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Properties of Fuzzy Norm of fuzzy Bounded Operators

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Abstract
The principal aim of this research is to use the definition of fuzzy normed space to define fuzzy bounded operator as an introduction to define the fuzzy norm of a fuzzy bounded linear operator then we proved that the fuzzy normed space FB(X,Y) consisting of all fuzzy bounded linear operators from a fuzzy norm space X into a fuzzy norm space Y is fuzzy complete if Y is fuzzy complete. Also we introduce different types of fuzzy convergence of operators.

Keywords: Fuzzy normed space, Fuzzy bounded operator, Fuzzy continuous operator, The fuzzy normed space FB(X, Y).

خواص القياس الضبابي لمؤثرات مقيدة ضبابيا

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الخلاصة
الهدف الاساسي في هذا البحث هو استخدام تعريف الفضاء القياسي الضبابي لتعريف التقيد الضبابي للمؤثرات كمقدمة لتقديم تعريف القياس الضبابي لمؤثر خطي مقيد ضبابيا بعد ذلك برهنا ان فضاء القياس الضبابي FB(X,Y) الذي يتالف من مجموعة كل المؤثرات الخطية والمقيدة ضبابيا من فضاء القياس الضبابي X الى فضاء القياس الضبابي Y يكون كامل اذا كانت Y كامل وكذلك قدمنا انواع مختلفة من التقارب الضبابي للمؤثرات.

1. Introduction

The theory of fuzzy set was introduced by Zadeh in 1965[1]. In 1984[2] Katsaras is the first one who introduced the notion of fuzzy norm on a vector space during his studying the notion fuzzy topological vector spaces. In 1984 Kaleva and Seikkala [3] introduced a fuzzy metric space. In [4] Felbin 1992 defined the notion of fuzzy norm on a vector space when the corresponding fuzzy metric is of Kaleva and Seikkala type Kramosil and Michalek introduced another definition of fuzzy metric space [5]. In 1994 Cheng and Mordeson [6] introduced the notion fuzzy norm on a vector space such that the corresponding fuzzy metric is of Kramosil and Michalek type. Bag and Samanta [7] in 2003 studied finite dimensional fuzzy normed spaces. In 2005 Saadati and Vaezpour [8] studied some results on fuzzy complete fuzzy normed spaces. In 2005 Bag and Samanta [9] studied fuzzy bounded linear operators and proved the fixed point theorems on fuzzy normed spaces. In 2009 Sadeqi and Kia [10] studied fuzzy normed space and its topological properties. In 2010 Si, Cao and Yang [11] studied the continuity of an intuitionistic fuzzy normed space.

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In 2015 Nadaban [12] studied properties of fuzzy continuous mapping on a fuzzy normed linear spaces. The concept of fuzzy norm has been used in developing the fuzzy functional analysis and its applications and many researches by distinct authors introduced for more details please see [13-23] .

In this paper we recall the definition of fuzzy norm duo to Bag and Samanta [7].Then we recall important properties of fuzzy normed space that's will be needed in the sequel. After that we define fuzzy bounded operator as an introduction to define the fuzzy norm of a fuzzy bounded linear operator. The structure of the present paper is as follows: In section two we recall basic definitions and properties of fuzzy norm. In section three we define the fuzzy norm of fuzzy bounded linear operator and functional then we proved the main results in this research.

2. Properties of Fuzzy normed space

In this section we recall basic properties of fuzzy normed space

Definition 2.1: [1]

Suppose that U is any set, a fuzzy set $\widetilde{A}$ in U is equipped with a membership function, $\mu_{\widetilde{A}}(u)$: U→ [0, 1]. Then $\widetilde{A}$ is represented by $\widetilde{A}=\{(u,\mu_{\widetilde{A}}(u)): u\in U, 0\le \mu_{\widetilde{A}}(u)\le 1\}$.

Definition 2.2: [7]

Let $*: [0,1]\times[0,1]\to[0,1]$ be a binary operation then $*$ is called a continuous **t -norm** (or **triangular norm**) if for all $\alpha,\beta,\gamma,\delta\in[0,1]$ it has the following properties

1. $\alpha*\beta=\beta*\alpha$
2. $\alpha*1=\alpha$
3. $(\alpha*\beta)*\gamma=\alpha*(\beta*\gamma)$
4. If $\alpha\le\beta$ and $\gamma\le\delta$ then $\alpha*\gamma\le\beta*\delta$

Examples 2.3: [7]

1. Define $\alpha*\beta=\alpha.\beta$ for all $\alpha,\beta\in[0,1]$ where $\alpha.\beta$ is usual multiplication in $[0,1]$. Then $*$ is a t-norm
2. Define $\alpha*\beta=\min\{\alpha,\beta\}$ for all $\alpha,\beta\in[0,1]$ then $*$ is a t-norm

Remark 2.4: [7]

1. If $\alpha>\beta$ then there is γ such that $\alpha*\gamma\ge\beta$
2. There is δ such that $\delta*\delta\ge\sigma$ where $\alpha,\beta,\gamma,\delta,\sigma\in[0,1]$

Definition 2.5: [7]

The triple $(V,L,*)$ is said to be a **fuzzy normed space** if V is a vector space over the field $\mathbb{F}$, $*$ is a t-norm and $L: V\times[0,\infty)\to[0,1]$ is a fuzzy set has the following properties for all a, b∈ V and $\alpha,\beta>0$.

1- $L(a,\alpha)>0$
2- $L(a,\alpha)=1 \Leftrightarrow a=0$
3- $L(ca,\alpha)=L\left(a,\frac{\alpha}{|c|}\right)$ for all $c\neq 0\in\mathbb{F}$
4- $L(a,\alpha)*L(b,\beta)\le L(a+b,\alpha+\beta)$
5- $L(a,.): [0,\infty)\to[0,1]$ is continuous
6- $\lim_{\alpha\to\infty} L(a,\alpha)=1$

Example 2.6: [8]

Suppose that $(X,||.||)$ is a normed space, put $\alpha*\beta=\alpha\beta$ or $\alpha*\beta=\min\{\alpha,\beta\}$.

Define $L_{||.||}(a,\alpha)=\frac{\alpha}{\alpha+\|a\|}$. Then $\left(X,L_{||.||},*\right)$ is a fuzzy normed space is known as the standard fuzzy norm depends on the norm $\|.\|$.

Corollary 2.7:[24]

Suppose that $(\mathbb{F},|.|)$ is a normed space then $\left(\mathbb{F},L_{|.|},*\right)$ is a fuzzy normed space where $L_{|.|}(a,t)=\frac{t}{t+|a|}$ for all $a\neq 0\in\mathbb{F}$.

Example 2.8: [9]

Let $(V,\|.\|)$ be a normed space, define $\alpha*\beta=\alpha.\beta$ for all $\alpha,\beta\in[0,1]$ and $L(a,\alpha)=\left[\exp(\frac{\|a\|}{\alpha})\right]^{-1}$ for all $a\in V$ and $t>0$. Then $(V,L,*)$ is a fuzzy normed space

Example 2.9: [12]

Assume $(V, ||.||)$ is a normed space, put $\alpha * \beta = \min\{\alpha, \beta\}$ for all $\alpha, \beta \in [0,1]$ and

$$L(x,t) = \begin{cases} 1 & \text{if } \|x\| < t \\ 0 & \text{if } \|x\| \geq t \end{cases}$$

Then $(V, L, *)$ is a fuzzy normed space

Lemma 2.10: [24]

Let $(V, L, *)$ be a fuzzy normed space then $L(a, .)$ is anon decreasing function of t for $a \in V$ that is if $0 < t < s$ then $L(a,t) < L(a,s)$

Remark 2.11: [24]

Assume that $(V, L, *)$ is a fuzzy normed space and let $a \in V, t > 0, 0 < q < 1$. If $L(a,t) > (1-q)$ then there is s with $0 < s < t$ such that $L(a,s) > (1-q)$.

Definition 2.12:[8]

Suppose that $(V, L, *)$ is a fuzzy normed space. Put

$B(a,p,t) = \{b \in X : L(a-b,t) > (1-p)\}$

$B[a,p,t] = \{b \in X : L(a-b,t) \geq (1-p)\}$

Then $B(a,p,t)$ and $B[a,p,t]$ is called **open and closed fuzzy ball** with the center $a \in V$and radius p, with $p > 0$.

Lemma 2.13:[8]

Suppose that $(V, L, *)$ is a fuzzy normed space then $L(x-y,t) = L(y-x,t)$ for all $x, y \in V$ and $t > 0$

Definition 2.14: [6]

Assume that $(V, L, *)$ is a fuzzy normed space. $A \subseteq V$ is called **fuzzy bounded** if we can find $t > 0$ and $0 < q < 1$ such that $L(a,t) > (1-q)$ for each $a \in A$.

Definition 2.15: [8]

A sequence (a_n) in a fuzzy normed space $(V, L, *)$ is called **converges to** $a \in V$ if for each $q > 0$ and $t > 0$ we can find N with $L[a_n - a, t] > (1-q)$ for all $n \geq N$. Or in other word $\lim_{n\to\infty} a_n = a$ or simply represented by $a_n \to a$, a is known the limit of (a_n).

Remark 2.16: [8]

A sequence (a_n) in a fuzzy normed space $(V, L, *)$ converges to $a \in V$ if $\lim_{n\to\infty} L[a_n - a, t] = 1$.

Definition 2.17: [8]

A sequence (a_n) in a fuzzy normed space $(V, L, *)$ is said to be a **Cauchy sequence** if for all $0 < q < 1$, $t > 0$ there is a number N with $L[a_m - a_n, t] > (1-q)$ for all $m, n \geq N$.

Lemma 2.18: [24]

If (a_n) is a sequence in a fuzzy normed space $(V, L, *)$ converges to $a \in V$ then (a_n) is a Cauchy sequence.

Definition 2.19: [4]

Suppose that $(V, L, *)$ is a fuzzy normed space and let A be a subset of V. Then the **closure of A** is written by $\overline{A}$ or $CL(A)$ and which is $\overline{A} = \bigcap\{B \subseteq V: B \text{ is closed}\}$.

Definition 2.20: [24]

Suppose that $(V, L, *)$ is a fuzzy normed space and $A \subseteq V$. Then A is called **dense** in V when $\overline{A} = V$.

Lemma 2.21:

Assume that $(V, L, *)$ is a fuzzy normed space and suppose that A is a subset of V. Then $y \in \overline{A}$ if and only if there is a sequence (y_n) in A with (y_n) converges to y.

Proof :

Suppose that $y \in \overline{A}$ when $y \in A$ then choose the sequence $(y, y, \ldots, y, \ldots)$ but when $y \notin A$ so it must be a limit point of A. Thus take the sequence $(y_n) \in A$ with $L[y_m - y, t] > \left(1 - \frac{1}{m}\right)$ for all $m = 1,2, \ldots$ which implies that the open fuzzy ball $B\left(y, \frac{1}{m}, t\right)$ contains $y_n \to y$ since $\lim_{n\to\infty} L[y_n - y, t] = 1$.

Conversely if (y_n) in A and $y_n \to y$. Then $y \in A$ or every open fuzzy ball of y contains $y_n \neq y$. That is y is a limit point of A hence $y \in \overline{A}$. ■

Theorem 2.22:

Suppose that $(V, L,*)$ is a fuzzy normed space and assume that A is a subset of V. Then A is dense in V if and only if for every $x \in V$ there is $a \in A$ such that $L[x - a, t] > (1 - \varepsilon)$ for some $0 < \varepsilon < 1$ and $t > 0$.

Proof:

Suppose that A is dense in V and $x \in V$ this implies that $x \in \bar{A}$ then by lemma 2.21 we can find a sequence $(y_n) \in A$ such that $y_n \to x$ that is for given $0 < \varepsilon < 1$ and $t > 0$ there is a number N such that $L[y_n - x, t] > (1 - \varepsilon)$ for all $n \geq N$. Take a= y_N so $L[a - x, t] > (1 - \varepsilon)$.

To prove the converse we will show that A is dense in V to prove that $V \subseteq \bar{A}$ suppose that $x \in V$ then we can find $a_j \in A$ with $L[a_j - x, t] > (1 - \frac{1}{j})$. Now take $0 < \varepsilon < 1$ with $\frac{1}{j} < \varepsilon$ for all j ≥N for some number N. Hence we have a sequence $(a_j) \in A$ such that $L[a_j - x, t] > \left(1 - \frac{1}{j}\right) > (1 - \varepsilon)$ for all $j \geq N$, that is $a_j \to x$ so $x \in \bar{A}$.■

Definition 2.23: [10]

A fuzzy normed space $(X, L,*)$ is said to be **fuzzy complete** if every Cauchy sequence in X converges to a point in X.

Definition 2.24: [8]

Suppose that $(V, L_V,*)$ and $(W, L_W,*)$ are two fuzzy normed spaces .The operator $S: V \to W$ is said to be **fuzzy continuous at $\mathbf{v_0}$** $\in V$ if for all $t > 0$ and for all $0 < \alpha < 1$ there is s[depends on t, α and v_0] and there is β [depends on t, α and v_0] with, $L_V[v - v_0, s] > (1 - p)$ we have $L_W[S(v) - S(v_0), t] > (1 - \alpha)$ for all v ∈V.

3.1 Fuzzy bounded linear operators

In this section we introduce the definition of fuzzy norm of fuzzy bounded operator in order to prove basic properties to the fuzzy normed space of all fuzzy bounded operators.

Definition 3.1:

Let $(X, L_X,*)$ and $(Y, L_Y,*)$ be two fuzzy normed spaces. An operator $T: D(T) \to Y$ is said to be **fuzzy bounded** if there exists r, $0 < r < 1$ such that $L_Y(Tx, t) \geq (1 - r) \star L_X(x, t)$ ..… (3.1)

for each $x \in D(T) \subseteq X$ and $t > 0$ where $\star$ is a continuous t-norm and D(T) is the domain of T.

Theorem 3.2:

Suppose that $(X, L_X,*)$ and $(Y, L_Y,*)$ are two fuzzy normed spaces. The operator $S: D(S) \to Y$ is fuzzy bounded if and only if S (A) is fuzzy bounded for every fuzzy bounded subset A of D(S).

Proof:

Suppose that S is fuzzy bounded then there is q, $0 < q < 1$ such that $L_Y(Sa, t) \geq (1 - q) \star L_x(a, t)$ for all $t > 0$ and $a \in D(S)$.

Let $A \subseteq D(S)$ be a fuzzy bounded so there is $0 < (1 - p) < 1$ such that $L_X(a, t) \geq (1 - p)$ for all $a \in A$ and $t > 0$. Now we can find, $0 < c < 1$ with $(1 - q) * (1 - p) \geq (1 - c)$

Hence $L_Y(S(a), t) \geq (1 - r)$ that is S(A) is fuzzy bounded.

Conversely let D(S) be fuzzy bounded then S(D) is fuzzy bounded.

Now there is q $0 < q < 1$ such that $L_Y(Sx, t) \geq (1 - q)$ for all $x \in D(S)$ and t>0. Hence we can find p $0 < p < 1$ with $(1-q) \geq (1-p) \star L_X(x, t)$ thus $L_Y(Sx, t) \geq (1 - p) \star L_X(x, t)$ for all $x \in D(S)$. ■

Notation:

Suppose that $(X, L_X,*)$ and $(Y, L_Y,*)$ are two fuzzy normed spaces. Put FB(X, Y) ={S: X→Y, S is a fuzzy bounded operator}

Lemma 3.3:

Let $(X, L_X,*)$ be a fuzzy normed space. If A and B are fuzzy bounded subset of X then A+B and αA are fuzzy bounded for every $\alpha \neq 0 \in \mathbb{F}$.

Proof:

Since A and B are fuzzy bounded so there is p, $0 < p < 1$ and

q, $0 < q < 1$ with $L_X(a,t) \geq (1-p)$ for all $a \in A$ and $t > 0$
also $L_X(b,s) \geq (1-q)$ for all $b \in B$ and $s > 0$. Now
$L_X(a+b,t+s) \geq (1-p) * (1-q)$
Put $(1-p) * (1-q) \geq (1-r)$ for some $r, 0 < r < 1$
Hence $L_X(a+b,t+s) \geq (1-r)$ that is A+B is fuzzy bounded also
$L_X(\alpha a,t) = L_X(a,\frac{t}{|\alpha|}) \geq (1-p)$ that is αA is fuzzy bounded.■

Lemma 3.4:

Let $(X, L_X,*)$ and $(Y, L_Y,*)$ be two fuzzy normed spaces then $T_1 + T_2 \in FB(X,Y)$ and $\alpha T \in FB(X,Y)$ for all $T_1, T_2 \in FB(X,Y)$ and $0 \neq \alpha \in \mathbb{F}$.

Proof:

Let $A \subseteq D(T_1) = D(T_2)$ be a fuzzy bounded then $T_1(A)$ and $T_2(A)$ are fuzzy bounded by Theorem 3.2 so $(T_1+T_2)(A) = T_1(A) + T_2(A)$ is fuzzy bounded by lemma 3.3 , hence $T_1 + T_2 \in F\,B(X,Y)$.
Let $A \subseteq D(T)$ be a fuzzy bounded so $T(A)$ is a fuzzy bounded by Theorem 3.2 $\alpha T(A)$ is fuzzy bounded for every $\alpha \neq 0 \in F$ by lemma 3.3 hence $\alpha T \in F\,B(X,Y)$.■

Theorem 3.5:

Suppose that $(X, L_X,*)$ and $(Y, L_Y,*)$ are two fuzzy normed spaces. Define:
$\mathbf{L\,(T,\,t) = \inf_{x \in D(T)} L_Y(Tx,t)}$ for all $T \in FB(X,Y)$ and $t > 0$ then $(FB(X,Y), L,*)$ is fuzzy normed space.

Proof:

(FN_1) Since $L_Y(Tx,t) > 0$ for all $x \in D(T)$ and $t > 0$ so $L(T,t) > 0$ for all $t > 0$.
(FN_2) For every $t > 0$, $L(T,t) = 1 \Leftrightarrow \inf_{x \in D(T)} L_Y(Tx,t)=1 \Leftrightarrow L_Y(Tx,t)=1 \Leftrightarrow T(x) = 0$ for all $x \in D(T) \Leftrightarrow T = 0$
(FN_3) For all $0 \neq \alpha \in \mathbb{F}$ we have

$$L(\alpha T,t) = \inf_{x \in D(T)} L_Y(\alpha Tx,t) = \inf_{x \in D(T)} L_Y(Tx,\frac{t}{|\alpha|}) = L\left(T,\frac{t}{|\alpha|}\right)$$

(FN_4)

$$\begin{aligned} L(T_1+T_2,t+s) &= \inf L_{Y_{x\in D(T_1)\cap D((T_2)}}\left((T_1+T_2)(x),t+s\right) \\ &= \inf L_{Y_{x\in D(T_1)\cap D((T_2)}}(T_1(x)+T_2(x),t+s) \\ &\geq \inf L_{Y_{x\in D(T_1)}}(T_1x,t) * \inf L_{Y_{x\in D(T_2)}}(T_2x,s) \\ &= L(T_1,t) * L(T_2,s) \end{aligned}$$

(FN_5)Let (t_n) be a continuous sequence in $[0,\infty)$ so $L(T,t_n) \to L(T,t)$ that is $(T,\bullet)$ is a continuous function.
(FN_6) $\lim_{t\to\infty} L(T,t) = \lim_{t\to\infty} \inf L_Y(Tx,t) = \inf \lim_{t\to\infty} L_Y(Tx,t) = 1$
Hence $(FB(X,Y), L,*)$ is a fuzzy normed space■

Example 3.6:

Let X be the vector space of all polynomials on [0,1] with $\|y\| = \max|y(t)|$, $t \in [0,1]$. Let

$$L(y,s) = \begin{cases} 1 & \text{if } \|y\| < s \\ 0 & \text{if } \|y\| \geq s \end{cases}$$

Then $(X, L,\wedge)$ is a fuzzy normed space. Let $T: X \to X$ be defined by $T[y(t)] = \acute{y}(t)$ then T is linear. Let $y_n(t) = t^n$ indeed $\|y_n\| = 1$ so

$$L[T(y),s] = \begin{cases} 1 & n < s \\ 0 & s \leq n \end{cases}$$

Hence there is no c, $0 < c < 1$ satisfies the inquality
$L_Y(T(y),s) \geq (1-c) \star L_X(y,t)$. Therefore T is not fuzzy bounded .

Remark 3.7:

Equation 3.1 can be written by:

$$L_Y[Tx,t] \geq L(T,t) \star L_X(x,t) \qquad \text{………(3.2)}$$

Theorem 3.8:

Let $(X, L_X, *)$ be a fuzzy normed space and let $(Y, L_Y, *)$ be a complete fuzzy normed space. Suppose that $S: D(S) \to Y$ is a fuzzy bounded linear operator. Then there is $T: \overline{D(S)} \to Y$ an extension of S such that T is fuzzy bounded linear operator with $L(T, q) = L(S, q)$ for all $q > 0$.

Proof:

Let $a \in \overline{D(S)}$ then by lemma 2.21 there is (a_n) in D(S) such that $a_n \to a$. Since S is linear and fuzzy bounded we have $L_Y(S(a), t) \geq (1 - p) \star L_Y(a, t)$ for each $a \in D(S)$ and $p > 0$ where $0 < p < 1$. Now

$L_Y[Sa_n - Sa_m, q] = L_Y[S(a_n - a_m), q] \geq L[S, q] \star L_X[a_n - a_m, q]$

This implies that $(S(a_n))$ is Cauchy but Y is complete so that $(S(a_n))$ converges to $b \in Y$. Define T(a) = b to prove that T does not depends by how we choose (a_n) in D(S) converging to a. Let $a_n \to a$ and $u_n \to a$ then $w_m \to a$ where $(w_m) = (a_1, u_1, a_2, u_2, \ldots\ldots)$. Hence (Sw_m) converges and the two subsequences (Sa_n) and (Su_n) of (Sw_m) must have the same limit. This proves that T is uniquely defined at

every $a \in \overline{D(S)}$. Clearly T is linear and S(a)=T(a) for every $a \in D(S)$ so that T is an extension of S. This implies that $L_Y[Sa_n, q] \geq L[S, q] \star L_x[a_n, q]$ let $n \to \infty$ then $Sa_n \to T(a) = y$ and since $a \to L_x(a, q)$ is a continuous mapping thus we obtain

$L_Y[Ta, q] \geq L[S, q] \star L_X[a, q]$. Hence T is fuzzy bounded and $L[T, q] \geq L[S, q]$ but $L[T, q] \leq L[S, q]$ because the fuzzy norm being defined by infimum cannot be increase in a extension together we have $L[T, q] = L[S, q]$.■

Theorem 3.9:

Suppose that $(X, L_X, *)$ and $(Y, L_Y, *)$ are two fuzzy normed spaces with $S: D(S) \to Y$ is a linear operator where $D(S) \subseteq X$. Then S is fuzzy bounded if and only if S is fuzzy continuous.

Proof:

Assume that S is fuzzy bounded and consider any $x \in D(S)$. Let $\varepsilon, 0 < \varepsilon < 1$ be given and $t > 0$ then for every $z \in D(S)$ such that $L_X[z, s] \geq (1 - r)$ we have $L_Y[Sz, t] \geq (1 - \varepsilon)$. Now let $y \in D(S)$ then $L_x[x - y, s] \geq (1 - r)$ implies

$L_Y[S(x - y), t] = L_Y[Sx - Sy, t] \geq (1 - \varepsilon)$. Thus S is a fuzzy continuous at x since x was an arbitrary. Hence S is fuzzy continuous.

Conversely suppose that S is a fuzzy continuous at an arbitrary point $x \in D(S)$. Then given $\varepsilon, 0 < \varepsilon < 1$ and $t > 0$ there is $r, 0 < r < 1$ and $s > 0$ such that $L_Y[Sx - Sy, t] > (1 - \varepsilon)$ for all $y \in D(S)$ satisfying $L_X[x - y, s] > (1 - r)$. Let $A \subseteq D(S)$ be a fuzzy bounded and take any $z \neq 0 \in A$. Set $y = x + z$, hence for all $t > 0$

$L_Y(Sz, t) = L_Y[S(y - x), t] = L_Y[Sy - Sx, t] > (1 - \varepsilon)$. This implies that S(A) is fuzzy bounded . Hence S is fuzzy bounded by Theorem 3.2.■

Corollary 3.10:

Let $(X, L_X, *)$ and $(Y, L_Y, *)$ be two fuzzy normed spaces. Assume that $T: D(T) \to Y$ is a linear operator where $D(T) \subseteq X$. Then T is a fuzzy continuous if T is a fuzzy continuous at $x \in D(T)$.

Proof:

If T is continuous at $x \in D(T)$ then by Theorem 3.9 T is fuzzy continuous .■

Corollary 3.11:

If $T: (X, L_X, *) \to (Y, L_Y, *)$ is a fuzzy bounded operator where $(X, L_X, *)$ and $(Y, L_Y, *)$ are two fuzzy normed space. Then

1)$x_n \to x$ [where $x_n, x \in D(T)$] implies $Tx_n \to Tx$

2)N(T) the kernel of T is closed .

Proof:

1)By Definition 3.1 we can find $r, 0 < r < 1$ such that

$L_Y[T(x), t] \geq (1 - r) \star L_X[x, t]$ for each $x \in D(T)$ and $t > 0$. Now

$L_Y[Tx_n - Tx, t] = L_Y[T(x_n - x), t] \geq (1 - r) \star L_X[x_n - x, t]$

But $x_n \to x$ so for any given $\varepsilon, 0 < \varepsilon < 1$ we can find N with

$L_X[x_n - x, t] \geq (1 - \varepsilon)$ for all $n \geq N$. We have $L_Y[Tx_n - Tx, t] \geq$

$(1 - r) \star (1 - \varepsilon)$ for all $n \geq N$. Let $(1 - r) \star (1 - \varepsilon) \geq (1 - q)$ for some q,

$0 < q < 1$. Hence $L_Y[Tx_n - Tx] \geq (1 - q)$ therefore $Tx_n \to Tx$.

2) For every $x \in \overline{N(T)}$ then there is a sequence (x_n) in N(T) such that $x_n \to x$ by Theorem 2.21. Hence $Tx_n \to Tx$ by part (1) . Also T(x)=0 since $T(x_n) = 0$ so that x ∈ N(T). Since $x \in \overline{N(T)}$ was arbitrary so N(T) is closed.■

Theorem 3.12:

Let $(X, L_X,*)$ and $(Y, L_Y,*)$ be two fuzzy normed spaces. If Y is fuzzy complete then FB(X, Y) is fuzzy complete.

Proof:

Suppose that (S_n) is a Cauchy sequence in FB(X,Y). That is for every r, r ∈[0,1] and $t > 0$ we can find N with $L[S_n - S_m, t] \geq (1 - r)$ for all $m, n \geq N$. Now for $a \in X$ and $m, n \geq N$ we have

$L_Y[S_n a - S_m a, t] \geq L_Y[(S_n - S_m)(a), t] > (1 - r) \star L_x[a, t]$ ……(3.3)

Now for any fixed x , we given r_a, $0< r_a < 1$ and we have from (3.3)

$L_Y[S_n a - S_m a, t] > (1 - r_x) \star L_x[a, t]$ this implies that $(S_n a)$ is a Cauchy sequence in Y since Y is fuzzy complete we have $(S_n a)$ converges to $b \in Y$ or $S_n a \to b$. Thus we can define an operator $S: X \to Y$ as S(a) = b. The operator S is linear since

$S[\alpha x + \beta z] = \lim_{n\to\infty} S_n[\alpha x + \beta z] = \alpha \lim_{n\to\infty} S_n x + \beta \lim_{n\to\infty} S_n z = \alpha S(x) + \beta S(z)$

Now we show that S is fuzzy bounded and $S_n \to S$ since equation (3.3) holds for every $m \geq N$ and $S_m a \to Sa$ we may let $m \to \infty$ we obtain from equation (3.3) for every $n \geq N$, $t > 0$ and all $a \in X$,

$L_Y[S_n a - Sa, t] = L_Y[S_n a - \lim_{m\to\infty} S_m a] = \lim_{m\to\infty} L_Y[S_n a - S_m a] > (1 - r) \star L_x[a, t]$ …….(3.4)

This shows that $(S_n - S)$ with $n \geq N$ is fuzzy bounded linear but S_n is fuzzy bounded so $S = S_n - (S_n - S)$ is fuzzy bounded that is S ∈ FB(X,Y) furthermore in equation (3.4) we take the infimum for all a we obtain,

$L(S_n - S, t) \geq (1 - r) \star L_x(a, t)$. Now put $L_x(a, t) = (1 - p)$ for some $0 < p < 1$ and $(1 - r) \star (1 - p) = (1 - q)$ for some $0 < q < 1$ so we get $L[S_n - S, t] \geq (1 - q)$ for all $n \geq N$ and $t > 0$ that is $S_n \to S$. ■

Now we introduce the definition of fuzzy bounded linear functional

Definition 3.13:

A linear functional f from a fuzzy normed space $(X, L_X,*)$ into the fuzzy normed space $(F, L_F,*)$ is said to be **fuzzy bounded** if there exists r, $0 < r < 1$ such that $L_F[f(x), t] \geq (1 - r) \star L_X[x, t]$ for all $x \in D(f)$ and $t > 0$. Furthermore, the fuzzy norm of f is $L(f, t) = \inf L_F(f(x), t)$ and $L_F(f(x), t) \geq L(f, t) \star L_X(x, t)$.

The prove of the following theorem follows from theorem 3.9.

Theorem 3.14:

A linear functional f with D(f) lies in the fuzzy normed space $(X, L_X,*)$ is fuzzy continuous if and only f is fuzzy bounded.

Definition 3.15:

Suppose that $(X, L_X,*)$ is a fuzzy normed space. Then the vector space FB(X,𝔽) ={f:X→ 𝔽, f is fuzzy bounded linear function } with a fuzzy norm defined by

$\mathbf{L(f, t) = \inf L_F\,(f(x), t)}$ forms a fuzzy normed space which is called the fuzzy dual space of X.

The prove of the following theorem follows from theorem 3.12

Theorem 3.16:

If $(X, L_X,*)$ is a fuzzy normed space then fuzzy dual space FB(X,ℝ) is complete

4. Convergence of sequence of operators and functions

In this section we introduce some types of fuzzy convergence of operators in the fuzzy normed space FB(X,Y).

Definition 4.1:

Suppose that $(X, L,*)$ is a fuzzy normed space then a sequence (a_n) in X is said to **weakly fuzzy convergent** if we can find an $a \in X$ such that for every f ∈FB(X,𝔽) $\lim_{n\to\infty} f(a_n) = f(a)$. Or simply written $a_n \to^w a$. a is known as weak limit to (a_n).

Theorem 4.2:

Suppose that (a_n) is a sequence in X then

1. If $a_n \to a$, a ∈X then $a_n \to^w a$
2. If dim X is finite then $a_n \to^w a$ implies $a_n \to a$ ∈X

Proof:

1. Since $a_n \to a$ so for given $t > 0$, $0 < p < 1$ we can find N such that $L[a_n - a, t] > (1 - p)\ for\ all\ n \geq N$. Now for every f ∈ FB(X,𝔽) $L_R[f(a_n) - f(a), t] = L_R\big[f\big(a_n - a\big), t\big] \geq L[f, t] \star L_X[a_n - a, t]$. Put $L[f, t] = (1 - \varepsilon)$ then we can find $(1 - q)$ for some $0 < q < 1$ with $(1 - \varepsilon) \star (1 - p) > (1 - q)$. Hence $L_R[f(a_n) - f(a), t] > (1 - q)$. This shows that $a_n \to^w a$.
2. Suppose that $x_n \to^w x$ and dimX=m let$\{e_1, e_2, \dots \dots e_m\}$ be a basis for X so $a_n = \alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \cdots \dots \dots + \alpha_m^{(n)} e_m$ and $a = \alpha_1\, e_1 + \alpha_2\, e_2 + \cdots \ldots + \alpha_m\, e_m$

but $f(a_n) \to f(a)$ for every $f \in FB(X, R)$. we take $f_1, f_2, \dots, f_m$defined by: $f_j\big(e_j\big) = 1\ and\ f_j(e_k) = 0\ k \neq j$ then $f_j(a_n) = \alpha_j^{(n)}$, $f_j(a) = \alpha_j$ hence $f_j(a_n) \to f_j(a)$ implies $\alpha_j^{(n)} \to \alpha_j$. Now for n ≥N

$$L_X[a_n - a, t] = L_X[\sum_{j=1}^{m} \Big(\alpha_j^{(n)} - \alpha_j\Big) e_j]$$

$$\geq L_X\left[e_1, \frac{t}{m|\alpha_1{}^{(n)} - \alpha_1|}\right] * L_X\left[e_2, \frac{t}{m|\alpha_2{}^{(n)} - \alpha_2|}\right] * \dots * L_X\left[e_m, \frac{t}{m|\alpha_m{}^{(n)} - \alpha_m|}\right]$$

when $n \to \infty$ we have $\lim_{n\to\infty} L_X\,[a_n - a, t] = 1 * 1 * \dots * 1 = 1$.

This show that $a_n \to a$ ∎

Definition4.3:

Suppose that $(X, L_X, *)$ and $(Y, L_Y, *)$ are two fuzzy normed spaces. A sequence (S_n) with $S_n \in$ FB(X,Y) is called

1. **Uniformly operator fuzzy convergent** if (S_n) converges in the fuzzy normed space (FB(X,Y), L,∗)
2. **Strong operator fuzzy convergent** if $(S_n a)$ converges in fuzzy normed space $(Y, L_Y, *)$ for every $a \in X$.
3. **Weakly operator fuzzy convergent** if $(S_n a)$ converges weakly in fuzzy normed space $(Y, L_Y, *)$ for every $a \in X$.

This means that there is operator $T: X \to Y$ such that

1. $L[S_n - S, t] \to 1\ for\ all\ t > 0$ and n≥N.
2. $L_Y[S_n a - Sa, t] \to 1\ for\ all\ t > 0\ and\ for\ all\ a \in X$ and n ≥ N.
3. $L_R[f(S_n a) - f(Sa), t]\ for\ all\ t > 0\ and\ f \in FB(Y, \mathbb{F})$ and n≥N.

Respectively, T is called uniform. Strong and weak operator limit of (T_n)

Remark 4.4:

It is not difficult to show that $(1) \to (2) \to (3)$. In general $(3) \nrightarrow (2) \nrightarrow (1)$

Definition4.5:

Let $(X, L_X, *)$ be a fuzzy normed space. A sequence (f_n) of functional $f_n \in$ FB(X,𝔽) is said to be

1) **Strong fuzzy converges** in the fuzzy norm on FB(X,𝔽) then we can find f ∈ FB(X,𝔽) with $L[f_n - f, t] \to 1\ for\ all\ t > 0$ or $f_n \to f$.

2) **Weak fuzzy converges** in the fuzzy normed space $(\mathbb{F}, L_{\mathbb{F}}, *)$ that is we can find $f \in$ FB(X,𝔽) with $f_n(a) \to f(a)$ for all $a \in X$ or $\lim_{n\to\infty} f_n(a)$ = f(a).

Conclusion

In the present paper our aim is to define L(S,t) of a fuzzy bounded linear operator S in order to prove that (FB(X, Y), L,∗) is a complete fuzzy normed space where FB(X,Y) ={S:X→Y, S is a fuzzy bounded operator} with $(X, L_X, *)$ and $(Y, L_Y, *)$ are two fuzzy normed spaces also for a linear operator fuzzy bounded is equivalent to fuzzy continuous. Finally we introduce different types of fuzzy convergence of operators.

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