_2$

Theorem 3.15: A space X is gpα-T_1 if and only if X has a singleton subset {x} is gpα-closed in X.

Proof: Let x be any point on X. Let $y \in \{x\}^c$ is true, then $y \neq x$, as X is gpα-T_1 and y differs from x, a gpα-open set G_y must exist such that $y \in G_y$ but $x \notin G_y$. This implies, for each $y \in \{x\}^c$, there exists gpα-open set G_y such that $y \in G_y \subseteq \{x\}^c$. Hence $\cup\{y: y \neq x\} \subseteq \cup\{ G_y: \neq x\} \subseteq \{x\}^c$. So $\{x\}^c \subseteq \cup\{G_y : y \neq x\} \subseteq \{x\}^c$. Hence, $\{x\}^c = \cup\{G_y : y \neq x\}$. As, G_y is gpα-open set, the union of gpα-open sets is also a gpα-open. Thus $\{x\}^c$ is gpα-open in X, and {x} is gpα-closed in X.
Conversely, let $x, y \in X$ be gpα-closed in X and $x \neq y$ be gpα-closed in X. Then $\{x\}^c$ and $\{y\}^c$ are gpα-open in X, with $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}$ but $y \in \{y\}^c$. There are gpα-open sets $\{x\}^c$ and $\{y\}^c$ such that $x \in \{y\}^c$, $y \in \{y\}^c$ and $y \in \{x\}^c$, $x \notin \{x\}^c$. Thus X is gpα-T_1 space.

Corollary 3.16: Every finite subset of X is gpα-closed if and only if the space X is gpα-T_1.

Definition 3.17: [16] A topological space (X, τ) is said to be $T_{gp\alpha}$-space if every gpα-closed set is closed in (X, τ).

Theorem 3.18: Let f: X → Y is bijective, gpα-open with X is gpα-T_1 and $T_{gp\alpha}$-space, and then Y is also gpα-T_1 space.

Proof: Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. As f is bijective and $T_{gp\alpha}$-space, there are unique points x_1, x_2 of X such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, and hence gpα-open sets U and V such that $x_1 \in U$, $x_2 \notin U$ and $x_1 \notin V$, $x_2 \in V$. Thus $y_1 = f(x_1) \in f(U)$, $y_2 = f(x_2) \notin f(U)$ and $y_2 = f(x_2) \in f(V)$, $y_1 = f(x_1) \notin f(V)$. U and V are open sets in X. Since, f(U) and f(V) are gpα-open subsets in Y, f is gpα-open. Thus, in Y there are gpα-open sets f(U) and f(V) such that $y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $m_1 \notin f(V)$. Thus Y is gpα-T_1 space.

Remark 3.19: If p: M → N is gpα -I, injective and N is gpα-T_1 then M is also gpα-T_1 space.

Theorem 3.20: Let p: M → N be gpα-c, injective function and N is T_1-space. Then M is gpα-T_1 space.

Proof: Let $x_1, x_2 \in X$ where $x_1 \neq x_2$. Then y_1, y_2 of Y exist, with $y_1 \neq y_2$ such that $f(x_1) = y_1$ and $p(m_2) = n_2$. As Y is T_1, there are gpα-open sets U and V in Y so that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$, that is $f(x_1) \in U$, $f(x_2) \notin U$ and $f(x_1) \notin V$, $f(x_2) \in V$. Hence $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$, $x_1 \notin f^{-1}(V)$, $x_2 \in f^{-1}(V)$. $f^{-1}(U)$ and $f^{-1}(V)$ are gpα-open sets in X, defined by gpα-c. Thus, for every $x_1, x_2 \in X$ with $x_1 \neq x_2$, there exist gpα open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$ and $x_1 \notin f^{-1}(V)$, $x_2 \in f^{-1}(V)$. As a result, X is gpα-T_1 space

Definition 3.21: Let (X, τ) be a topological space. The space (X, τ) is said to be gpα-T_2 if there are disjoint gpα-open sets U, V such that $x \in U$ and $y \in V$ for each $x, y \in X$ with $x \neq y$.

Example 3.22: Let $X = \{a_1, a_2, a_3\}$ and $\tau = \{X, \varphi, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$. Then (X, τ) is gpα-$T_2$ space.

Remark 3.23: Every gpα-T_2 space is gpα-T_1 space.

Example 3.24: From Example 3.22, it is clear that the space X is gpα-T_2 not gpα-T_1 space.

Theorem 3.25: The intersection of all gpα-closed neighborhoods of each point of X is singleton set if and only if the topological space X is gpα-T_2.

Proof: Let us say x, y ∈ X with x ≠ y. There are gpα-open sets U and V such that x ∈ U, y ∈ V and U ∩ V = φ, that is x ∈ U ⊆ X −V according to the definition 3.21. As a result, X − V is a gpα-closed neighborhood of x that excludes y. So, y does not form part of the intersection of all gpα-closed neighborhoods of x. Since y is an arbitrary point, the singleton set {x} is the intersection of all gpα-closed neighborhoods of x. In the alternative, consider x to be the intersection of all gpα-closed neighborhoods of any arbitrary point x ∈ X and y to be any arbitrary point of X such that x ≠ y. Since y does not belong to the intersection, there exists a gpα-closed neighborhood Y of x such that y ∉ Y. Then there is a gpα-open set U such that x ∈ U ⊆Y. Consequently, U and X − N are gpα-open sets such that x ∈ U and y ∈ X − Y such that U ∩ (X −Y) = φ. As a result, X is gpα-T_2 space.

Theorem 3.26: Let f: X → Y is gpα-c, injective and Y is T_2- space. Then X is gpα-T_2 space.

Proof: Consider any two distinct points of X to be x_1 and x_2. Because f is injective, disjoint points y_1 and y_2 of Y exist with $y_1 = f(x_1)$ and $y_2 = f(x_2)$. There exist disjoint open sets U and V such that y_1 ∈ U, y_2 ∈ V due to a property of T_2-space. Thus, in X, $x_1 \in f^{-1}$ (U) and $x_2 \in f^{-1}$ (V) are gpα-open sets. Consider f^{-1} (U) ∩ f^{-1} (V) = f^{-1}(U ∩V) = f^{-1}(φ) = φ. Thus for each x_1, x_2 ∈ X with $x_1 \neq x_2$, there are gpα-open sets f^{-1} (U) and f^{-1} (V) with f^{-1} (U) ∩ f^{-1} (V) = φ such that $x_1 \in f^{-1}$ (U), $x_2 \in f^{-1}$ (V). Thus X is gpα-T_2 space.

Remark 3.27: If f: X → Y is gpα -I, injective and Y is gpα-T_2 then X is also gpα-T_2 space.

Theorem 3.28: For any topological space X, the following properties are equivalent.
(i) X is gpα-T_2,
(ii) There is gpα-open set U such that x ∈ U and y ∉ gpα-cl (U) for each distinct points x, y,
(iii) For each point x∈ X, we have {x} =∩{gpα-cl(U):U is gpα-open in X such that x ∈ U}.

Proof: (i) → (ii) Let us say x, y ∈ X with x≠ y. Then there are gpα-open sets U and V with U ∩V = φ such that x ∈ U, y ∈ V exist. X-V is gpα-closed in X. Since U ∩V = φ, U ⊆ X-V is the result. So, gpα-cl (U) ⊆gpα-cl (X−V) because X-V is gpα-closed. Since y∉ X-V, implies y ∉ gpα-cl (U).
(ii) → (iii) Let x, y ∈ X with x ≠ y. Then there is a gpα-open set U such that x ∈ U, y ∉ gpα-cl(U) exists. Hence, y ∉ {gpα-cl (U) : U is gpα-open in X and x ∈ U} = {x}.
(iii) → (i) Let x, y ∈ X with x ≠ y. Then there is a gpα-open set U such that x ∈ U, y ∉gpα-cl(U) exists. Then there exists a gpα-closed V such that y ∈ V that is y ∈ X −V, where X -V is gpα-open. So, gpα-open sets U and X -V such that x ∈ U, y ∈ X − V and U ∩ (X −V) = φ. Hence X stands for gpα-T_2 space.

Theorem 3.29: Let X be gpα-T_2, $T_{gp\alpha}$-space and Y be a subspace of X. Then Y is also gpα-T_2 space.

Proof: If x, y ∈ Y then x, y ∈ X as Y ⊂ X. Then gpα-open sets U and V such that U ∩ V = φ. Since X is $T_{gp\alpha}$-space, U and V are open sets in X while U ∩ Y and V ∩ Y are open sets in Y. Therefore, U ∩ Y and V ∩ Y are gpα-open sets in Y. In addition, we have x ∈ U and x ∈Y, implying that x ∈ U ∩ Y. In a similar way y ∈ V ∩ Y. Consider (U ∩ Y) ∩ (V ∩ Y)=Y ∩ (U ∩ V)= Y ∩ φ = φ. Thus for each x, y ∈ Y with x ≠ y, there exist disjoint gpα-open sets U ∩ Y and V ∩ Y such that x∈ U ∩ Y and y ∈ V ∩ Y have (U ∩ Y) ∩ (V ∩ Y) = φ. Hence, Y is gpα-T_2 space.

4. Characterizations of Generalized Pre α -Difference Sets

In this section, new types of separation axioms are defined and studied in topological spaces called gpα-D_k for k = 0, 1, 2, and also some properties of these spaces are explained.

Definition 4.1: A subset A of a space X is called a gpα-difference set (gpα-D Set) if there exist two gpα-open sets U and V in X with $U \neq M$ and $A = U \setminus V$.

Remark 4.2: Every proper gpα-open set is gpα-D set.

Proof: Assume A is a proper gpα-open subset of X. Put U = A and V = φ. Then $A = U \setminus V$ with $U \neq X$. Hence, A is gpα-D set.

Definition 4.3: A space X is said to be a gpα-D_0, if there exists gpα-D set of X including x but not y or gpα-D set of X containing y but not x for any distinct points x, y in X.

Definition 4.4: A space X is said to be gpα-D_1, if there exists gpα-D set of X including x but not y and a gpα-D set of X containing y but not x for any pair of distinct points x, y in X.

Definition 4.5: A space X is gpα-D_2, if there are disjoint gpα-D sets U and V of X containing x and y for any pair of different points x, y of X.

Remark 4.6: Every gpα-T_k space is gpα-D_k space.

Proof: As a sample, we prove for k = 0. Assume that X is gpα-T_0 space and then for each pair of distinct points x, y ∈ X, there exists a gpα-open set U with $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$. From Remark 4.2 we know that every proper gpα-open set is gpα-D set. Thus U is gpα-D set in X. So for each x, y ∈ X with $x \neq y$ there exists a gpα-D set U with $x \in U$, $y \notin U$ or $x \notin U$, $y \in U$. Hence, (X, τ) is gpα-D_0 space.
Similarly, if we take k = 1, 2 we may establish the result. Hence, every gpα-T_1 space is gpα-D_1 space and every gpα-T_2 space is gpα-D_2 space.

Remark 4.7: Every gpα-D_k space is gpα-D_{k-1} space.

Proof: Let us take k = 2. Let (X, τ) be gpα-D_2 space. Then there are gpα-D sets U and V containing x and y respectively for each x, y ∈ X with $x \neq y$. It follows, for each x, y ∈ X with $x \neq y$, a gpα-D set U including x but not y and gpα-D set V containing y but x is not exists, indicating that (X, τ) is gpα-D_1 space. Similarly we can show that gpα-D_1 space is gpα-D_0 space.

Remark 4.8: Every pre-D_k space is gpα-D_k space.

Proof: Since we know that every pre-T_k space is gpα-T_k space and every gpα-T_k space is gpα-D_k space. So, every pre-D_k space is gpα-D_k space.

Definition 4.9: A space X is said to be gpα-symmetric if for each x, y in X, x ∈ gpα-cl ({y}) then y ∈ gpα-cl ({x}).

Theorem 4.10: In a space X. The space X is gpα-symmetric if and only if {x} is gpα-closed, for each x ∈ X.

Proof: Let us consider {x} ⊆ U ∈gpα-O(X) but gpα-cl ({x}) ⊆U. So gpα-cl ({x}) ∩ (X\U) ≠φ. Now consider y ∈gpα-cl ({x})∩(X \U) from hypothesis, x ∈gpα-cl({y}) ⊆ (X \ U) and x∉ U, which is a contradiction. Thus {x} is gpα-closed, for each point x ∈ X.
Conversely, assume that x ∈ gpα-cl({n}) where y ∉ gpα-cl({x}). Then {y} ⊆ X \ gpα-cl({x}) and so gpα-cl({y}) ⊆ X \ gpα-cl({x}). Thus x ∈ X \ gpα-cl({x}) which contradicts the hypothesis. Thus y ∈ gpα-cl({x}). So X is gpα-symmetric space.

Corollary 4.11: A space X is gpα-T_1, and then it is a gpα-symmetric.

Proof: Theorem 3.15, shows that every singleton sets are gpα-closed, and from Theorem 4.10, X is gpα-symmetric. Hence X is gpα-symmetric space.

Theorem 4.12: Every gpα-symmetric and gpα-T_0 space is gpα-T_1 space.

Proof: Let X be a gpα-T_0 and gpα-symmetric space, and x, y∈ X with x ≠y. As X is gpα-T_0, assume that x ∈ U ⊆ X \ {y}, where U ∈gpα-O(X). So x ∉ gpα-cl ({y}) and hence y ∉gpα-cl ({n}) as X is gpα-symmetric. Then there exists gpα-open set V in X such that y ∈ V ⊆ X \ {f}. Hence, X is gpα-T_1 space.

Corollary 4.13: Every gpα-D_1 space is gpα-T_0 space.

Definition 4.14: Let A be a subset of a space X. Then, gpα-ker (A) denotes the gpα-kernel of A which is defined as gpα-ker (A) =∩ {U ∈gpα-O(X): A ⊆ U}.

Theorem 4.15: Let X is a space and x ∈ X. Then y ∈gpα-ker(x) if x ∈gpα-cl ({y})

Proof: Let y ∉ gpα-ker(x). Then there exists gpα-open set V containing x with y ∉V. Thus x ∉gpα-cl({y}). Hence, y ∈gpα-ker(x) if x ∈gpα-cl({y}).

Theorem 4.16: Let X is a space and A⊂ X. Then gpα-ker (A) ={x ∈ X: gpα-cl({x}) ∩ A ≠φ}.

Proof: Let x ∈ gpα-ker (A), and gpα-cl({x})∩A = φ}. Hence x ∉gpα-cl ({x}), which is a gpα-open set containing A, which is impossible, since x ∈gpα-ker(A). So gpα-cl({x})∩A ≠ φ}. On the other hand, let x∈ X with gpα-cl({x}) ∩ A ≠ φ} and assume that x ∉gpα-ker(A). Then there exists gpα-open set V containing x with x ∉ V such that y ∈gpα-cl ({x}) ∩ A}. Then, V is gpα-nbd of y which does not contain the point x, which is a contradiction to the fact that x ∈gpα-ker (A).

Theorem 4.17: If a singleton set {x} is gpα-D set in (X, τ). Then gpα - ker{x} ≠ φ.

Proof: Let {x} is gpα-D set. Then there exist two subsets U, V∈gpα-O(X, τ) with {x} = U \ V, where {x} ⊆ U and U ≠ X. So, gpα-ker{x} ⊆ $_1$ ≠ X. Thus gpα - ker{x} ≠ φ.

5. Generalized Pre α -Regular Spaces

The concept of gpα-regular spaces and their properties are studied in this section

Definition 5.1: A space X is said to be gpα-regular if there are disjoint open sets U and V such that F ⊆ U, x ∈ V for every gpα-closed set F and a point x ∉ F.

Remark 5.2: Every gpα-regular space is regular. However, as the following example shows, the converse is not always true.

Example 5.3: Let $X = \{a_1, a_2, a_3\}$ and $\tau = \{M, \varphi, \{a_1\}, \{a_2, a_3\}\}$. Then the space X is regular but not gpα-regular.

Theorem 5.4: Let X is gpα-regular, and Y is a gpα-closed and open subset of X. Then a subspace Y is gpα-regular.

Proof: Let F be any gpα-closed subset of Y and $y \in Y - F$. Then F is gpα-closed set in X, by the Theorem 3.21 of [18]. As X is gpα-regular, then there exist disjoint open sets U and V such that $y \in U$ and $F \subseteq V$. So $U \cap Y$ and $V \cap Y$ are disjoint open sets in the subspace Y, such that $y \in U \cap Y$ and $F \subseteq V \cap Y$. Thus Y is gpα-regular.

Theorem 5.5: Consider the topological space (X, τ). The following statements are identical in this case:
(i) X is gpα-regular.
(ii) There exists an open set U of x such that cl (U) ⊆ W, for each $m \in M$ and each gpα-open neighborhood W of m.
(iii) There exists an open set V such that $cl (V) \cap F = \varphi$, for each point $x \in X$ and each gpα-closed set F such that $x \notin F$.

Proof: (i) → (ii) Allow W to be any gpα-open neighborhood of x in X. Then there is a gpα-open set G and W such that $x \in G \subseteq W$, and because $X - G$ is gpα-closed and $x \in X - G$, there must exist open sets U and V such that $X - G \subseteq U$, $x \in V$ with $U \cap V = \varphi$. As a result $V \subseteq X - U$. Now, $cl(V) \subseteq cl(X - U) = X - U$ and $X - G \subseteq U$ implies that $X - U \subseteq G \subseteq W$. Hence, clU) ⊆ W.
(ii)→ (i) Let F be any gpα-closed set in X that has the property $x \notin F$. Then $x \in X - F$, where $X - F$ is gpα-open and hence $X - F$ is gpα-neighborhood of x. According to the hypothesis, there exists an open set V such that $x \in V$ and $cl(V) \subseteq X - F$, that is $F \subseteq cl(X - V)$. Then $cl(X - V)$ is an open set containing F with $V \cap cl(X - V) = \varphi$. Hence X is gpα-regular.
(ii)→ (iii) Let $x \in X$ and F be any gpα-closed set in X such that $x \notin F$. Then $X-F$ is a gpα-neighborhood of x. According to the hypothesis, there is an open set V such that $cl (V) \subseteq M-F$. As a result $cl (V) \cap F = \varphi$.
(iii) → (ii) Let $x \in X$ and W be any gpα-neighborhood of m. Then there is an open set G in which $x \in G \subseteq W$ exists. Because $X - G$ is gpα-closed and $x \notin X - G$. Then, according to the hypothesis, there exists an open set U for which $cl (U) \cap X-G = \varphi$. Hence $cl(U) \subseteq G \subseteq W$.

Theorem 5.6: A gpα-regular is hereditary property.

Proof: Let X be a gpα-regular space and $Y \subset X$. Let $x \in Y$ and F be a gpα-closed set in Y with $x \notin F$. Then, there exists gpα-closed set A in X with $F=Y \cap A$ and $x \notin A$. Therefore, $x \in X$ and A is gpα-closed in X with $x \notin A$. Then, there are open sets G and H such that $x \in G$, $A \subseteq H$ and $G \cap H = \varphi$. Here, $Y \cap G$ and $Y \cap H$ are open sets in Y. Also, we have $x \in G$ and $x \in Y$ that is $x \in Y \cap G$ and $A \subseteq H \Rightarrow (Y \cap A) \subset (Y \cap H) \Rightarrow F \subset Y \cap H$ and $(Y \cap G) \cap (Y \cap H) = \varphi$. So, for each $x \in Y$ and each closed set F in Y, there exists an open sets $Y \cap G$ and $Y \cap H$ such that $x \in Y \cap G$ and $F \subseteq Y \cap H$ with $(Y \cap G) \cap (Y \cap H) = \varphi$. Hence Y is gpα-regular.

Theorem 5.7: If $f: X \rightarrow Y$ is gpα-I, bijective, open and M is gpα-regular, then N is also gpα-regular.

Proof: In Y, let F be a gpα-closed set with y ∉ F. For some point x ∈ X, Put y = f(x). Then $f^{-1}(y)$ in X, f^{-1} (F) is gpα-closed since f is gpα-I with x ∉ f^{-1}(F). There are disjoint open sets U and V such that x ∈ U, f^{-1} (F) ⊆ V exist as X is gpα-regular. So, y = f(x) ∈ f(U) and F ⊆ f(V). Thus, there exist open sets f (U) and f(V) such that y ∈ f(U) and F ⊆ f(V) with f(U) ∩ f(V) = φ for all y ∈ Y and for each gpα-closed set F in Y with y ∉ F. Thus Y is gpα-regular.

Theorem 8.8: If f: X → Y is pre gpα-open closed injective and Y is gpα-regular, then X is gpα-regular.

Proof: Let x∈ X and F be gpα-closed set in X, with x ∉ F. As f is pre gpα-open, f(F) is gpα-closed in Y such that f(x) ∉ f(F). Then as Y is gpα-R, there exist an open sets U and V such that f(x) ∈ U and f(F) ⊆ V, that is x ∈ f^{-1}(U) and F ⊆ f^{-1}(V) with f^{-1}(U) ∩ f^{-1}(V) = φ. Therefore for all x ∈ X and for every gpα-closed set F in X with x ∉ F, there exist disjoint open sets f^{-1}(U) and f^{-1}(V) such that f ∈ f^{-1} (U) and F ⊆ f^{-1} (V) with f^{-1} (U) ∩ f^{-1}(V) = φ. Hence X is gpα-regular.

6. Generalized Pre α -Normal Spaces

The concept of gpα-normal spaces and their properties are studied in this section**.**

Definition 6.1: A space X is said to be gpα-normal if for any pair of distinct gpα-closed sets A and B in X there exist disjoint open sets U and V in X such that A⊆U, B⊆V.

Example 6.2: Let X = {a_1, a_2, a_3, a_4} and τ = {X, φ, {a_1}, {a_2}, {a_1, a_2}, {a_2, a_4}, {a_1, a_2, a_4}, {a_2, a_3, a_4}}. Then X is gpα-normal space.

Remark 6.3: Every gpα-normal space is normal. However, as in the following example, the converse is not true in general.
Proof: Let A, B ∈ C(X) with A∩B= φ. Since every closed set is gpα-closed. A, B ∈ gpα -C(X). Then by gpα-normality, there exists U, V ∈ O(X) with U∩V= φ such that A ⊆ U and B ⊆ V. Hence, X is normal.

Example 6.4: Let X = {a_1, a_2, a_3} and τ = {X, φ, {a_1}, {a_2, a_3}}. Then the space X is normal but not gpα-normal.

Theorem 6.5: For a topological space X is normal if and only if there exists gpα-open set U in X such that A ⊆ U ⊆ cl(U) ⊆ V for each closed set A and any open set V containing A.
Proof: Let A be any closed set and V be an open set containing A. Then A, X − V are disjoint closed sets in X. Then, there exist disjoint gpα-open sets U and W such that A ⊆ U, X − V ⊆ W. As X-V is closed, then X-V is gpα-closed. Hence X – V ⊆ int (W) and U ∩int(W) = φ and so cl(U) ∩int(W) = φ. Thus A ⊆ U ⊆cl(U) ⊆ X −int(W) ⊆ V that is A ⊆ U ⊆ cl(U) ⊆ V .
Conversely, let us consider disjoint closed sets A and B in X. Then A⊆ X −B, where X −B open in X. Then by (ii), there exists gpα-open set G such that A ⊆ G ⊆cl(G) ⊆ X − B. As A is closed, then A is gpα-closed. Thus A ⊆int(G). Put U = int(int(G)) and V = int (int (X−int (G))). Thus U and V are disjoint open sets in X such that A ⊆ U, B ⊆V. Hence X is normal.

Theorem 6.6: Let f: X →Y be gpα-I, bijective, open mapping and X is gpα-normal. Then Y is also gpα-normal space.
Proof: Let A and B be two gpα-closed sets in Y that are disjoint. In X, f^{-1}(A) and f^{-1}(B) are disjoint gpα-closed sets since f is gpα-I. Then there exist disjoint open sets U and V such that

$f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$, as f is open and bijective f(U) and f(V) are open sets in Y such that $A \subseteq f(U)$ and $B \subseteq f(V)$ with $f(U) \cap f(V) = \varphi$. Hence Y is gpα-normal.

Theorem 6.7: Let X is gpα-normal, Y is gpα-closed subset of X. Then the subspace Y is gpα-normal.
Proof: Let A and B be two gpα-closed sets in Y that are disjoint. Then A and B are disjoint gpα-closed sets in X. There are disjoint open sets U and V such that $A\subseteq U$ and $B \subseteq V$ exist as X is gpα-normal. As a $U \cap Y$ and $V \cap Y$ are disjoint open subsets of the subspace Y with $A \subseteq U \cap Y$ and $B \subseteq V \cap Y$ respectively. Thus, Y is gpα-normal.

Theorem 6.8: If f: X → Y is pre gpα-closed, continuous, and injective, Y is gpα-normal. Then X is gpα-normal.
Proof: Let A and B be any two gpα-closed disjoint sets in X. f(A) and f(B) are disjoint gpα-closed sets in Y since f is pre gpα-closed. There are disjoint open sets U and V such that $f(A) \subseteq U$ and $f(B) \subseteq V$ exist since Y is gpα-normal. As a result $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$ with $f^{-1}(U)\cap f^{-1}(V) = \varphi$, as f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ in X are open sets. There are disjoint open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$ for any two disjoint gpα-closed sets A and B in X. Hence X is gpα-normal.

Remark 6.9: If f: X → Y be gpα-I, bijective, open map from a gpα-normal space X onto the space Y. Then Y is gpα-normal.

7. Generalized Pre α -Closed graphs

We say that a function f: X → Y has a closed graph if the graph of a function f, that is the set $\{(x, y) \in X \times Y: y - f(x), x \in X\}$ is a closed subset of the product $X \times Y$.

Definition 7.1: A function f: X → Y is said to be a strongly gpα-closed graph if for each $\{(x, y) \in X \times Y - G(f)\}$, there is a gpα-open set U and a gpα-open set V in Y containing x and y respectively such that $(U \times V) \cap G(f) = \varphi$. Where G(f) is graph of a function f.

Lemma 7.2: For a function f: X → Y, the graph G(f) is said to be strongly in $X \times Y$, if and only if there exist gpα-open sets U, V such that $f(U) \cap V= \varphi$ for each $\{(x, y)\in X \times Y - G(f)\}$.
Proof: Let f be strongly gpα-closed graph. Then, for each $x \in X$ and $y \in Y$ with $y \neq f(x)$, there are gpα-open sets U and V containing x and y respectively such that $(U \times V) \cap G(f) = \varphi$, that is for each $x \in X$ and $y \in Y$ with $y \neq f(x)$. Hence $f(U) \cap V = \varphi$.
Conversely, let $(x, y) \notin G(f)$. Then there are gpα-open sets U and V each having x and y such that $f(U) \cap V = \varphi$. Then we have $f(x) \neq y$ for each $x \in X$ and $y \in Y$. Hence $(U \times V) \cap G(f) = \varphi$. Thus, p has a strongly gpα-closed graph.

Theorem 7.3: Let f: X → Y be injective with strongly gpα-closed graph G(f). Then X is gpα-T_1 space.
Proof: Let $x, y \in X$ with $x \neq y$ that is $f(x) \neq f(y)$. Then $(x, f(y)) \in X \times Y - G(f)$. As G (f) is strongly gpα-closed graph, there exist gpα-open sets U and V containing x and f(y) respectively such that $f(U)\cap V = \varphi$, and so $y \neq U$. Similarly there exist gpα-open sets Q and R containing y and f(x) with $f(Q) \cap R = \varphi$, so $x \in Q$. Hence, for each $x, y \in X$ with $x \neq y$, there exists a gpα-open set U containing x but not y and gpα-open set Q containing y but not x. Hence, X is gpα-T_1 space.

Theorem 7.4: If f: X → Y is surjective with strongly gpα-closed graph G (f), then Y is gpα-T_1 space.

Proof: Let us say x, y ∈ Y where x ≠ y. There exists o ∈ X with f(o) = y since f is surjective. As a result of Lemma 7.2 (o, x) ∈G(f). There are gpα-open sets U and V containing o and x ,respectively with f(U)∩V = φ, so y ∉ V . Similarly, there exists w ∈ X with f (w) = y and hence (w, x) ∈ G (f). Then there are gpα-open sets Q and R, each containing w and y, such that f (Q) ∩ Y = φ and x ∉ R. As a result, there exist gpα-open sets V and R with x ∈ V and y ∉ V and x ∉R and y ∈ R for any x, y ∈ Y with x ≠ y. Hence Y stands for gpα-T_1 space.

Theorem 7.5: Let f: X → Y be bijective with strongly gpα-closed graph G(f). Then both X and Y are gpα-T_1.
Proof: Follows from Theorem 7.3 and Theorem 7.4

Theorem 7.6: Let f: X → Y is gpα-c and Y is gpα-T_2 space. Then G (f) is strongly gpα-closed.
Proof: Let (x, y) ∉G(f), and hence (x, y) ∈ (X × Y) − G(f), as Y is gpα-T_2 space, y ∉ f(x). There are two gpα-open sets U and V that have f(x) ∈ U, y ∈ V in Y and U ∩ V = φ. There is a gpα-open neighborhood W of X with f(w) ⊂ U according to gpα-open neighborhood W of X with f(w) ⊂ U according to gpα-continuity. Hence, f(W) ∩ V = φ. Thus (W × V) ∩ G (f) = φ. Hence f is strongly gpα-closed graph.

Definition 7.7: [16] A function f: X → Y is said to be pre gpα-open if the image of every gpα-open set in X is gpα-open in Y.

Theorem 7.8: Let f: X → Y be pre gpα-open surjective function with strongly gpα-closed graph G (f). Then Y is gpα-T_2 space.
Proof: Let y, w ∈ Y with y ≠ w. As f is surjective, there exist x, o ∈ X such that f(x) = y, f (o) = w, and so (x, w) ∉ G(f). Because G (f) is strongly gpα-closed graph, there are gpα-open sets U and V containing x and w respectively such that f(U) ∩ V = φ. However, because f(U) is gpα-open, it must include a point y. Hence, Y is gpα-T_2 space.

Theorem 79: If f: X → Y is injective, gpα-c with strongly gpα-closed graph G (f) and Y is gpα-T_2 space. Then X is gpα-T_2 space.
Proof: Let x, y ∈ X with x ≠ y. There are disjoint open sets U and V in Y with f(x) ∈ U, f(y) ∈ V since Y is gpα-T_2 space. Then, because f is gpα-c, f^{-1} (U) and f^{-1} (V) are gpα-open sets in X containing x and y respectively. As a result, f^{-1} (U) ∩ f^{-1} (V) = φ. Therefore, for each x, y ∈ M with x ≠ y, there exist gpα-open sets f^{-1} (U) and f^{-1} (V) containing x and y respectively such that f^{-1} (U) ∩ f^{-1} (V) = φ. Thus x ≠ y is gpα-T_2 space.

8. Conclusions

The class of generalized closed sets has an important role in general topology. In this work we introduced and study new types of separation axioms, namely gpα-T_k (k = 0, 1, 2) spaces, using gpα-closed sets. Several characterizations and properties of these concepts are provided. Their implication with other separation axioms have also been examined and emphasized which extends the future scope of normal and regular topological spaces. Also, we have investigated some results on gpα-Difference sets and gpα-closed graphs.

9. Acknowledgment
The second author is grateful to Karnatak University Dharwad for the financial support to research work under URS scheme.

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