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Dynamics and an Optimal Policy for A Discrete Time System with Ricker Growth

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Abstract

The goal of this paper is to study dynamic behavior of a sporadic model (prey-predator). All fixed points of the model are found. We set the conditions that required to investigate the local stability of all fixed points. The model is extended to an optimal control model. The Pontryagin's maximum principle is used to achieve the optimal solutions. Finally, numerical simulations have been applied to confirm the theoretical results.

Keywords: Ricker function; Local stability; Discrete system; Optimal harvesting; Hamiltonian function

ديناميكية و سياسة مثلى لنظام متقطع مع دالة ريكر للنمو

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الخلاصة

الهدف من هذا البحث هو دراسة السلوك الديناميكي لنموذج متقطع (الفريسة المفترسة). تم يجاد نقاط الاتزان للنموذج المقترح. كذلك تم وضع الشروط اللازمة لتحقيق الاستقرار المحلي لجميع النقاط الاتزان. لقد تم توسيع النموذج إلى مسألة سيطرة مثلى. حيث استخدم مبدأ بونترياجين الأعظم للحصول على الحل الامثل للنظام. اعطيت امثلة عددية لتأكيد النتائج النظرية.

1. Introduction

In real world there is nothing surprising that organisms track spatial and environomental variation. In fact there are many studies to try understand the dynamics behavior of the population or to study the interrelation between species see [1,2]. The Lotka-Volterra model is the earliest ecological model. It is formed by using a set of differential equations to investigate the interaction between predator-prey species. The model is also used to describe a chemical reaction as well as to describe the animals. Many authors have been given a modification for the system using nonlinear difference equations or partial differential equations [3-5]. Some authors have been noted in their work that a chaotic population dynamic can be raised [6, 7] in multispecies continuous time predator-prey models as well as in discrete time model [8]. Their works have been carried out examine large competitions models with many species [9-12]. A simple nonlinear difference equation $x_{t+1} = f(x_t)$ is to describe the growth of single population with discrete nonoverlapping generations. This model has been investigated of is known to possess complicated dynamics [13, 1]. For the two-dimensional model, system of first order difference equations widely used are used

$$x_{t+1} = f(x_t, y_t)$$
$$y_{t+1} = g(x_t, y_t)$$

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Such systems are especially relevant to the study of arthropod predator-prey as well as host-parasitic interaction. This paper will give a modification of a discrete time predator prey model specified by Neubert et. al. [1]. Their model is defined as follows:

$$\begin{aligned} x_{t+1} &= x_t e^{r(1-x_t-y_t)} \\ y_{t+1} &= c x_t y_t \end{aligned} \qquad (1)$$

Where x_t represents the current prey-population, the Ricker function is used to describe the growth of the prey population, and y_t represents the current predator population with predator growth being directly proportional to the number of prey present. The parameter r is the growth rate of the prey which is also the consumption rate of the prey in to predator, and c is the conversion rate of predator. Our modified model is defined as follows:

$$\begin{aligned} x_{t+1} &= x_t e^{r(1-x_t-y_t)} \\ y_{t+1} &= b x_t y_t - a y_t \end{aligned} \qquad (2)$$

Here the parameter a represents the decay rate of predator in the absence of prey. This system has three fixed points.

All fixed points of the system (2) can be determined by solving the following algebraic equations:

$$\begin{aligned} x &= x e^{r(1-x-y)} \\ y &= bxy - ay \end{aligned}$$

After simple calculations we have the following lemma.

Lemma 1: The system (2) has three fixed points for all parameters values, namely $E_i, i = 0,1,2$. These are:

1) $E_0 = (0,0)$ is always exist.
2) $E_1 = (1,0)$ is always exist.
3) $E_2 = (x^*, y^*) = \left(\frac{1+a}{b}, 1 - x^*\right)$ is exist if $1 + a < b$.

We will discuss the local stability analysis of system (2) around each fixed point. So that the general Jacobain matrix of the system (2) at point (x, y) that is given by:

$$J(x,y) = \begin{bmatrix} J_{11}(x,y) & J_{12}(x,y) \\ J_{21}(x,y) & J_{22}(x,y) \end{bmatrix}$$

$J_{11}(x,y) = e^{r(1-x-y)} - rxe^{r(1-x-y)} = e^{r(1-x-y)}(1 - rx)$
$J_{12}(x,y) = -rxe^{r(1-x-y)}$
$J_{21}(x,y) = by$
$J_{22}(x,y) = bx - a$

Therefore

$$J(x,y) = \begin{pmatrix} e^{r(1-x-y)}(1-rx) & -rxe^{r(1-x-y)} \\ by & bx - a \end{pmatrix}$$

Thus the characteristic polynomial of the Jacobain matrix $J(x, y)$ of system (2) is

$$F(\lambda) = \lambda^2 + p\lambda + q \qquad (3)$$

Where $p = -trac(J)$ and $q = \det(J)$

In order study stability analysis of the fixed point E_0 of the system (2) we have the following theorem.

Theorem 2: For the fixed point E_0 we have:

i- E_0 is never to be sink.
ii- E_0 is source if $a > 1$.
iii- E_0 is saddle point if $a < 1$.
iv- E_0 is non-hyperbolic if $a = 1$.

Proof:

The Jacobain matrix at E_0 can be written as follows:

$$J_{E_0} = \begin{pmatrix} e^r & 0 \\ 0 & -a \end{pmatrix}$$

Then the eigenvalues of J_{E_0} are $\lambda_1 = e^r$ and $\lambda_2 = -a$, because of the e^r is always greater than 1 so that E_0 is never to be sink. It is clear that if $a > 1$ the E_0 is source and if $a < 1$ the point E_0 is saddle point. If $a = 1$ then $\lambda_2 = -1$ and the point E_0 is non-hyperbolic point

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By the same way we can study the local stability of E_1, the Jacobain matrix at E_1 is:

$$J_{E_1} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$J_{11} = e^{r(1-1)}(1-r) = 1-r$
$J_{12} = -re^{r(1-1)} = -r$
$J_{21} = 0$
$J_{22} = b-a$
So that

$$J_{E_1} = \begin{pmatrix} 1-r & -r \\ 0 & b-a \end{pmatrix}$$

The eigenvalues of the Jacobain matrix J_{E_1} are $\lambda_1 = 1-r$ and $\lambda_2 = b-a$, the next theorem gives the local stability of the fixed point E_1

Theorem 3: For the fixed point E_1 of the system(2) we have the following
1- E_1 is sink if the following conditions are holds:
i. $r \in (0,2)$
ii. $b \in (a-1, 1+a)$
2- E_1 is source if the following conditions are holds:
i. $r \in (0,2)$
ii. $b \in (a-1, 1+a)$
3- E_1 is saddle point either $r \in (0,2)$ and $b \notin (a-1, a+1)$ or $r \in (2,\infty)$ and $b \in (a-1, 1+a)$
4- E_1 is non-hyperbolic point if one of the following conditions are holds:
i. $r = 2$
ii. $b = 1+a$
iii. $b = a-1$

Proof(1): Let $|\lambda_1| < 1$ if and only if $-1 < 1-r < 1$ if and only if $0 < r < 2$.
Now $|\lambda_2| < 1$ if and only if $|b-a| < 1$ if and only if $-1 < b-a < 1$ if and only if $a-1 < b < 1+a$ so that E_1 is sink point if $r \in (0,2)$ and $b \in (a-1, 1+a)$. It is clear that the proof of (2) and proof (3) are coming from proof (1).
Proof (4): if $r = 2$ then $\lambda_1 = -1$ and E_1 is non-hyperbolic point. If $b = 1+a$ or $b = a-1$ then $\lambda_2 = -1$ or $\lambda_2 = 1$ and E_1 is non-hyperbolic point.

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For studying the local stability of the unique positive fixed point E_2 the next lemma gives the stability criterions which are needed to the local stability of the unique positive fixed point E_2. This is found in [14] so that the proof is omitted.

Lemma 4: Let $F(\lambda) = \lambda^2 + p\lambda + q$ suppose that $F(1) > 0$, and λ_1, λ_2 are the roots of F then:
1- $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $q < 1$.
2- $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $q > 1$.
3- $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$),if and only if $F(-1) < 0$.
4- $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0,2$.

we will find the Jacobain matrix at E_2. This is given by:

$$J_{11} = e^{r(1-\frac{1+a}{b}-1+\frac{1+a}{b})}\left(1 - r\frac{1+a}{b}\right) = 1 - r\left(\frac{1+a}{b}\right)$$

$$J_{12} = -r\left(\frac{1+a}{b}\right)e^{r\left(1-\frac{1+a}{b}-1+\frac{1+a}{b}\right)} = -r\left(\frac{1+a}{b}\right)$$

$$J_{21} = b\left(1 - \frac{1+a}{b}\right) = b-1-a$$

$$J_{22} = b\left(\frac{1+a}{b}\right) - a = 1$$

So that

$$J_{E_2} = \begin{bmatrix} 1 - r\left(\frac{1+a}{b}\right) & -r\left(\frac{1+a}{b}\right) \\ b-1-a & 1 \end{bmatrix}$$

The corresponding characteristic polynomial of J_{E_2} is $F(\lambda) = \lambda^2 + p\lambda + q$ where

$$F(\lambda) = \left(1 - r\left(\frac{1+a}{b}\right) - \lambda\right)(1-\lambda) + \left[r\frac{1+a}{b}(b-1-a)\right]$$

$$= 1 - \frac{r}{b} - \frac{ra}{b} - \lambda - \lambda + \frac{r}{b}\lambda + \frac{ra}{b}\lambda + \lambda^2 + r + ra - \frac{r}{b} - \frac{ra}{b} - \frac{ra}{b} - \frac{ra^2}{b}$$
$$F(\lambda) = \lambda^2 + \left(\frac{r}{b} + \frac{ra}{b} - 2\right)\lambda + \left(1 + r + ra - \frac{2r}{b} - \frac{3ra}{b} - \frac{ra^2}{b}\right)$$

$p = \frac{r}{b} + \frac{ra}{b} - 2$ and $q = 1 + r + ra - \frac{2r}{b} - \frac{3ra}{b} - \frac{ra^2}{b}$

The next theorem gives the dynamics behavior of the positive fixed point.

Theorem 5: For the unique fixed point E_2 we have:

1. E_2 is sink if $a \in (-1, b-1) \cap (b-2, \infty) \cap (k_1, k_2)$ where $k_1 = \frac{(rb-4r)-\sqrt{(rb-2r)^2+1}}{2r}$ and $k_2 = \frac{(rb-4r)+\sqrt{(rb-2r)^2+1}}{2r}$
2. E_2 is source if $a \in (k_1, k_2) \cap (-1, b-1)$
3. E_2 is saddle point if $a \in (-1, b-1) \cap [(-\infty, k_1) \cup (k_2, \infty)]$
4. E_2 is non-hyperbolic if and only if $a = k_1$ or $a = k_2$ and $a \neq \frac{2b}{r} - 1$ or $a \neq \frac{4b}{r} - 1$

Proof

1- We will apply lemma 4, so that $F(1) = 1 + p + q > 0$ if and only if $1 + \frac{r}{b} + \frac{ra}{b} - 2 + 1 + r + ra - \frac{2r}{b} - \frac{3ra}{b} - \frac{ra^2}{b} > 0$ if and only if $r + ra - \frac{r}{b} - \frac{2ra}{b} - \frac{ra^2}{b} > 0$ if and only if $b + ba - 1 - 2a - a^2 > 0$ if and only if $a^2 - (b-2)a - (b-1) < 0$. The solution of this inequality for a gives $F(1) > 0$ if and only if $a \in (-1, b-1)$.

Now, we have to show that $F(-1) > 0$ and $q < 1$

$F(-1) = 1 - \frac{r}{b} - \frac{ra}{b} + 2 + 1 + r + ra - \frac{2r}{b} - \frac{3ra}{b} - \frac{ra^2}{b}$. Then $F(-1) > 0$ if and only if $4 + r + ra - \frac{3r}{b} - \frac{4ra}{b} - \frac{ra^2}{b} > 0$ if and only if $4b + rb + rab - 3r - 4ra - ra^2 > 0$ if and only if $ra^2 - (rb-4r)a - (4b - 3r + rb) < 0$. After solving this inequality for a we have $F(-1) > 0$ if and only if $a \in (k_1, k_2)$.

Now, $q < 1$ if and only if $1 + r + ra - \frac{2r}{b} - \frac{3ra}{b} - \frac{ra^2}{b} < 1$ if and only if $r + ra - \frac{2r}{b} - \frac{3ra}{b} - \frac{ra^2}{b} < 0$ if and only if $b + ba - 2 - 3a - a^2 < 0$ if and only if $a^2 - (b-3)a - (b-2) > 0$.

The solution of this inequality for a gives $q < 1$ if and only if $a \in (-\infty, -1) \cup (b-2, \infty)$

By lemma 4 (1) E_2 is sink if $a \in (-1, b-1) \cap (b-2, \infty) \cap (k_1, k_2)$

2- From proof (1) we have $F(1) > 0$ if and only if $a \in (-1, b-1)$ and $F(-1) > 0$ if and only if $a \in (k_1, k_2)$. Now we have to prove $q > 1$. It is clear that $q > 1$ if and only if $a \in (-1, b-2)$. According to lemma 4 (2), we have that E_2 is source if and only if $a \in (k_1, k_2) \cap (-1, b-2) \cap (-1, b-1)$.

3- From proof (1) we have $F(1) > 0$ if and only if $a \in (-1, b-1)$, and one can easily to get $F(-1) < 0$ if and only if $a \in (-\infty, k_1) \cup (k_2, \infty)$ so that , according to lemma 4 (3) E_2 is saddle point if and only if $a \in (-1, b-1) \cap [(-\infty, k_1) \cup (k_2, \infty)]$.

4- By applying (4) in lemma 4 we have $F(-1) = 0$ if and only if $4 + r + ra - \frac{3r}{b} - \frac{4ra}{b} - \frac{ra^2}{b} = 0$ if and only if $4b + rb + rab - 3r - 4ra - ra^2 = 0$ if and only if $ra^2 - (rb-4r)a - (4b - 3r + rb) = 0$. So that $F(-1) = 0$ if and only if $a = k_1$ or $a = k_2$. Now $p \neq 0$ if and only if $\frac{r}{b} + \frac{ra}{b} - 2 = 0$ if and only if $a \neq \frac{2b}{r} - 1$. If $p \neq 2$ if and only if $\frac{r}{b} + \frac{ra}{b} - 2 = 2$ if and only if $a \neq \frac{4b}{r} - 1$. Therefore E_2 is non-hyperbolic point..

2. An optimal harvesting policy

This section concerns with an optimal harvesting problem. The aim of the problem is to maximize the amount of money that one can earn by selling the harvesting on the market, so that the system (2) can be extended, including the harvesting, to the following system

$$x_{t+1} = x_t e^{r(1-x_t-y_t)}$$
$$y_{t+1} = bx_t y_t - ay_t - h_t y_t$$

where $x_t, y_t, r, a,$ and b are defined as before. The control variable is h_t since that $0 \le h_t < M < 1$, for $t = 0,1,\cdots,T-1$. For a given time T which represents the harvesting amount and M is maximum removing amount.

Therefore objective functional will be given by the following

$$J(h_t) = \sum_{t=0}^{T-1} c_1 h_t y_t - c_2 y_t - c_3 {h^2}_t$$

The term $c_2 y_t$ is associated with the cost of supporting the fish, $c_3 {h^2}_t$ is associated with the cost of catching the fish. The quadratic term of control is to penalize the amount of harvesting [15, 16]. While $c_1 h_t y_t$ represents the amount of money that one has to earn. The pontryagin's maximum principle is used to get the optimal harvesting solution, so that the adjoints variables λ_1 and λ_2 are introduced as well as Hamiltonian function which defined by

$$H_t = c_1 h_t y_t - c_2 y_t - c_3 {h^2}_t + \lambda_{1,t+1}\left(x_t e^{r(1-x_t-y_t)}\right) + \lambda_{2,t+1}(b x_t y_t - a y_t - h_t y_t)$$

For $= 0,1,2,\ldots$. According to the pontryagin's maximum principle, the necessary conditions are

$$\lambda_{1,t} = \frac{\partial H}{\partial x_t} = \lambda_{1,t+1}\left(e^{r(1-x_t-y_t)} - r x_t e^{r(1-x_t-y_t)}\right) + \lambda_{2,t+1} b y_t$$

$$\lambda_{2,t} = \frac{\partial H}{\partial y_t} = c_1 h_t + c_2 y_t + \lambda_{1,t+1}\left(-r x_t e^{r(1-x_t-y_t)}\right) + \lambda_{2,t+1}(b x_t - a - h_t)$$

and $\lambda_1(T) = \lambda_2(T) = 0$, and the optimality condition which is given by

$\left.\frac{dH}{dh_t}\right|_{h_t = {h_t}^*} = c_1 y_t - 2c_3 {h_t}^* - \lambda_{2,t+1} y_t = 0$, then the characterization of the optimal harvesting policy is

$$h_t^* = \begin{cases} 0 & if \ \frac{(c_1 - \lambda_{2,t+1}) y_t}{2c_3} \le 0 \\ \frac{(c_1 - \lambda_{2,t+1}) y_t}{2c_3} & if \ 0 < \frac{(c_1 - \lambda_{2,t+1}) y_t}{2c_3} < M \\ M & if \ M \le \frac{(c_1 - \lambda_{2,t+1}) y_t}{2c_3} \end{cases}$$

The optimal harvesting h_t^* at time t will be determined numerically by maximizing the Hamiltonian function at that t.

3. Numerical results and discussions

To confirm the theoretical analysis of system (2) we use a different set of parameters values that shows the stability of the fixed point E_1 as well as the unique positive fixed point E_2. For the fixed point E_1, we choose the values of parameters as follows $r = 0.9, b = 0.75$ and $a = 0.1$, with the initial condition (0.23, 0.6). Therefore the (i) in Theorem 3 is satisfied. Figure-1 shows that the fixed point E_1 is locally stable. For the positive fixed point E_2 we choose the following set of values, $r = 0.7, b = 1.75$ and $a = 0.2$, with these values the condition (i) in Theorem 5 is satisfied. Figure-2 illustrates the local stability of E_2. Other set of values of parameters may be given. For the optimal control problem we apply which is found in [2]. For that we choose the set of values of parameters as follows:

$r = 0.7, b = 1.75, a = 0.2$ and $T = 60$, we get the total optimal harvesting $J = 0.7972$. In Figure-3 the prey population is plotted according to the system (4). The dotted line shows the prey species without control while the solid line represents the prey species with control. Figure-4 shows the effect of the harvesting on the predator species according to the system (4). Figure-5 illustrates the optimal harvesting solution which is plotted as a function of time.

One can see that the optimal solution of this problem takes three phases; the first phase is a time of recovering the population from low levels. This phase depends on the initial values of population, then the removing at optimal rate and the final phase, the unrestricted harvesting sets in Table a comparison is given between the optimal harvesting result and other harvesting strategies by using the same values of parameters with the same condition.

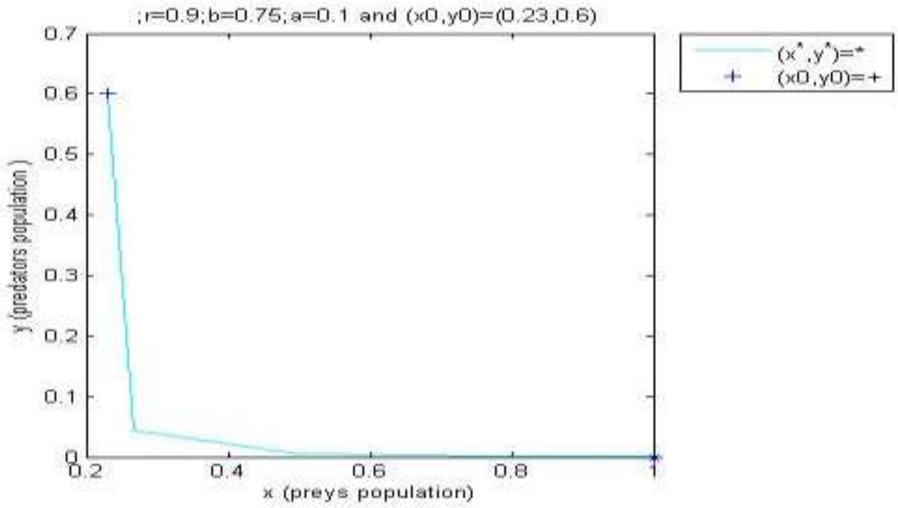

Figure 1-This figure shows the local stability of the fixed point fixed point E_1

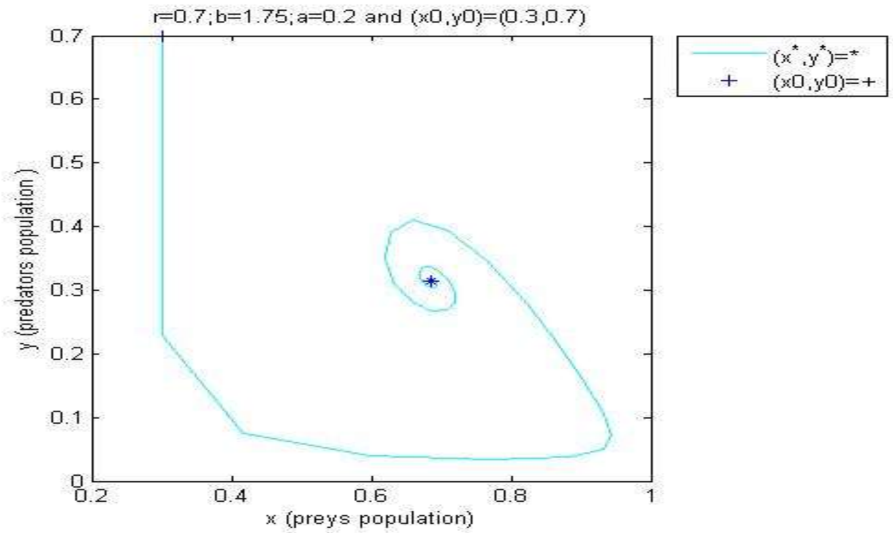

Figure 2-This figure shows the local stability of the unique positive fixed point E_2

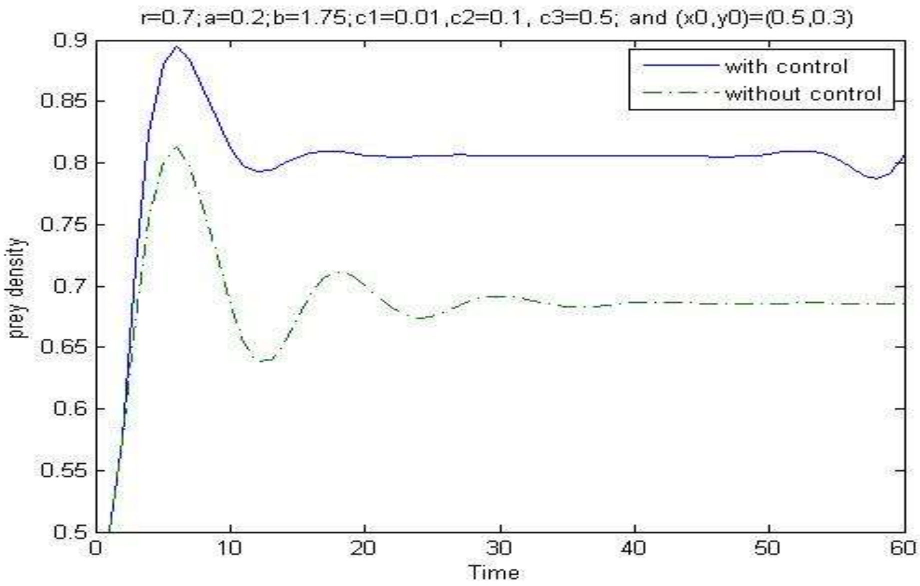

Figure 3-This shows illustrates the prey density population with and without control

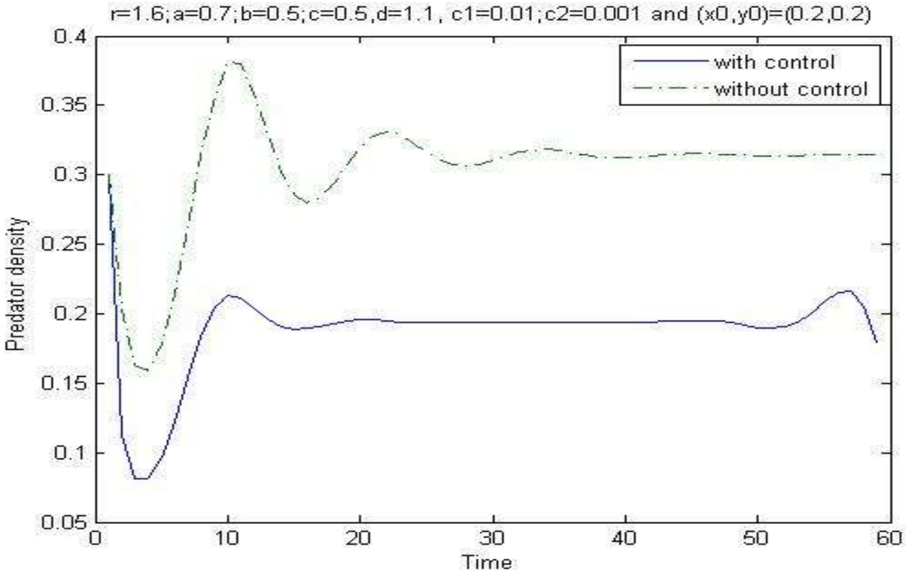

Figure 4- This shows illustrates the affect of harvesting on the predator population with and without harvesting

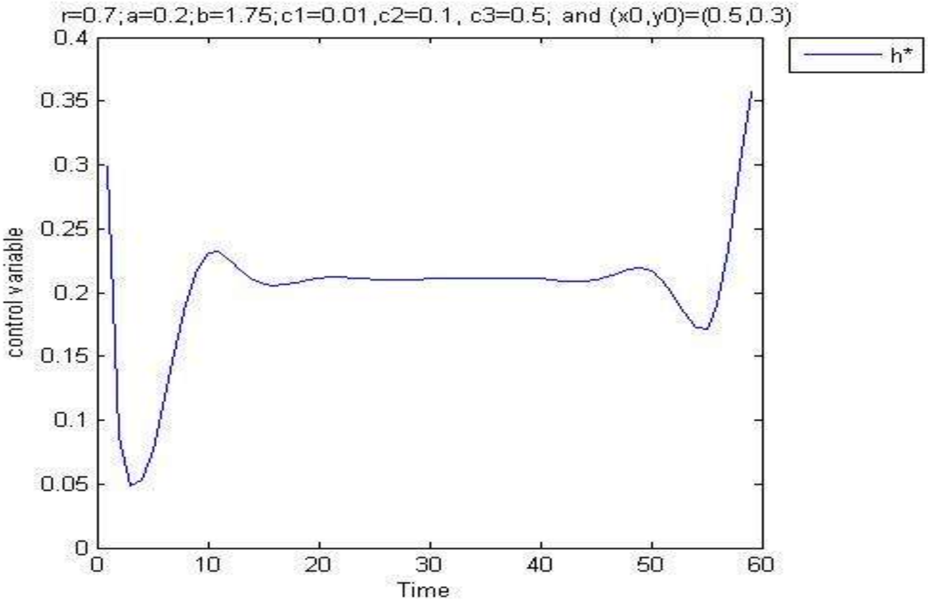

Figure 5-The optimal harvesting is plotted as a function of time

Table 1-The results of optimal harvesting with other strategies. All values of parameters are the same.

The control variable	Total net harvesting(J)
$h_t = h^*$	$J = 0.7972$
$h_t = 0.1$	$J = 0.5225$
$h_t = 0.15$	$J = 0.7010$
$h_t = 0.2$	$J = 0.7645$
$h_t = 0.3$	$J = 0.5482$

4. Conclusions

A discrete time prey-predator model with Ricker function growth has been studied. The model has three fixed points. The trivial fixed point is always exist and the other fixed points are exist for some values of parameters. The conditions for the local stability of all fixed points are discussed. An optimal harvesting policy is investigated to the model. The pontryagin's maximum principle is applied to determine the optimal strategy. Numerical analysis confirms and indicates the theoretical results.

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