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Full Transformation Semigroup of A Free Left *S*-Act on *N*-Generators

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Abstract

It is well known that the wreath product $S \wr_n \mathcal{T}_n$ is the endmorphism monoid of a free S-act with n-generators. If S is a trivial semigroup then $S \wr_n \mathcal{T}_n$ is isomorphic to $\mathcal{T}_n$. The extension for $\mathcal{T}_n$ to $\mathcal{T}_{\mathbf{A}}$, where **A** is an independent algebra has been investigated. In particular, we consider **A** is to be $\mathbf{F_n}(\boldsymbol{S})$, where $F_n(S) = \dot{\cup}_{i=1}^{n} Sx_i$ is a free left S-act of n-generators. The eventual goal of this paper is to show that $\mathcal{T}_{\mathbf{F_n(S)}}$ is an endomorphism monoid of a free left S-act of n-generators and to prove that $\mathcal{T}_{\mathbf{F_n(S)}}$ is embedded in the wreath product $S \wr_n \mathcal{T}_n$.

Keywords: *Semigroups, Transformation semigroups, Endomorphism monoid, Wreath product.*

تحويلات شبة الزمرة الكاملة للمجموعة S الحرة اليسارية على nمن المولدات

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الخلاصة

من المعروف ان الضرب الاكليلي $S \wr_n \mathcal{T}_n$ هي شبة زمرة تشاكلات شاملة ذات محايد للمجموعة S الحرة اليسارية ذات n من المولدات. اذا كانت S مجموعة تافهة هذا يعني ان $S \wr_n \mathcal{T}_n$ هي تشاكل مع $\mathcal{T}_n$. توسيع النتائج من T_n الى $\mathcal{T}_n$ هي مجموعة جبرية مستقلة قد درست وبرهنت. في الخصوص تم اعتبار ان **A** هي $\mathrm{F_n}(S)$ عندما Sx_i $\dot{\cup}_{i=1}^{n}$= $F_n(S)$ هي مجموعة حرة يسارية على n من المولدات. الهدف الرئيسي لهذا البحث هو برهان ان $\mathcal{T}_{\mathbf{F_n(S)}}$ هي شبة زمرة تشاكلات شاملة ذات محايد على شبة الزمرة الحرة اليسارية على n من المولدات وكذلك برهان ان $\mathcal{T}_{\mathbf{F_n(S)}}$ هي مجموعة مغمورة في الضرب الاكليلي $S \wr_n \mathcal{T}_n$.

1. Introduction

Transformation semigroups such as the symmetric group S_X, the full transformation semigroup $\mathcal{T}_X$, and the partial transformation semigroup $\mathcal{PT}_X$ are the most fundamental sets in semigroup theory due to the Cayley's theorem which states that every semigroup is isomorphic to a transformation semigroup $\mathcal{PT}_X$. Under the semigroup operation of composition, the full transformation on a non-empty set X is a semigroup $\mathcal{T}_X$ of all functions from the set X into itself. Throughout the paper, we write X_n for the finite set$\{1,2,\dots,n\}$, and $\mathcal{T}_n$ for the transformation on X_n.

The concept of an endomorphism of independent algebra was first investigated by Gould [1]. She obtained results by characterizing Green's relation on nd A, where A is an independence algebra. Narkiewicz [2] introduced independence algebra which is a class of universal algebras .

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The wreath product of semigroup theory is a specialized product of two semigroups, which is based on a semidirect product. Any two semigroups can be combined to construct a third semigroup which is known as a wreath product of the two semigroups. Many authors have been used the construction of wreath product in group theories for many years as well as in the middle of the last century is widely used in semigroups[3,4]. The abstract wreath product construction was extended to semigroups by B.H. Neumann in[5].

For a monoid S and the full transformation semigroup $\mathcal{T}_n$, the wreath product $S \wr_n \mathcal{T}_n$ can be defined as $S^n \times \mathcal{T}_n$ such that for $1 \leq i \leq n, s_i \in S$ and $\alpha \in \mathcal{T}_n$, then $(s_1, \dots, s_n, \alpha) \in S \wr_n \mathcal{T}_n$. If S is a trivial monoid that means $S = \{1_S\}$, then $S \wr_n \mathcal{T}_n \cong \mathcal{T}_n$, and if $\mathcal{T}_n = I_n$, then $S \wr_n \mathcal{T}_n \cong S^n$, where $I_n: X_n \to X_n$ the identity transformation on X_n. It has been known that the wreath product $S \wr_n \mathcal{T}_n$is the endomorphism monoid of a free left S-act with n generators.

In this paper, we found the structure of the endomorphism monoid of a full transformation semigroup T_{A}, where A is an independence algebra. First, we consider the semigroup $\mathcal{T}_{\mathbf{F_n}(S)}$, where $F_n(S) = \dot{\bigcup}_{i=1}^{n} Sx_i$ is a free left S-act of n-generator. It is clear that if $S = \{1_S\}$, then $\mathcal{T}_{\mathbf{F_n}(S)} \cong \mathcal{T}_n$. We also show that $\mathcal{T}_{\mathbf{F_n}(S)}$ is isomorphic to $EndF_n(S)$. After that, the description of Green's relation for $\mathcal{T}_{\mathbf{F_n}(S)}$ has been found. Finally, we prove that the semigroup $\mathcal{T}_{\mathbf{F_n}(S)}$ is embedded in the wreath product $S \wr_n \mathcal{T}_n$.

2. Preliminaries

Definition 2.2 [6,7] Let $\emptyset \neq X$ be a set and let

$$\mathcal{T}_X = \{\alpha: \alpha \; is \; a \; transformation \; on \; X\}$$

If $X = X_n = \{1, \dots, n\}$, then we will write $\mathcal{T}_n$ for $\mathcal{T}_X$. Under the semigroup operation of composition the set $\mathcal{T}_X$ be a semigroup i.e., for any $\alpha, \beta \in \mathcal{T}_X$ we have $x(\alpha \circ \beta) = x(\alpha\beta) = (x\alpha)\beta$ for all $x \in X$. We often write the operation of composition as multiplication. Note that a semigroup $\mathcal{T}_X$ is a monoid as it has the identity transformation I_X such that $I_X \circ \alpha = \alpha \circ I$, for any $\alpha \in \mathcal{T}_X$.

If $\alpha \in \mathcal{T}_n$, we often write α as

$$\alpha = \begin{pmatrix} 1 & \dots & n \\ 1\alpha & \dots & n\alpha \end{pmatrix}.$$

As not every element $\alpha \in \mathcal{T}_n$ is one-to-one the second row in α is not a permutation of the first row. It is easy to check that the number of elements in $\mathcal{T}_n$ is n^n, [7].

Definition 2.3 [7] Let S and T be semigroup. A mapping $\varphi: S \to T$ is called a semigroup morphism of S into T if $(ss')\varphi = (s)\varphi(s')\varphi$ for all $s, s' \in S$. A semigroup morphism between monoids S and T with $(1_S)\varphi = 1_T$is called a monoid morphism.

Definition 2.4[7] Let S be a monoid and A is a non-empty set. We call A a left-S-act and write sA, if we have a mapping $\mu: S \times A \to A$, $(s, a) \mapsto sa \coloneqq \mu(S, a)$ such that; $s(ta) = (st)a$ and $1_S a = a$ for all $a \in A$ and $s, t \in S$.

Definition 2.5[7] Let sA and sB be two left S-acts. A mapping $\alpha: sA \to sB$ is called an S-morphism if $(sa)\alpha = s(a\alpha)$ for all $a \in sA$ and $s \in S$. The identity mapping $I_S: sA \to sA$ is clearly an S-morphism. An S-morphism $\alpha: sA \to sA$ is called endomorphism of sA.The set which forms a monoid under composition of mappings is denoted by $End\ sA$.

2.1 The structure of the endomorphism monoid of a free-left S-act

Definition 2.6 [6] Let X be a non-empty set, $F_X(S)$ is called free S-act on X if

1. There is a map $\alpha: X \to F_X(S)$;
2. For every S-act A and every map $\phi: X \to A$ there exist a unique morphism $\psi: F_X(S) \to A$ such that the diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\phi"'] & F_X(S) \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

For a non-empty set X, we can define the set $S \times X$ into left S-act by defining an action of S on $S \times X$ by $s(t,x) = (st,x)$ and $1_S(t,x) = (t,x)$. It is clear that $S \times X$ is free left S-act on X.

Let $F_X(S)$ be the set of all expression of the form sx, $s \in S$, $x \in X$, $sx = ty$ for $s,t \in S$, $x,y \in X$ if and only if $s = t$ and $x = y$.
We can say $F_X(S)$ is a left S-act by putting $s(tx) = (st)x$ and $1_S x = x$. It is clear that $\times X \cong F_X(S)$, where $(s,x) \mapsto sx$.
We shall usually consider the set $F_X(S) = F_n(s)$, where $X = \{1, \dots, n\}$. Let S be a monoid and $F_n(S) = Sx_1 \dot{\cup} \dots \dot{\cup} Sx_n$ such that $Sx_i = \{sx_i : s \in S\}$, where $i \in \{1, \dots, n\}$, $1x_i = x_i$ and $sx_i = tx_j$ if and only if $i = j$ and $s = t$. An action of S on $F_n(S)$ is defined by

$$s(tx_i) = (st)x_i$$

for all $s \in S$, $tx_i \in F_n(S)$.
Definition 2.7 [9] The set of all morphism from the free left S-act into itself is called the endomorphism monoid of a free left S-act and denoted by $EndF_n(S)$ such that

$$EndF_n(S) = \{\alpha | \alpha : F_n(S) \to F_n(S), and\ \alpha\ is\ morphism\ \}.$$

Let $\alpha \in EndF_n(S)$, then α is a map defining by $x_i\alpha = s_i^{\alpha} x_{i\bar{\alpha}}$, where $1 \le i \le n$ and $\bar{\alpha} \in \mathcal{T}_n$, and $(s_1^{\alpha}, \dots, s_n^{\alpha}) \in S^n$.
Note that for each $\alpha \in EndF_n(S)$ it depends on its action on the free generators $\{x_i : 1 \le i \le n\}$ only. If $\alpha \in EndF_n(S)$, then α is an S-morphism and for $s \in S$ and $1 \le i \le n$ we have

$$(sx_i)\alpha = s(x_i\alpha) = ss_i^{\alpha} x_{i\bar{\alpha}}.$$

2.2 Wreath product multiplication
In this section, we define wreath product multiplication on a direct product of semigroup S^n and $\mathcal{T}_n$ by setting
$(s_1^{\alpha}, \dots, s_n^{\alpha}, \alpha)\left(s_1^{\beta}, \dots, s_n^{\beta}, \beta\right) = \left(s_1^{\alpha} s_{1\alpha}^{\beta}, \dots, s_n^{\alpha} s_{n\alpha}^{\beta}, \alpha\beta\right)$, where $\alpha, \beta \in \mathcal{T}_n$.
Under this multiplication $S^n \times \mathcal{T}_n$, the wreath product becomes a monoid with identity $(1, \dots, 1, I_n)$ where I_n is the identity transformation in $\mathcal{T}_n$ and it is denoted by $S \wr_n \mathcal{T}_n$.
The following lemma has been proved in [9].
Lemma 2.1 For $n \in \mathbb{N}$, $EndF_n(S)$ is isomorphic to $S \wr_n \mathcal{T}_n$.
For more details about the wreath product multiplication, we refer to [10,11] .
3. Semigroups T_A and $T_{\mathbf{A}}$
Let **A** be an algebra and A a universe of **A,** [12,13]. The semigroup of all maps from A to A, which is denoted by T_{A} and the semigroup of all morphisms $\mathbf{A} \to \mathbf{A}$ is denoted by $T_{\mathbf{A}}$.
The following lemma shows that $\mathcal{T}_{\mathbf{A}}$ is a submonoid of $\mathcal{T}_A$.
Lemma 3.1 The semigroup $\mathcal{T}_{\mathbf{A}}$ is a submonoid of $\mathcal{T}_A$.
Proof. Let $\alpha, \beta \in \mathcal{T}_{\boldsymbol{A}}$ such that $\alpha: \mathbf{A} \to \mathbf{A},$ and $\beta: \mathbf{A} \to \mathbf{A}$ be morphism. Since $Dom(\alpha\beta) = Dom\ \alpha = \mathbf{A}$ and $Im(\alpha\beta) = Im(\beta) = \mathbf{A}$. We define the composition of α and β as $x(\alpha\beta) = (x\alpha)\beta$ for all $x \in Dom\ \alpha\beta$. Thus $\alpha\beta$ is a map between two universes.
Now to show that $\alpha\beta \in \mathcal{T}_{\mathbf{A}}$which means for all $(a_1, \dots, a_n) \in Dom\ \alpha\beta$ and terms $\big(t(a_1, \dots, a_n)\big)(\alpha\beta) = t\big(a_1(\alpha\beta), \dots, a_n(\alpha\beta)\big)$.
Now,
$\big(t(a_1, \dots, a_n)\big)(\alpha\beta) = (t(a_1, \dots, a_n)\alpha)\beta$

$= \big(t(a_1\alpha, \dots, a_n\alpha)\big)\beta$ (as α is morphism)
$= t\big((a_1\alpha)\beta, \dots, (a_n\alpha)\beta\big)$ (as β is morphism)
$= t(a_1(\alpha\beta), \dots, a_n(\alpha\beta))$.

It is clear, the identity I_A of $\mathcal{T}_{\mathbf{A}}$.

4. Semigroup $\mathcal{T}_{\mathbf{F_n}(S)}$.

The semigroup of all morphism $\mathbf{A} \to \mathbf{A}$ is denoted by $\mathcal{T}_{F_n(S)}$. Let $\alpha \in \mathcal{T}_{\mathbf{F_n}(S)}$, then α can be described by

$$\alpha = \begin{pmatrix} x_{i_1} & \dots & x_{i_k} \\ s_{i_1}^{\alpha} x_{i_{1\bar{\alpha}}} & \dots & s_{i_k}^{\alpha} x_{i_{k\bar{\alpha}}} \end{pmatrix},$$

where $\bar{\alpha} \in \mathcal{T}_n$, $s_{i_1}^{\alpha}, \dots, s_{i_k}^{\alpha} \in S$ such that $1 \le i_1 < \dots < i_k \le n$, $k \ge 0$ and $x_{i_\ell}\alpha = s_{i_\ell}^{\alpha} x_{i_{\ell\bar{\alpha}}}$. For every choice of $\bar{\beta} \in \mathcal{T}_n$ with $Dom(\bar{\beta}) = \{j_1, \dots, j_t\}$ for $1 \le j_1 < \dots < j_t \le n$, $t \ge 0$ and $t_{j_1}^{\beta}, \dots, t_{j_t}^{\beta} \in S$ this gives

$$\beta = \begin{pmatrix} x_{j_1} & \dots & x_{j_t} \\ t_{j_1}^{\beta} x_{j_{1\bar{\beta}}} & \dots & t_{j_t}^{\beta} x_{j_{t\bar{\beta}}} \end{pmatrix} \in \mathcal{T}_{\mathbf{F_n}(S)}$$

Lemma 4.1 For $n \in \mathbb{N}$, $\mathcal{T}_{\mathbf{F_n}(S)}$ is morphic to $EndF_n(S)$.

Proof. Define $\gamma: \mathcal{T}_{\mathbf{F_n}(S)} \to EndF_n(S)$ by $\alpha\gamma = \bar{\alpha}$ where $(sx_i)\bar{\alpha} = (sx_i)\alpha$ for all $s \in S$. First, we need to show that $\bar{\alpha}$ is an S- act morphism, that means $s(a\bar{\alpha}) = (sa)\bar{\alpha}$ for all $s \in S$ and $a \in F_n(S)$. Assume $a = tx_i$. An element $x_i \in Dom(\alpha)$ if and only if $tx_i \in Dom(\alpha)$ for all $t \in S$. If $tx_i \in Dom\,(\alpha)$, then for each $s \in S$, $stx_i \in Dom\,(\alpha)$, we have
$(stx_i)\bar{\alpha} = (stx_i)\alpha = s\big((tx_i)\alpha\big) = s\big((tx_i)\bar{\alpha}\big)$
To prove γ is bijection. Let $\alpha\gamma = \beta\gamma$, for an element $sx_i \in F_n(S)$ we have $(sx_i)\alpha\gamma = (sx_i)\beta\gamma$ for all i , so that
$(sx_i)\alpha = (sx_i)\bar{\alpha} = (sx_i)\alpha\gamma = (sx_i)\beta\gamma = (sx_i)\bar{\beta} = (sx_i)\beta$, so $\alpha = \beta$. Therefore, γ is one-to-one. To prove γ is onto. Let $\bar{\mu} \in EndF_n(S)$ and define $\mu \in \mathcal{T}_{\mathbf{F_n}(S)}$ such that $\mu: F_n(S) \to F_n(S)$ by
$sx_i\mu = (sx_i)\bar{\mu}$, for all $sx_i \in F_n(S)$.
It is obvious that μ is an S-act morphism, since for any $sx_i \in Dom\,\mu = F_n(S)$and $t \in S$, we have $t\big((sx_i)\mu\big) = t\big((sx_i)\bar{\mu}\big) = (tsx_i)\bar{\mu} = (tsx_i)\mu$.

Now, $(sx_i)\mu = (sx_i)\bar{\mu} = (sx_i)\mu\gamma$, for all $sx_i \in F_n(S), s \in S$, hence $\mu\gamma = \bar{\mu}$, so that γ is onto.
To prove γ is homomorphism. Let $\alpha, \beta \in \mathcal{T}_{F_n(S)}$. We have to show that $(\alpha\beta)\gamma = \alpha\gamma\beta\gamma$. For all i we have

$$(sx_i)(\alpha\beta)\gamma = (sx_i)\overline{\alpha\beta} = (sx_i)\alpha\beta.$$

On the other hand, $(sx_i)(\alpha\gamma\beta\gamma) = (sx_i)\bar{\alpha}\,\bar{\beta}$. It is clear that $\bar{\alpha}, \bar{\beta} \in EndF_n(S)$, therefore $\bar{\alpha}$ and $\bar{\beta}$ are an S-act morphism .
Now, $(sx_i)\bar{\alpha}\,\bar{\beta} = \big((sx_i)\bar{\alpha}\big)\bar{\beta} = \big((sx_i)\alpha\big)\bar{\beta} = \big((sx_i)\alpha\big)\beta = (sx_i)\alpha\beta = (sx_i)\overline{\alpha\beta}$. Hence, $(\alpha\beta)\gamma = \overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta} = \alpha\gamma\,\beta\gamma$, as it required.

Lemma 4.2 For $n \in \mathbb{N}$, $\mathcal{T}_{\mathbf{F_n}(S)}$ is embeded in $S \wr_n \mathcal{T}_n$.

Proof. Let $\alpha \in \mathcal{T}_{\mathbf{F_n}(S)}$, then α is described by

$$\alpha = \begin{pmatrix} x_{i_1} & \dots & x_{i_k} \\ s_{i_1}^{\alpha} x_{i_{1\bar{\alpha}}} & \dots & s_{i_k}^{\alpha} x_{i_{k\bar{\alpha}}} \end{pmatrix},$$

where $\bar{\alpha} \in \mathcal{T}_n$, $s_{i_1}^{\alpha}, \dots, s_{i_k}^{\alpha} \in S$ for $1 \le i_1 < \dots < i_k \le n$, and $k \ge 0$.

Let $\psi: \mathcal{T}_{\mathrm{F_n}(S)} \to S \wr_n \mathcal{T}_n$ be given by $\alpha\psi = (s_1^\alpha, \dots, s_n^\alpha, \bar{\alpha})$. To prove ψ is an embeding, we need to show that ψ is one-to-one homomorphism map. To prove ψ is homomorphism, let $\alpha, \beta \in T_{F_n(S)}$. It is enough to prove $(\alpha\beta)\psi = \alpha\psi\, \beta\psi$. Notice that

$$\alpha\psi\, \beta\psi = (s_1^\alpha, \dots, s_n^\alpha, \bar{\alpha})\left(s_1^\beta, \dots, s_n^\beta, \bar{\beta}\right) = \left(s_1^\alpha s_{1\bar{\alpha}}^\beta, \dots, s_n^\alpha s_{n\bar{\alpha}}^\beta, \bar{\alpha}\bar{\beta}\right)$$

and

$$(\alpha\beta)\psi = (s_1^{\alpha\beta}, \dots, s_n^{\alpha\beta}, \overline{\alpha\beta})$$

If $\alpha, \beta \in T_{F_n(S)}$, then α, β are morphism that take $F_n(S)$ into itself. From that we obtain for all i the following :

$x_i\alpha\beta = (x_i\alpha)\beta = (s_i^\alpha x_{i\bar{\alpha}})\beta = s_i^\alpha(x_{i\bar{\alpha}}\beta) = s_i^\alpha s_{i\bar{\alpha}}^\beta x_{i\bar{\alpha}\bar{\beta}}$ and $x_i\alpha\beta = s_i^{\alpha\beta} x_{i\overline{\alpha\beta}}$. Therefore, for all i we have

$$s_i^{\alpha\beta} x_{i\overline{\alpha\beta}} = s_i^\alpha s_{i\bar{\alpha}}^\beta x_{i\bar{\alpha}\bar{\beta}},$$

so that we obtain

$s_i^{\alpha\beta} = s_i^\alpha s_{i\bar{\alpha}}^\beta$ and $\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}$.

Hence, ψ is homomorphism.

To prove ψ is one-to-one. Let $\alpha, \beta \in \mathcal{T}_{\boldsymbol{F_n(S)}}$ such that $\alpha\psi = \beta\psi$, this implies

$(s_1^\alpha, \dots, s_n^\alpha, \bar{\alpha}) = \left(s_1^\beta, \dots, s_n^\beta, \bar{\beta}\right)$.

Then for any $sx_i \in F_n(S)$, we have

$$(sx_i)\alpha = s(s_i^\alpha x_{i\bar{\alpha}}) = s\left(s_i^\beta x_{i\bar{\beta}}\right) = s(x_i\beta) = (sx_i)\beta,$$

and this means $\alpha = \beta$, and ψ is one-to-one, as it required.

5. Description of Green's relations on $\mathcal{T}_{\mathbf{F_n}(S)}$.

Let **A** and **B** be algebras of the same type [12,13], and $\theta: \mathbf{A} \to \mathbf{B}$ be a homomorphism, then

$Im\,(\theta) = \{a\theta : a \in \mathbf{A}\}$,

$Ker(\theta) = \{(x, y) \in \mathbf{A} \times \mathbf{A} : x\alpha = y\alpha\}$.

In our case we have $\mathbf{A} = \mathbf{B} = \mathbf{F_n}(\boldsymbol{S})$.

Lemma 5.1 A subset B is a subalgebra of $\mathbf{F_n}(\boldsymbol{S})$ if and only if $B = Sx_{i_1} \dot{\cup}\ Sx_{i_2} \dot{\cup} \dots \dot{\cup}\, Sx_{i_m}$ for some $1 \le i_1 < i_2 < \cdots < i_m \le n$ and $0 \le m \le n$.

Proof. Let B be a subalgebra of $F_n(S)$. Claim that $B = Sx_{i_1} \dot{\cup}\ Sx_{i_2} \dot{\cup} \dots \dot{\cup}\, Sx_{i_m}$ for $1 \le i_1 < i_2 < \cdots < i_m \le n$ and $0 \le m \le n$. Let $sx_k \in B$; as B is a subalgebra we obtain that $(h1_S)(sx_k) \in B$ for all $h \in S$, hence $(hs)x_k \in B$. Therefore, one direction of our claim is proved. It is already known that for any $1 \le i_1 < i_2 < \cdots < i_m \le n$, $A = Sx_{i_1} \dot{\cup}\ Sx_{i_2} \dot{\cup} \dots \dot{\cup}\, Sx_{i_m}$ is a subalgebra of $\mathrm{F_n}(S)$. To show that, let $sx_{i_j} \in B$, $s' \in S$ this implies $s'\left(sx_{i_j}\right) = s'sx_{i_j} \in B$.

Remark 5.2:

1. For a subalgebra $B = Sx_{i_1} \dot{\cup}\ Sx_{i_2} \dot{\cup} \dots \dot{\cup}\, Sx_{i_m}$ for $1 \le i_1 < i_2 < \cdots < i_m \le n$ and $0 \le m \le n$, we say that $rank\ B$, $\rho(B)$ is m .
2. For $\alpha \in \mathcal{T}_{\boldsymbol{F_n(S)}} = EndF_n(S)$, we define $\rho(\alpha)$ be $\rho(Im\,(\alpha))$ such that $X_n = \{x_1, \dots, x_n\}$ and $Im\ \alpha = \dot{\cup}_{y \in Y} S_y$, $Y \subseteq X_n$ then $\rho(\alpha) = \rho(Im(\alpha)) = \left|\{S_y : y \in Y\}\right| = |Y|$.

The next example explain the procedure of the Remark 5.2.:

Example 5.3

Let $X = \{x_1, x_2, x_3\}$ and $\alpha, \beta \in T_{F_3(S)}$ such that

$$\alpha = \begin{pmatrix} x_1 & x_2 & x_3 \\ sx_2 & x_2 & tx_3 \end{pmatrix},\ \beta = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & ux_2 & x_2 \end{pmatrix}$$

$Im(\alpha) = (Sx_1 \dot{\cup} Sx_2 \dot{\cup} Sx_3)\alpha = S(x_1\alpha) \dot{\cup} S(x_2\alpha) \dot{\cup} S(x_3\alpha)$
$= S(sx_2) \dot{\cup} S(x_2) \dot{\cup} S(tx_3) = S(x_2) \dot{\cup} S(x_3).$
By the same way we have
$Im(\beta) = (Sx_1 \dot{\cup} Sx_2 \dot{\cup} Sx_3)\beta$
$= S(x_1\beta) \dot{\cup} S(x_2\beta) \dot{\cup} S(x_3\beta)$
$= S(x_1) \dot{\cup} S(ux_2) \dot{\cup} S(x_2) = S(x_1) \cup S(x_2),$

As $\rho(\alpha) = \rho(Im(\alpha)) = |\{S_y : y \in Y\}| = |Y|,$ Where $Y \subseteq X_3$ then $\rho(\alpha) = \rho(Im(\alpha)) = 2$ and $\rho(\beta) = 2$.

Definition 5.4[6] Green's relations characterize the elements of a semigroup in terms of the principal ideals they generate. They are five equivalence relations, namely for $a, b \in S,$ we have

1- $a \leq_{\mathcal{R}} b$ if and only if $a \in bS^1$
2- $a \leq_{\mathcal{L}} b$ if and only if $a \in S^1 b,$
3- $a \leq_{\mathcal{J}} b$ if and only if $a \in S^1 b S^1,$
4- The Green's relations by $a\,\mathcal{R}\,b$ if and only if $aS^1 = bS^1$ as well as $a\,\mathcal{L}b$ if and only if $S^1 a = S^1 b$
5- $a\,\mathcal{J}\,b$ if and only if $S^1 a S^1 = S^1 b S^1$,

whereas S^1 denotes S with identity element adjoined ,unless S already has one. In addition, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, where the join of the equivalences $\mathcal{R}$ and $\mathcal{L}$ is $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. It is clear that for finite semigroups, we usually have $\mathcal{D} = \mathcal{J}$, while the general inclusions $\mathcal{H} \subseteq \mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ is satisfied.

The characterization of Green's relations has been obtained by V. Gould [1], on End**A** where **A** is an independent algebra. While A. Alaadhmi [9] obtained a description of Green's relations on the special case of End**A** where $\mathbf{A} = \boldsymbol{F_n(G)}$ and **G** is a finite group.

Our main results in this section are to give an explicit description of [1,9] for $\leq_{\mathcal{L}}, \leq_{\mathcal{R}}, \leq_{\mathcal{J}}$, $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D},$ and $\mathcal{J}$, where $\mathbf{A} = \mathbf{F_n}(\boldsymbol{S}),$ and $\boldsymbol{S}$ is a finite semigroup with identity 1_S **.**

Before state the main results, we give the following proposition:

Proposition 5.5: For arbitrary $\alpha, \beta \in \mathcal{T}_n$, the following is true,
1. $Dom(\alpha\beta) \subseteq Dom(\alpha)$;
2. $Im(\alpha\beta) \subseteq Im(\beta)$,

Proof:
1. $x \in Dom(\alpha\beta) \Leftrightarrow x \in Dom(\alpha)$ and $x \in Dom(\beta)$. Therefore, $Dom(\alpha\beta) \subseteq Dom(\alpha)$.
2. If $x \in Im(\alpha\beta) \Rightarrow x = y(\alpha\beta),$ for some $y \in Dom(\alpha\beta)$, i.e., $x = (y\alpha)\beta \in Im(\beta)$, then $Im(\alpha\beta) \subseteq Im(\beta)$.

Now we state the main result of this section:

Lemma 5.6: For $\alpha, \beta \in EndF_n(S)$, where S be any monoid, we have the following:
i. $\alpha \leq_{\mathcal{L}} \beta$ if and only if $Im(\alpha) \subseteq Im(\beta)$,
ii. $\alpha \leq_{\mathcal{R}} \beta$ if and only if $ker(\beta) \subseteq ker(\alpha)$,
iii. $\rho(\alpha\beta) \leq \rho(\alpha), \rho(\beta)$.

Proof.

i. Clearly, if $\alpha \leq_{\mathcal{L}} \beta$ in $EndF_n(S)$, then $\alpha \leq_{\mathcal{L}} \beta$ in $\mathcal{T}_{F_n(S)}$ which means $\mathcal{T}_{F_n(S)}\alpha \subseteq \mathcal{T}_{F_n(S)}\beta$ and this implies $\alpha = \gamma\beta$ for some $\gamma \in EndF_n(S)$ by using Proposition 5.5 we obtain $Im(\alpha) = Im(\gamma\beta) \subseteq Im(\beta)$.

Conversely, suppose that $Im(\alpha) \subseteq Im(\beta)$, for $1 \leq i \leq n$ we have $x_i\alpha \in Im(\alpha) \subseteq Im(\beta)$, so we choose $b_i \in F_n(S)$ such that $x_i\alpha = b_i\beta$. Now, define $\mu \in EndF_n(S)$ by $x_i\mu = b_i$ for $1 \leq i \leq n$. This gives $x_i\mu\beta = b_i\beta = x_i\alpha$. From this, we obtain $\mu\beta = \alpha$, which means $\alpha \leq_{\mathcal{L}} \beta$.

ii. If $\alpha \leq_{\mathcal{R}} \beta$ in $EndF_n(S)$, this means $\alpha \leq_{\mathcal{R}} \beta$ in $\mathcal{T}_{\boldsymbol{F_n(S)}}$, which implies $\alpha = \beta\nu$, for some $\nu \in EndF_n(S)$.
Now, let $(x,y) \in \ker(\beta)$, so that $x\beta = y\beta$, and then $x\alpha = x(\beta\nu) = (x\beta)\nu = (y\beta)\nu = y(\beta\nu) = y\alpha$. Hence, $(x,y) \in \ker(\alpha)$, so we obtain $\ker(\beta) \subseteq \ker(\alpha)$.

Conversely, suppose $\ker(\beta) \subseteq \ker(\alpha)$. Define $\mu \in F_n(S)$, $\mu: F_n(S) \to F_n(S)$, by let $Im(\beta) = Sx_{i_1} \dot{\cup} Sx_{i_2} \dot{\cup} \dots \dot{\cup} Sx_{i_m}$ and define $x_{i_j}\mu = w_j\,\alpha$, where $w_i\beta = x_{i_j}$, for $i \notin \{i_1, \dots, i_m\}$.
In order to prove that $\alpha \leq_{\mathcal{R}} \beta$, first we need to show that μ is well-define and an S-morphism. If $w_j\beta = w_j'\beta = x_{i_j}$, then we have $(w_j, w_j') \in \ker(\beta) \subseteq \ker(\alpha)$, so we have $w_j\alpha = w_j'\alpha$, hence μ is well-define.
Now, as $F_n(S)$ is a free-left S- act on n generators over S, which means $F_n(S)$ has a basis $\{x_1, \dots, x_n\}$. So, μ must be an S-morphism.
Let $w \in F_n(S)$ be such that $w = sx_k$, $s \in S$ and $x_k\beta = tx_{i_j}$, $t \in S$, then we obtain
$w\beta = (sx_k)\beta = s(x_k\beta) = s\left(tx_{i_j}\right) = st\left(x_{i_j}\right) = st(w_j\beta)$.
Hence, $w\beta\mu = \left(stx_{i_j}\right)\mu = (st)\left(x_{i_j}\mu\right) = (st)(w_j\alpha) = (stw_j)\alpha$.
Now, if $w\beta = (stw_j)\beta$, and $\ker(\beta) \subseteq \ker(\alpha)$, then we have $w\alpha = (stw_j)\alpha = w\beta\mu$. This gives $\alpha = \beta\mu$ which means $\alpha \leq_{\mathcal{R}} \beta$.
iii. We claim for any $\gamma, \mu \in EndF_n(S)$, that $\rho(\gamma\mu) \leq \rho(\gamma)$ and $\rho(\gamma\mu) \leq \rho(\mu)$. We know that $\rho(\gamma) = \rho(Im(\gamma))$ and let $Im(\gamma) = \dot{\cup}_{y\in Y} S_y$, where $Y \subseteq X_n$, so that $\rho(\gamma) = |Y|$. Since $Im(\gamma\mu) = Im(\gamma)\mu = \dot{\cup}_{y\in Y} (S_y)\mu = \dot{\cup}_{y\in Y} S_{y\mu}$. From this we obtain $\rho(\gamma\mu) \leq |Y| = \rho(\gamma)$.
Now, $\rho(\mu\gamma) = \rho Im(\mu\gamma) = \rho(Im(\mu)\gamma)$.
Since $Im(\mu\gamma) \subseteq Im(\gamma)$, we have $\rho Im(\mu\gamma) \leq \rho\, Im(\gamma)$ which implies $\rho(\mu\gamma) \leq \rho(\gamma)$. So that $\rho(\alpha\beta) \leq \rho(\alpha), \rho(\beta)$.

Remark 5.7: Given sets A, and B, we say that A is of cardinality at most that of B, and write $|A| \leq |B|$ if there is an injective one-one function$f: A \to B$. We say that A and B has cardinality, $|A| = |B|$ if and only if there is a bijective function $f: A \to B$. If A has cardinality strictly less than $|A| < |B|$ if and only if there is a one-one function $f: A \to B$ by there is no bijection $f: A \to B$.
Lemma 5.8 For $\alpha, \beta \in End\ F_n(S) = \mathcal{T}_{\mathbf{F_n}(S)}$, we have the following
i. $\alpha\ \mathcal{L}\ \beta$ if and only if $Im(\alpha) = Im(\beta)$;
ii. $\alpha\ \mathcal{R}\ \beta$ if and only if $\ker(\alpha) = \ker(\beta)$;
iii. $\alpha\ \mathcal{H}\ \beta$ if and only if $Im(\alpha) = Im(\beta)$; $\ker(\alpha) = \ker(\beta)$;
iv. $\alpha\ \mathcal{D}\ \beta$ if and only if $\rho(\alpha) = \rho(\beta)$;
v. $\alpha \leq_{\mathcal{J}} \beta$ if and only if $\rho(\alpha) \leq \rho(\beta)$;
vi. $\alpha\ \mathcal{J}\ \beta$ if and only if $\rho(\alpha) = \rho(\beta)$;
vii. $\mathcal{D} = \mathcal{J}$.
Proof: It is easy to prove (i) and (ii) by using pervious Lemma.
For (iii), this immediate consequence of (i) and (ii).
By using previous lemma, we can say that
$\rho(\alpha) = \rho(\mu\beta\nu) \leq \rho(\mu\beta)$ and $\rho(\beta) = \rho(\gamma\alpha\delta) \leq \rho(\gamma\alpha) \leq \rho(\alpha)$, so that $\rho(\alpha) = \rho(\beta)$.
(iv) Let $\rho(\alpha) = \rho(\beta)$. $Im(\alpha) = \dot{\cup}_{y\in Y} S_y$, $Im(\beta) = \dot{\cup}_{z\in Z} S_z$ for some $Y, Z \subseteq X_n$ with $|Y| = |Z| = \rho(\alpha) = \rho(\beta)$.
Suppose $\eta: Y \to Z$ is a bijection and define $\bar{\eta}: Im(\alpha) \to Im(\beta)$ by $(sy)\bar{\eta} = s(y\eta)$ for all $s \in S$ and $y \in Y$. It is clear that $\bar{\eta}$ is one-to-one. Since $(sy)\bar{\eta} = (ty')\bar{\eta}$ this implies that $s(y\eta) = t(y'\eta)$ for all $s, t \in S$ and $y, y' \in Y$. As $Im(\beta)$ is free S-act with basis Z, then this

forces $y\eta = y'\bar{\eta}$ and $s = t$. Since η is bijective, we obtain $y = y'$. Furthermore, $\bar{\eta}$ is onto and from the definition of $\bar{\eta}$ we have $(sy)\bar{\eta} = s(y\eta)$ for all $s \in S$ and $y \in Y$, and since or all η is bijection $sz \in Im(\beta)$, pick $y \in Y$ with $y\eta = z$, then $sy \in Im(\alpha)$ and we get $sz = s(y\eta) = (sy)\bar{\eta}$.
Let $= \alpha\bar{\eta}$, then $\gamma \in EndF_n(S)$. Note that $Im(\gamma) = Im(\alpha\bar{\eta}) = \big(Im(\alpha)\big)\bar{\eta} = Im(\beta)$, so that $\beta\mathcal{L}\gamma$. By letting $u, v \in F_n(S)$, it is clear $u\alpha = v\alpha$ if and only if $(u\alpha)\bar{\eta} = (v\alpha)\bar{\eta}$ as $\bar{\eta}$ is one-to-one, so $\ker\alpha = \ker\alpha\,\bar{\eta} = \ker\gamma$. Therefore, $\alpha\mathcal{R}\gamma$ and we get $\alpha\mathcal{D}\beta$.
Conversely, suppose $\alpha\mathcal{D}\beta$, then $\alpha\ \mathcal{R}\ \mu\ \mathcal{L}\ \beta$ for some $\mu \in EndF_n(S)$. From (i) and (ii) we have $\ker(\alpha) = \ker(\mu)$, and $Im(\gamma) = Im(\beta)$. Now since $Im(\alpha) \cong \frac{F_n(S)}{\ker(\alpha)} = \frac{F_n(S)}{\ker(\mu)} = Im(\mu)$, by the Fundamental Theorem of Semigroup theory and since $\rho(\alpha) = \rho(Im(\alpha)) = \rho(Im(\mu)) = \rho(\mu)$ we obtain $\rho(\alpha) = \rho(\mu)$, moreover, $\rho(\gamma) = \rho(Im(\mu)) = Im(\beta) = \rho(\beta)$ so that we have $\rho(\alpha) = \rho(\beta)$.
(v) If $\alpha \leq_{\mathcal{J}} \beta$, then $\alpha = \nu\beta\mu$, so by Lemma 5.6 (iii) we have $\rho(\alpha) = \rho(\nu\beta\mu) \leq \rho(\nu\beta) \leq \rho(\beta)$.
Conversely, suppose $\rho(\alpha) \leq \rho(\beta)$ and let $Im(\alpha) = \dot{\cup}_{y\in Y} S_y$ and $Im(\beta) = \dot{\cup}_{z\in Z} S_z$, for some $Y, Z \in X_n$; form this we have $\rho(\alpha) = |Y|$ and $\rho(\beta) = |Z|$. As $\rho(\alpha) \leq \rho(\beta)$ so that there is one-to-one map $\phi: Y \to Z$. By letting $W = Im(\phi)$, so $W \subseteq Z$ and $|Y| = |W|$. Fix $w_0 \in W$and define $\kappa: Z \to W$ by $z\kappa = z$, for all $z \in W$, $z\kappa = w_0$, for all $z \in Z\backslash W$, so $Im\kappa = W$. Now define $\mu: Im(\alpha) = \dot{\cup}_{z\in Z} S_z \to \dot{\cup}_{w\in W} S_w$ by $z\mu = z\kappa$. It is clear that μ extends to an S-act morphism so $\beta\mu \in EndF_n(S)$. Now since
$Im(\beta\mu) = (Im(\beta))\mu = (\dot{\cup}_{z\in Z} S_z)\mu = \dot{\cup}_{z\in Z} S_{z\mu} = \dot{\cup}_{z\in Z} S_{z\kappa} = \dot{\cup}_{w\in W} G_w$ we obtain $\rho(\beta\mu) = |W| = |Y| = \rho(\alpha)$. Hence $\rho(\beta\mu) = \rho(\alpha)$. By (iv) we have $\beta\mu\mathcal{D}\alpha$ this means $\beta\mu\mathcal{J}\alpha$ as $\mathcal{D} \subseteq \mathcal{J}$, and hence $\alpha \leq_{\mathcal{J}} \beta$.
(vi) If $\rho(\alpha) = \rho(\beta)$, then by (iv) we have $\alpha\mathcal{D}\beta$ that means $\alpha\mathcal{J}\beta$ as $\mathcal{D} \subseteq \mathcal{J}$. Conversely, suppose $\alpha\mathcal{J}\beta$, then $\alpha = \gamma\beta\delta$, $\beta = \mu\alpha\nu$, for $\mu, \nu, \gamma, \delta \in EndF_n(S)$. Using Lemma 5.6 we have $\rho(\alpha) = \rho(\gamma\beta\delta) \leq \rho(\gamma\beta) \leq \rho(\beta)$ and $\rho(\beta) = \rho(\mu\alpha\nu) \leq \rho(\gamma\alpha) \leq \rho(\alpha)$, so $\rho(\alpha) = \rho(\beta)$
(vii) This is an immediate sequence of (iv) and (vi).

Conclusions

In this work, The extension for $\mathcal{T}_n$ to $\mathcal{T}_{\mathbf{A}}$, where **A** is an independent algebra has been discussed and studied. By considering that **A** is to be $\mathbf{F_n}(\boldsymbol{S})$, the $\mathcal{T}_{\mathbf{F_n(S)}}$ is an endomorphism monoid of a free left *S*-act of *n*-generators has been shown, as well as the $\mathcal{T}_{\mathbf{F_n(S)}}$ is embedded in the wreath product $S \wr_n \mathcal{T}_n$ is shown. Many results are also given and we find the description of Green's relation for $\mathcal{T}_{\mathbf{F_n}(S)}$. In the end of this paper, the semigroup $\mathcal{T}_{\mathbf{F_n}(S)}$ is embedded in the wreath product $S \wr_n \mathcal{T}_n$ is proved.

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