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# Certain Subclasses of Meromorphic Functions Involving Differential Operator

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**Abstract**
We obtain the coefficient estimates, extreme points, distortion and growth boundaries, radii of starlikeness, convexity, and close-to-convexity, according to the main purpose of this paper.

**Keywords:** Multivalent Function, Convex Function, Starlike Function, Coefficient Estimates, Hadamard product..

# فئات جزئيه معينه من الدوال الميرومورفيه التي تنطوي على عامل تفاضلي

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**الخلاصة**
نحصل على تقديرات المعامل ، والنقاط المتطرفة ، وحدود التشوه والنمو ، وأنصاف أقطار التشبه بالنجوم ، والتحدب ، والقرب من التحدب ، وفقًا للغرض الرئيسي من هذه الورقة.

## 1. Introduction

Let $\sum$ denote by class of meromorphic functions of the form

$$f(w) = \frac{1}{w} + \sum_{k=1}^{\infty} a_k w^k, \qquad a_k \geq 0. \tag{1}$$

which are analytic in unit disk

$$U^* = \{w: w \in \mathbb{C}, 0 < |w| < 1\} = U\backslash\{0\}. \tag{2}$$

Let $g \in \sum$ be given by

$$g(w) = \frac{1}{w} + \sum_{k=1}^{\infty} b_k w^k, \qquad b_k \geq 0. \tag{3}$$

Then the Hadamard product of $f$ and $g$ is given by

$$(f * g)(w) = \frac{1}{w} + \sum_{k=1}^{\infty} a_k b_k w^k, \ a_k b_k \geq 0. \tag{4}$$

A function $f * g$ in $\sum$ is said to be meromorphically starlike of order $\varepsilon$ if

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$$Re\left\{-\frac{w(f*g)'(w)}{(f*g)(w)}\right\} > \varepsilon, w \in U^*, (0 \leq \varepsilon < 1). \quad (5)$$

We denote by $\sum^*(\varepsilon)$ class of all meromorphically starlike functions of order $\varepsilon$. Similarly, a function $f*g$ in $\sum$ is said to be meromorphically convex of order $\varepsilon$ if

$$Re\left\{-\left(1+\frac{w(f*g)''(w)}{(f*g)'(w)}\right)\right\} > \varepsilon, w \in U, (0 \leq \varepsilon < 1). \quad (6)$$

And we denote by $\sum_k^*(\varepsilon)$ the class of meromorphically convex functions of $\varepsilon$. The class $\sum^*(\varepsilon)$ and $\sum_k^*(\varepsilon)$ were introduced and studied by pommerenke [1], Miller [2], Mogra et al. [3], Aouf et al. [4,5], El-Ashwah et al. [6], Mostafa et al. [7] and Venkateswarlu et al. [8,9,10].

Let us consider the second order linear homogenous differential equation (see, Baricz [11],[12]):

$$w^2 z''(w) + swz' + [tw^2 - u^2 + (1-h)]z(w) = 0, (h, t, u \in \mathbb{C}). \quad (7)$$

The function $z_{u,h,t}(w)$ which is called the generalized Bessel function of the first kind of order $\varepsilon$ where $\varepsilon$ is an unrestricted (real or complex) number, is defined a particular solution of (7). The function $z_{u,h,t}(w)$, has the representation

$$z_{u,h,t}(w) = \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)} \left(\frac{w}{2}\right)^{2k+u}.$$

Assume

$$\mathcal{J}_{u,h,t}(w) = \frac{2^u \Gamma\left(u+\frac{h+1}{2}\right)}{w^{\frac{u}{2}+1}} z_{u,h,t}\left(w^{\frac{1}{2}}\right)$$
$$= \frac{1}{w} + \sum_{k=1}^{\infty} \frac{(-t)^k \Gamma\left(u+\frac{h+1}{2}\right)}{4^k \Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)} w^k.$$

**Definition 1.1. [**13]**.** The operator $\mathcal{J}_{u,h,t}$ is a modification of the operator introduced by Deniz [5].

$$(\mathcal{J}_{u,h,t})(w) = \frac{1}{w} + \sum_{k=1}^{\infty} \frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)} w^k, \quad (h, t, u \in \mathbb{C}).$$

**Definition 1.2. [14,15].** For $(f*g)(w) \in \mathcal{A}$ the linear operator $\mathcal{T}_\tau^m : \sum \longrightarrow \sum$ is defined by

$$\mathcal{T}_\tau^m (f*g)(w) = \frac{1}{w} + \sum_{k=1}^{\infty} [1+\tau(k+1)]^m a_k b_k w^k, \quad (8)$$

for $m \in N_0 = \{0,1,2,...\}$. It can easily be observed that

$$\mathcal{T}_\tau^0 (f*g)(w) = (f*g)(w)$$

$$\mathcal{T}_\tau^1 (f*g)(w) = (1-\tau)(f*g)(w) + \tau \frac{\left(w^2 (f*g)(w)\right)'}{w}, \ \tau \geq 0$$
$$= (1+\tau)(f*g)(w) + \tau w (f*g)'(w) = \mathcal{T}_\tau (f*g)(w)$$

$$\mathcal{T}_\tau^2(f*g)(w)=\mathcal{T}_\tau\big(\mathcal{T}^1(f*g)(w)\big).$$

**Definition 1.3.** For $(f*g)(w)\in\mathcal{A}$ the operator $\Im_{\varsigma,u,h,t}^{m,k}(f*g)(w):\sum\longrightarrow\sum$ is defined by Hadamard product of operator $\mathcal{T}_\tau^m$ and the operator $\mathcal{J}_{u,h,t}$

$\Im_{\tau,u,h,t}^{m,k}(f*g)(w)=\mathcal{T}_\tau^m(f*g)(w)*\big(\mathcal{J}_{u,h,t}\big)(w),$

$$\Im_{\tau,u,h,t}^{m,k}(f*g)(w)=\frac{1}{w}+\sum_{k=1}^{\infty}\frac{(\frac{-t}{4})^k\Gamma\left(v+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}a_kb_kw^k. \qquad (9)$$

We note that

$$\Im_{\tau,u,1,1}^{0,k}(f*g)(w)=\Im_u(f*g)(w).$$

**Definition 1.4. [16].** For $0\le\alpha<1$, $\mathbf{0}<\delta\le1$, $\frac{1}{2}<\mathcal{Y}\le1$ if $\alpha=0,\frac{1}{2}<\mathcal{Y}\le\frac{1}{2\alpha}$ *if* $\alpha\neq0$, let $\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$ denote a Subclass of $\sum$ consisting functions of the form (4) that satisfy the requirement

$$\left|\frac{\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)'}{\Im_{\tau,u,h,t}^{m,k}(f*g)(w)}+1}{2\mathcal{Y}\left[\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)'}{\Im_{\tau,u,h,t}^{m,k}(f*g)(w)}+\alpha\right]-\left[\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)'}{\Im_{\tau,u,h,t}^{m,k}(f*g)(w)}+1\right]}\right|<\delta, \qquad (10)$$

where $\Im_{\tau,u,h,t}^{m,k}(f*g)(w)$ is given by (9). In addition, we state that a function $(f*g)(w)\in\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$, whenever $(f*g)(w)$ is of the form (4).

We can see that by specializing the parameters in the operator and the class, we get the classes examined by Aouf [5], Kulkarni and Joshi [17], Mogra et al., [3].

## 2. Coefficient inequality

We get the coefficient boundaries function in this part $(f*g)(w)$ for the class $\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$.

**Theorem 2.1.** A function $(f*g)(w)$ in the class $\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$ if

$$\sum_{k=1}^{\infty}(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}|a_k||b_k| \le 2\delta\mathcal{Y}(1-\alpha). \qquad (11)$$

**Proof.** If $(f*g)(w)\in\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$, then by (10) we get

$$\left|\frac{\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)'}{\Im_{\tau,u,h,t}^{m,k}(f*g)(w)}+1}{2\mathcal{Y}\left[\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)'}{\Im_{\tau,u,h,t}^{m,k}(f*g)(w)}+\alpha\right]-\left[\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)'}{\Im_{\tau,u,h,t}^{m,k}(f*g)(w)}+1\right]}\right|<\delta$$

$$\left|\frac{w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)' + \Im_{\tau,u,h,t}^{m,k}(f*g)(w)}{2\mathcal{Y}w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)' + 2\alpha\mathcal{Y}\,\Im_{\tau,u,h,t}^{m,k}(f*g)(w) - w\left(\Im_{\tau,u,h,t}^{m,k}(f*g)(w)\right)' - \Im_{\tau,u,h,t}^{m,k}(f*g)(w)}\right| < \delta$$

$$\left|\frac{\sum_{k=1}^{\infty}\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}(k+1)a_kb_kw^k}{2\mathcal{Y}(\alpha-1)\frac{1}{w}+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\sum_{k=1}^{\infty}\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}a_kb_kw^k}\right| < \delta$$

$$= \sum_{k=1}^{\infty}\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}(k+1)|a_k||b_k||w^k|$$
$$+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\sum_{k=1}^{\infty}\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}|a_k||b_k||w^k|$$
$$\leq 2\delta\mathcal{Y}(1-\alpha)|w|^{-1},$$

when $w\longrightarrow 1^-$, we obtain

$$\sum_{k=1}^{\infty}(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}|a_k||b_k|$$
$$\leq 2\delta\mathcal{Y}(1-\alpha).$$

Then

$$\sum_{k=1}^{\infty}(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}|a_k||b_k|$$
$$\leq 2\delta\mathcal{Y}(1-\alpha).$$

Thus, $(f*g)(w) \in \sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$.

## 3. Distortion Theorems

We get the sharp for the distortion theorems in this section.

**Theorem 3.1.** If $(f*g)(w)$ in the class $\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$. Then for $0<|w|=r<1$, we have

$$\frac{1}{r}-\left(\frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}\right)r$$
$$\leq |(f*g)(w)|, \qquad (12)$$

and

$$|(f*g)(w)| \leq \frac{1}{r} + \left( \frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\dfrac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} \right) r. \quad (13)$$

The function $(f*g)(w)$ from equation in (12) and (13) are obtained

$$(f*g)(w) = \frac{1}{w} + \left( \frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\dfrac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} \right) w. \quad (14)$$

**Proof.** Via Theorem 2.1, we have

$$\sum_{k=1}^{\infty}(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}|a_k||b_k| \leq 2\delta\mathcal{Y}(1-\alpha),$$

for $|w| = r < 1$, we get

$$|(f*g)(w)| \geq r^{-1} - \sum_{k=1}^{\infty}|a_k||b_k|r \geq r^{-1} - r\sum_{k=1}^{\infty}|a_k||b_k|$$

$$\geq r^{-1} - \left( \frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\dfrac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} \right) r.$$

Also,

$$|(f*g)(w)| \leq r^{-1} + \sum_{k=1}^{\infty}|a_k||b_k|r \leq r^{-1} + r\sum_{k=1}^{\infty}|a_k||b_k|$$

$$\leq r^{-1} + \left( \frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\dfrac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} \right) r.$$

As a result, the proof is complete.

## 4. Growth Theorems

In this section we get the sharp for the growth theorems

**Theorem 4.1** If $(f*g)(w)$ in the class $\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$. Then for $0 < |w| = r < 1$, we have

$$r^{-2}-\left(\frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}\right)\leq|(f*g)'(w)|,(15)$$

and

$$|(f*g)'(w)|\leq r^{-2}+\frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}. \qquad (16)$$

The function $(f*g)'(w)$ from equation (15) and (16) are obtained

$$(f*g)'(w)=z^{-2}-\left(\frac{2\delta\mathcal{Y}(1-\alpha)}{(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}.\right). \quad (17)$$

**Proof.** Since

$$|(f*g)'(w)|\leq\left|w^{-2}-\sum_{k=1}^{\infty}ka_kb_kw^{k-1}\right|.$$

Form Theorem 2.1, we get

$$\sum_{k=1}^{\infty}(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}|a_k||b_k|\leq 2\delta\mathcal{Y}(1-\alpha),$$

for $|w|=r<1$, we obtain

$$|(f*g)'(w)|\geq r^{-2}-\sum_{k=1}^{\infty}k|a_k||b_k|$$

$$\geq r^{-2}-\left(\frac{2\delta\mathcal{Y}(1-\alpha)}{\big[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\big]\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}\right).$$

We can do the same thing,

$$|(f*g)'(w)|\leq r^{-2}+\sum_{k=1}^{\infty}k|a_k||b_k|$$

$$\leq r^{-2} + \left( \frac{2\delta\mathcal{Y}(1-\alpha)}{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} \right).$$

We have now completed the proof of the theorem.

The function $(f * g)(w)$ meets the radii of starlikeness, result convexity, and close-to-convex to convexity criteria, as shown by [18].

**Theorem 4.2.** Let $(f * g)(w) \in \sum(\alpha, \delta, \mathcal{Y}, \tau, u, h, t)$. Then the function $(f * g)$ is of starlikeness order $\varepsilon, (0 \leq \varepsilon < 1)$, in $|w| < r_1$, where
$r_1(\alpha, \delta, \mathcal{Y}, \tau, u, h, t)$

$$= \inf_{k\geq 1} \left\{ \frac{1-\varepsilon}{(k-\varepsilon+2)} \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) \right\}^{\frac{1}{k+1}}.$$

**Proof.** We must demonstrate this

$$\left| \frac{w(f * g)'(w)}{(f * g)(w)} + 1 \right| \leq 1 - \varepsilon,$$

$$\left| 1 + \frac{-w^{-1} + \sum_{k=1}^{\infty} k a_k b_k w^k}{w^{-1} + \sum_{k=1}^{\infty} a_k b_k w^k} \right| \leq 1 - \varepsilon \rightarrow. \quad (18)$$

From (14) holds if

$$\sum_{k=1}^{\infty} (k+1)|a_k||b_k||w|^{n+1} \leq (1-\varepsilon) - (1-\varepsilon) \sum_{k=1}^{\infty} |a_k||b_k||w|^{n+1}.$$

Then

$$\sum_{k=1}^{\infty} \frac{(k-\varepsilon+2)}{1-\varepsilon} |a_n||b_k||w|^{k+1} \leq 1. \quad (19)$$

From Theorem 2.1, we get

$$\sum_{k=1}^{\infty} \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) |a_k|\,|b_k| \leq 1. \quad (20)$$

By combining (19) and (20) we get.

$$\frac{(k-\varepsilon+2)}{1-\varepsilon}|w|^{k+1} \leq \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right]\frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right),$$

that is

$$|w|^{k+1} \leq \frac{1-\varepsilon}{(k-\varepsilon+2)} \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right]\frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right).$$

Therefore,

$$|w| \leq \left\{ \frac{1-\varepsilon}{(k-\varepsilon+2)} \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right]\frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) \right\}^{\frac{1}{k+1}}.$$

**Theorem 4.3.** Let $(f*g)(w) \in \sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$. Then the function $(f*g)$ is of convex order $\varepsilon,(0 \leq \varepsilon < 1)$, in $|w| < r_1$, where

$$r_2(\alpha,\delta,\mathcal{Y},\tau,u,h,t) = \inf_{k\geq 1} \left\{ \frac{1-\varepsilon}{k(k-\varepsilon+2)} \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right]\frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) \right\}^{\frac{1}{k+1}}$$

**Proof.** We must demonstrate this

$$\left|2+\frac{w(f*g)''(w)}{(f*g)'(w)}\right| \leq 1-\varepsilon,$$

$$\left|2+\frac{2w^{-2}+\sum_{k=1}^{\infty}k(k-1)a_kb_kw^{k-1}}{-w^{-2}+\sum_{k=1}^{\infty}k\,a_kb_kw^{k-1}}\right| \leq 1-\varepsilon. \tag{21}$$

From (17) holds if

$$\frac{\sum_{k=1}^{\infty}k(k+1)\,|a_k||b_k||w|^{k+1}}{1-\sum_{n=1}^{\infty}k|a_k||b_k||w|^{k+1}} \leq 1-\varepsilon,$$

then

$$\sum_{k=1}^{\infty} \frac{k(k+2-\varepsilon)}{1-\varepsilon} |a_k||b_k||w|^{k+1} \le 1. \tag{22}$$

From Theorem 2.1, we get

$$\sum_{k=1}^{\infty} \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right) [1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) |a_k||b_k| \le 1. \tag{23}$$

By combining (22) and (23) we get

$$\left(\frac{k(k+2-\varepsilon)}{1-\varepsilon}\right)|w|^{k+1} \le \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right) [1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right)$$

$$|w|^{k+1} \le \left(\frac{1-\varepsilon}{k(k+2-\varepsilon)}\right) \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right) [1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right).$$

Hence

$$|w| \le \left\{ \left(\frac{1-\varepsilon}{k(k+2-\varepsilon)}\right) \left( \frac{\left[(k+1)+\delta\big((1-k)+2\mathcal{Y}(k-\alpha)\big)\right] \frac{\left(\frac{-t}{4}\right)^k \Gamma\left(u+\frac{h+1}{2}\right) [1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) \right\}^{\frac{1}{k+1}}.$$

As a result, the evidence is now complete.

The close-to-convexity property of the considered subclass functions is demonstrated in the following theorem.

**Theorem 4.4**. Let $(f*g)(z) \in \sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$. Then the function $(f*g)$ is of close-to-convex order $\varepsilon, (0 \le \varepsilon < 1)$, in $|z| < r_1$, where

$r_3(\alpha,\beta,\mathcal{Y},\tau,u,h,t)$

$$= \inf_{k\geq 1}\left\{\frac{\left(|w|^{-2}-(2-\varepsilon)\right)}{k|w|^{-2}}\left(\frac{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^{k}\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)}\right)\right\}^{\frac{1}{k+1}}$$

**Proof.** We must demonstrate this

$$|(f*g)'(w)-1|\leq 1-\varepsilon,$$

that is

$$|(f*g)'(w)-1|\leq |w|^{-2}+\sum_{k=1}^{\infty}k|a_k||b_k||w|^{n-1}-1\leq 1-\varepsilon$$

$$|(f*g)'(w)-1|\leq |w|^{-2}+\sum_{k=1}^{\infty}k|a_k||b_k||w|^{n-1}\leq 2-\varepsilon.$$

From Theorem 2.1, we get

$$\sum_{k=1}^{\infty}|a_k||b_k|\leq \frac{2\delta\mathcal{Y}(1-\alpha)}{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^{k}\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}},$$

then

$$\sum_{k=1}^{\infty}\left(\frac{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^{k}\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)}\right)|a_k||b_k| \leq 1. \quad (24)$$

Observe that (24) is true if

$$\frac{k|w|^{k-1-2+2}}{|w|^{-2}-(2-\varepsilon)} \leq \left(\frac{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^{k}\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)}\right),$$

that is

$$|w|^{K+1}\leq \frac{\left(|w|^{-2}-(2-\varepsilon)\right)}{k|w|^{-2}}\left(\frac{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^{k}\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)}\right).$$

Hence

$$\leq \left\{ \frac{(|w|^{-2}-(2-\varepsilon))}{k|w|^{-2}} \left( \frac{[(k+1)+\delta((1-k)+2\mathcal{Y}(k-\alpha))] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} \right) \right\}^{\frac{1}{k+1}} |w| .$$

As a result, the evidence is now complete.

## 5. Extreme points Theorems

**Theorem 5.1.** Let $(f*g)_0(w)=\frac{1}{w}$ with $(k\geq 1)$ and

$$(f*g)_k(w)= \frac{1}{w}+\frac{2\delta\mathcal{Y}(1-\alpha)}{[(k+1)+\delta((1-k)+2\mathcal{Y}(k-\alpha))] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} w^k.$$

Then $(f*g)\in\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$, if and only if it can be expressed in this way

$$(f*g)(w)=\sum_{k=0}^{\infty}\mu_k (f*g)_k(w), \quad \left(\mu_k\geq 0, \sum_{k=0}^{\infty}\mu_k=1\right). \tag{25}$$

**Proof.** Suppose $(f*g)$ can be written as in (25). Then

$$(f*g)(w)=\sum_{k=0}^{\infty}\mu_k (f*g)_k(w)=\mu_0 (f*g)_0(w)+\sum_{k=1}^{\infty}\mu_k (f*g)_k(w)$$

$$(f*g)_k(w)= \frac{1}{w}+\sum_{k=1}^{\infty}\mu_k\frac{2\delta\mathcal{Y}(1-\alpha)}{[(k+1)+\delta((1-k)+2\mathcal{Y}(k-\alpha))] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}} w^k.$$

Therefore,

$$\sum_{k=1}^{\infty}\mu_k\frac{2\delta\mathcal{Y}(1-\alpha)}{[(k+1)+\delta((1-k)+2\mathcal{Y}(k-\alpha))] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}$$

$$\times\frac{[(k+1)+\delta((1-k)+2\mathcal{Y}(k-\alpha))] \frac{\left(\frac{-t}{4}\right)^{k} \Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^{m}}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)} w^k$$

$$=\sum_{k=1}^{\infty}\mu_k-1=1-\mu_0\leq 1.$$

So by Theorem 2.1 we have $(f*g)\in\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$.

Conversely, suppose $(f*g)\in\sum(\alpha,\delta,\mathcal{Y},\tau,u,h,t)$. Since

$$a_kb_k\le\frac{2\delta\mathcal{Y}(1-\alpha)}{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}},k\ge 1.$$

We set,

$$\mu_k=\frac{\left[(k+1)+\delta\left((1-k)+2\mathcal{Y}(k-\alpha)\right)\right]\frac{\left(\frac{-t}{4}\right)^k\Gamma\left(u+\frac{h+1}{2}\right)[1+\tau(k+1)]^m}{\Gamma(k+1)\Gamma\left(k+u+\frac{h+1}{2}\right)}}{2\delta\mathcal{Y}(1-\alpha)}a_kb_k,k\ge 1$$

and $\mu_0=1-\sum_{k=1}^{\infty}\mu_k$. Hence,

$$(f*g)(w)=\sum_{k=0}^{\infty}\mu_k(f*g)_k(w)=\mu_0(f*g)_0(w)+\sum_{k=1}^{\infty}\mu_k(f*g)_k(w).$$

As a result, below are the outcomes.

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