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Approximate Solution for advection dispersion equation of time Fractional order by using the Chebyshev wavelets-Galerkin Method

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Abstract
The aim of this paper is adopted to give an approximate solution for advection dispersion equation of time fractional order derivative by using the Chebyshev wavelets-Galerkin Method . The Chebyshev wavelet and Galerkin method properties are presented. This technique is used to convert the problem into the solution of linear algebraic equations. The fractional derivatives are described based on the Caputo sense. Illustrative examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords*:* Fractional derivative, Advection dispersion equation of time Fractional order, Chebyshev wavelet, Operational matrix of the fractional integration , Galerkin Method.

الحل التقريبي لمعادلة الانتقال والتشتت ذات رتبة الزمن الكسرية باستخدام طريقة مويجات تشيبشيف – كاليركن

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الخلاصة
ان الهدف الرئيسي لهذا البحث هو ايجاد حل تقريبي لمعادلة الانتقال والتشتت ذات رتبة الزمن الكسرية باستخدام طريقة مويجات تشيبيشيف – كاليركن. تم عرض خصائص مويجات تشيبيشيف وطريقة كاليركن هذه التقنية تعتمد تحويل المسالة الى حل معادلات جبرية خطية . المشتقات الكسرية هنا هي من نوع كابوتو. تم اعتماد عدد من الامثلة التوضيحية لاثبات صحة وامكانية تطبيق التقنية المقترحة .

1. Introduction

Fractional calculus is a generalization of classical calculus, which provides an excellent tool to describe memory and hereditary properties of various materials and processes. Fractional calculus has found diverse applications in different scientific and technological fields[1–5], such as thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, edge detection, turbulence, signal processing, information sciences, communications, and many other physical processes and also in medical sciences.

Differential equations of fractional order are generalizations of ordinary differential equations to an arbitrary (non integer) order. They have attracted considerable interest because of their ability to

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model complex phenomenas. These equations capture nonlocal relations in space and time with power-law memory kernels. Due to the extensive applications of differential equations of fractional order in engineering and science, research in this area has grown significantly all around the world [6]. Partial differential equations of fractional order, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance, physical and biological processes and systems [7-15].

As a special type of partial differential equations of fractional order, fractional order advection-dispersion equation have been applied to many problem. For example, as mention in [16-18], in practical physical applications, dispersion or diffusion problems such as mixing in inlandand coastal waters [19], transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations [20], and groundwater hydrology to model the transport of passive tracers carried by fluid flow are modeled by the advection-dispersion equation of fractional order.

There are several methods to solve the advection–dispersion equation such as variable transformation [21], the Green function [22], the implicit and explicit difference method [23–26], and the Adomian decomposition method [27].

In this paper, we consider the following advection dispersion equation of time fractional order of the form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \qquad (1)$$

$$0 < x < 1,\ 0 < t < 1,\ 0 < \alpha \le 1$$

subject to the initial condition as following:

$$u(x,0) = \vartheta(x),\ 0 \le x \le 1 \qquad (2)$$

and boundary conditions

$$\begin{cases} u(0,t) = \ell_0(t) \\ u(1,t) = \ell_1(t), \quad 0 \le t \le 1 \end{cases} \qquad (3)$$

The organization of the rest of this article is as follows. In section 2 we introduce some necessary definitions of the fractional calculus theory, in section 3 the Chebyshev wavelet function, as well as, its properties are introduced. While in section 4 we illustrate how Chebyshev wavelet function with Galerkin method may be used to replace problem (1)–(3) by an explicit system of linear algebraic equations. In section 5, we present some numerical examples to demonstrate the effectiveness of the proposed method, concluding remarks are given in the final section.

2. Fractional Derivative and Integration

In this section, we shall review the basic definitions and properties of fractional integral and derivatives, which are used further in this paper [28].

Definition (1):- The Riemann-Liouville fractional integral operator of order $v > 0$, is defined as

$$I^v f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \quad v > 0, x > 0. \qquad (4)$$

$$I^0 f(x) = f(x)$$

Definition (2):-The Riemann-Liouville fractional derivative operator of order $v > 0$, is defined as

$${}_0D_x^v f(x) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-v-1} f(t)dt, \quad v > 0, x > 0. \qquad (5)$$

Where n is an integer and $n-1 < v \le n$.

Definition (3):- The Caputo fractional derivative operator of order $v > 0$, is defined as

$${}^cD_x^v f(x) = \frac{1}{\Gamma(n-v)} \int_0^x (x-t)^{n-v-1} \frac{d^n}{dx^n} f(t)dt, \quad v > 0, x > 0 \qquad (6)$$

Where n is an integer and $n-1 < v \le n$.

The relation between Caputo fractional derivative and Riemann-Liouville:

$$I^v\ {}^cD_x^v f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!} \qquad (7)$$

Where n is an integer and $n-1 < v \le n$.

Also, for the Caputo fractional derivative we have

$$ {}^{c}D_x^{v}x^{\beta} = \begin{cases} 0 & \text{for } \beta \in N_0 \text{ and } \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil v \rceil \text{ or } \beta \notin N \text{ and } \beta > \lfloor v \rfloor. \end{cases} \quad (8) $$

We use the ceiling function $\lceil v \rceil$ to denote the smallest integer greater than or equal to v, and the floor function $\lfloor v \rfloor$ to denote the largest integer less than or equal to v. Also $N = \{1,2,\dots\}$ and $N_0 = \{0,1,2,\dots\}$.

Recall that for $v = 0$, the Caputo differential operator concides with the usual identity differential operator of an integer order. Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operator; i.e.

$$ {}^{c}D_x^{v}\big(\lambda f(x) + \mu g(x)\big) = \lambda\, {}^{c}D_x^{v}f(x) + \mu\, {}^{c}D_x^{v}g(x) $$

Where λ and μ are constants.

3. Chebyshev Wavelets[29]:

Wavelets are family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as

$$ \psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right) \quad a, b \in \mathbb{R} \quad a \neq 0 $$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}, b = nb_0a_0^{-k},\ a_0 > 1,\ b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$ \psi_{n,k}(t) = |a_0|^{\frac{k}{2}}\psi\big(a_0^{k}t - nb_0\big) $$

where $\psi_{n,k}$ form a wavelet basis for $L^2(\mathbb{R})$. For $a_0 = 2$ and $b_0 = 1$, then $\psi_{n,k}(t)$ forms an orthonormal basis.

Chebyshev wavelets $\phi_{n,m}(t) = \phi(s, n, m, t)$ have four arguments: k can assume any positive integer, m is the degree of Chebyshev polynomials and t is the normalized time. They are defined on the interval $(0,1)$ by

$$ \phi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\widehat{T}_m(2^s t-(2n-1)) & \frac{n-1}{2^{s-1}} \leq t < \frac{n}{2^{s-1}} \\ 0 & \text{otherwise} \end{cases} \quad (9) $$

Where

$$ \widehat{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} &, \quad m = 0 \\ \sqrt{\frac{2}{\pi}}\, T_m, & m > 0 \end{cases} \quad \text{and } m = 0,1,\cdots, M-1 \text{ and } n = 0,1,\cdots, 2^{s-1},\ s \in \mathbb{N}. $$

where $T_m(t)$ be the Chebyshev polynomials which are orthogonal with respect to the weighted function $w(t) = \frac{1}{\sqrt{1-t^2}}$ defined on the interval $[-1,1]$, and satisfy the following recursive formula:

$$ T_0(t) = 1, T_1(t) = t, T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,3,\dots, $$

The Chebyshev wavelets $\phi_{n,m}(t)$ form an orthonormal basis for $L^2[0,1]$ with respect to weighted function $w_n(t) = w(2^k t - (2n-1))$, where L^2 is the space of square integrable function over [0,1].
A function f(t) defined over $L^2[0,1]$ can be expanded in the terms of Chebyshev wavelets as

$$ f(t) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} c_{pq}\phi_{pq}(t), \quad (10) $$

where $c_{pq} = \langle f(t), \phi_{pq}(t)\rangle_{w_n}$, in which $\langle , \rangle$ denotes the inner product. If the infinite series in (10) is truncated, then it can be written as

$$ f(t) = \sum_{p=1}^{2^{s-1}}\sum_{q=0}^{M-1} c_{pq}\phi_{pq}(t) = C^T\Phi(t), \quad (11) $$

where C and $\Phi(t)$ are $2^{s-1}(M) \times 1$ matrices given by

$$ C = c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, c_{2,1}, \dots c_{2M-1}, \dots, c_{(2^{s-1}),0}, c_{(2^{s-1}),1}, \dots, c_{(2^{s-1}),M-1})^T .. (12) $$

$$ \Phi(t) = (\phi_{1,0}, \phi_{1,1}, \dots, \phi_{1,M-1}, \phi_{2,0}, \phi_{2,1}, \dots \phi_{2M-1}, \dots, \phi_{(2^{s-1}),0}, \phi_{(2^{s-1}),1}, \dots, \phi_{(2^{s-1}),M-1})^T . (13) $$

Taking the collocation points as follows:

$$ t_i = \frac{(2i-1)}{2^{s-1}M},\ i = 1,2,\dots, 2^{s-1}M. $$

Let we define the Chebyshev wavelet matrix $\Psi_{m\times m}$ as:

$$ \Psi_{m\times m} = \left[\Phi\left(\frac{1}{2m}\right), \Phi\left(\frac{3}{2m}\right), \dots, \Phi\left(\frac{2m-1}{2m}\right)\right] \quad (14) $$

An arbitrary function of two variables $u(x,t) \in L^2(\mathbb{R}\times\mathbb{R})$ defined over $[0,1]\times[0,1]$, can expanded into Chebyshev wavelets basis as,

$$u(x,t) = \sum_{l=1}^{2^{s_1-1}} \sum_{r=0}^{M-1} \sum_{k=1}^{2^{s_2-1}} \sum_{d=0}^{N-1} c_{lrkd}\, \phi_{lr}(x)\phi_{kd}(t), \tag{15}$$

Let

$$u(x,t) = \sum_{i=1}^{\widehat{m}} \sum_{j=1}^{\hat{n}} u_{ij}\phi_i(x)\phi_j(t) = \Phi^T(x)U\Phi(t) \tag{16}$$

where $U = (u_{ij})$ and $u_{ij} = \langle \phi_i(x), \langle u(x,t), \phi_j(t)\rangle\rangle_{w_n}$, U is unknown $(\widehat{m}\times\hat{n})$ matrix where $\widehat{m} = 2^{s_1-1}(M)$ and $\hat{n} = 2^{s_2-1}(N)$, the elements of the matrix U can be calculated from

$$u_{ij} = \int_0^1\int_0^1 \phi_i(x)\phi_j(t)u(x,t)dtdx, \quad i = 1,2,\dots,\widehat{m}, \;\; j = 1,2,\dots,\hat{n}.$$

3.1 Operational matrix of the fractional integration

The integration of the vector $\Phi(t)$ defined in Eq.(13) can be approximated by Chebyshev series with Chebyshev coefficient matrix P_{Cw} as:

$$\int_0^x \Phi(\tau)d\tau = P_{Cw}\,\Phi(x), \tag{17}$$

where a $\widehat{m}\times\hat{n}$ squre matrix P_{Cw} is called the Chebyshev wavelets operational matrix of integration.
Because the Chebyshev wavelets are piecewise constant, it may be expanded into m – term Block Pulse Function (BPF) as:

$$\Phi_m(x) = \Psi_{m\times m}B_m(x) \tag{18}$$

where

$$B_m(x) \triangleq [b_0(x), b_1(x) \dots b_i(x) \dots b_{m-1}(x)]^T$$

With
$$b_i(x) = \begin{cases} 1 & \frac{i}{m} \le x < \frac{i+1}{m} \\ 0, & otherwise \end{cases}, \;\; where\ i = 0,1,2,\dots m-1. \tag{19}$$

The function $b_i(x)$ are disjoint and orthogonal, that is:

$$b_i(x)b_l(x) = \begin{cases} 0 & i \ne l \\ b_i(x), & i = l \end{cases}$$

Next, we shall derive the Chebyshev wavelets operational matrix of the fractional order integration by letting:

$$(I^\alpha\Phi_m)(x) = P_{Cw}{}^{\alpha}_{m\times m}\,\Phi_m(x) \tag{20}$$

where the matrix $P_{Cw}{}^{\alpha}_{m\times m}$ is called the Chebyshev wavelets operational matrix of the fractional integration
Kilicman and Al-Zhour in [30] have given the Block Pulse operational matrix of the fractional integration is

$$I^\alpha B_m(x) = F^\alpha B_m(x), \tag{21}$$

where

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & & 1 \end{pmatrix}, \tag{22}$$

With $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Now using (18) we have

$$(I^{\alpha}\Phi_m)(x) = (I^{\alpha}\Psi_{m\times m}B_m)(x) = \Psi_{m\times m}(I^{\alpha}B_m)(x)$$

$$\approx \Psi_{m\times m}F^{\alpha}B_m(x)$$

Then

$$P_{Cw}{}^{\alpha}_{m\times m}\,\Phi_m(x) = P_{Cw}{}^{\alpha}_{m\times m}\Psi_{m\times m}B_m$$

$$= \Psi_{m\times m}F^{\alpha}B_m(x)$$

So the Chebyshev wavelet operational matrix of the fractional integration $P_{Cw}{}^{\alpha}_{m\times m}$ is given by

$$P_{Cw}{}^{\alpha}_{m\times m} = \Psi_{m\times m}F^{\alpha}\Psi^{-1}_{m\times m} \tag{23}$$

4.Function approximation

Consider advection dispersion equation of time fractional order derivative of the form:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \tag{24}$$
$$0 < x < 1,\ 0 < t < 1,\ 0 < \alpha \le 1$$

subject to the initial condition as following:

$$\mathrm{u}(\mathrm{x},0) = \vartheta(\mathrm{x}),\ \ 0 \le \mathrm{x} \le 1 \tag{25}$$

and boundary conditions

$$\begin{cases}\mathrm{u}(0,\mathrm{t}) = \ell_0(\mathrm{t}) \\ \mathrm{u}(1,\mathrm{t}) = \ell_1(\mathrm{t}), \quad 0 \le t \le 1\end{cases} \tag{26}$$

By applying the Riemann–Liouville fractional integration of order α with respect to t on both sides of Eq. (24) and using the initial condition in Eq. (25), we obtain:

$$\mathrm{u}(\mathrm{x},\mathrm{t}) - \vartheta(\mathrm{x}) = \left(\mathrm{I}^{\alpha}\frac{\partial^2\mathrm{u}}{\partial \mathrm{x}^2}\right)(\mathrm{x},\mathrm{t}) - \left(\mathrm{I}^{\alpha}\frac{\partial \mathrm{u}}{\partial \mathrm{x}}\right)(\mathrm{x},\mathrm{t}) + (\mathrm{I}^{\alpha}\mathrm{f})(\mathrm{x},\mathrm{t}) \tag{27}$$

Now, we approximate $\frac{\partial^2\mathrm{u}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}^2}$ by the Chebyshev wavlet as:

$$\frac{\partial^2\mathrm{u}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}^2} = \Phi^{\mathrm{T}}(\mathrm{x})\mathrm{U}\Phi(\mathrm{t}) \tag{28}$$

Where $\mathrm{U} = [\mathrm{u}_{\mathrm{ij}}]_{\hat{\mathrm{m}}\times\hat{\mathrm{n}}}$ is an unknown matrix which should be computed and $\Phi(\mathrm{x})$ is defined in Eq.(13).
Now integrating two times Eq. (28) with respect to x, we get

$$\frac{\partial \mathrm{u}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}} \simeq \left.\frac{\partial \mathrm{u}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}}\right|_{\mathrm{x}=0} + \Phi^{\mathrm{T}}(\mathrm{x})P_{Cw}{}^{\mathrm{T}}\mathrm{U}\Phi(\mathrm{t}) \tag{29}$$

$$\mathrm{u}(\mathrm{x},\mathrm{t}) \simeq \mathrm{u}(0,\mathrm{t}) + \mathrm{x}\left(\left.\frac{\partial \mathrm{u}(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}}\right|_{\mathrm{x}=0}\right) + \Phi^{\mathrm{T}}(\mathrm{x})\left(P_{Cw}{}^{\mathrm{T}}\right)^2\mathrm{U}\Phi(\mathrm{t}) \tag{30}$$

and by putting $x = 1$ in Eq. (30), and considering Eq.(26), we obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} \simeq \ell_1(t) - \ell_0(t) - \Phi^T(1)\left(P_{Cw}{}^{T}\right)^2 U\Phi(t) \tag{31}$$

Let we expand $\ell_0(t)$ and $\ell_1(t)$ by the Chebyshev wavlets as follows:

$$\ell_0(t) \simeq \mathcal{L}_0^T\Phi(t) \quad \text{and} \quad \ell_1(t) \simeq \mathcal{L}_1^T\Phi(t) \tag{32}$$

Where G_0 and $G_1$are the Chebyshev vectors.
By substituting Eq.(32) into Eq.(31), we obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} \simeq \left(\mathcal{L}_1^T - \mathcal{L}_0^T - \Phi^T(1)\left(P_{Cw}{}^T\right)^2 U\right)\Phi(t) \triangleq \widetilde{U}^T\Phi(t) \tag{33}$$

Now, by substituting Eq.(33) into Eq.(29) and Eq.(30), we have

$$\frac{\partial u(x,t)}{\partial x} \simeq E\widetilde{U}^T + P_{Cw}{}^T U = \Phi^T(x)\forall\Phi(t) \tag{34}$$

$$u(x,t) \simeq \Phi^T(x)\left[E\mathcal{L}_0^T + X\widetilde{U}^T + \left(P_{Cw}{}^T\right)^2 U\right] = \Phi^T(x)\Omega\Phi(t) \tag{35}$$

Where X and E are the Chebyshev wavelets coefficient vectors for x and the unit function (or Heaviside function) respectively.
Furthermore, we expand $s(x)$ and $f(x,t)$ by the Chebyshev wavelet as:

$$\vartheta(x) \simeq \Phi^T(x)\Theta, \qquad f(x,t) = \Phi^T(x)F\Phi(t) \tag{36}$$

Then by substituting Eqs.(28), (34)-(36) into Eq.(27), and using operational matrices of fractional integration of Chebyshev wavelets, we can write the residual function $R(x,t)$ for equation (24) as follows:

$$R(x,t) = \Phi^T(x)[\Omega - UP_{Cw}{}_{m\times m}^{\alpha} + \forall P_{Cw}{}_{m\times m}^{\alpha} - \Theta\mathrm{E}^{\mathrm{T}} - FP_{Cw}{}_{m\times m}^{\alpha}]\Phi(t).$$

The expansion coefficients $u_{ij}, i = 1,2,\dots,\widehat{m}, j = 1,2,\dots,\hat{n}$ are determined by Galerkin equations:

$$\langle R(x,t), \phi_i(x)\phi_j(t)\rangle_{\mathrm{w_n}} = 0 \tag{37}$$

Where $\langle . \rangle$ denotes inner product defined as

$$\langle R(x,t), \phi_i(x)\phi_j(t)\rangle_{\mathrm{w_n}} = \int_0^1\int_0^1 R(x,t)\phi_i(x)\phi_j(t)\mathrm{dxdt}, \quad i = 1,2,\dots,\widehat{m}, j = 1,2,\dots,\hat{n} \tag{38}$$

Galerkin equations (37) give a system of equations that can be solved for the elements of u_{ij}, $i = 1,2,\dots,\widehat{m}, j = 1,2,\dots,\hat{n}$

4. Numerical Examples:

In this section, we will examine the accuracy and efficiency of the proposed method by the two following examples.

Example 1[31]: Consider the following advection dispersion equation of time fractional order:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} - k\frac{\partial u(x,t)}{\partial x} + f(x,t), \tag{39}$$

$$0 < x < 1,\ 0 < t < 1,\ 0 < \alpha \le 1$$

subject to the initial condition as following:

$$\mathrm{u(x,0)} = 0,\ 0 \le x \le 1 \tag{40}$$

and boundary conditions

$$\begin{cases}\mathrm{u(0,t)} = 0\\ \mathrm{u(1,t)} = 0, \quad 0 \le t \le 1\end{cases} \tag{41}$$

Where $\mathrm{f(x,t)} = \frac{6\mathrm{t}^{3-\alpha}}{\Gamma(4-\alpha)}(\mathrm{x}^3 - \mathrm{x}^4) + \mathrm{t}^3(-6\mathrm{x} + 15\mathrm{x}^2 - 4\mathrm{x}^3)$,

The exact solution of this problem is $u(x,t) = t^3x^3(1-x)$.

We Apply the present method for solving Eq. (39), the diagram of the comparison between the exact and approximate solution for $\widehat{\mathrm{m}} = \hat{\mathrm{n}} = 12$ ($s_1 = s_2 = 1, M = N = 12$), and $\alpha = 0.5$ is presented in Figure-1 and the error for different values of alpha is presented in Figure-2 also the absolute errors when solving this problem are listed in Table- 1 for different values of x and t with various value of α for $\widehat{\mathrm{m}} = \hat{\mathrm{n}} = 24$ ($s_1 = s_2 = 2, M = N = 12$).

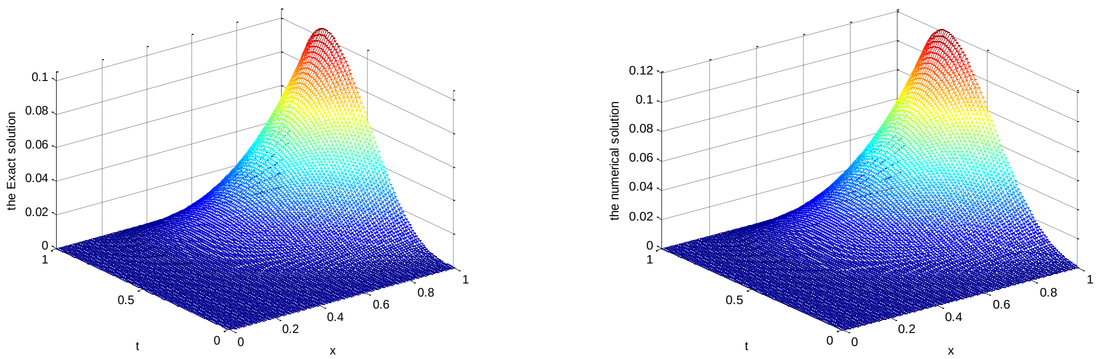

Figure 1- The comparison between the exact and approximate solution for $\widehat{m} = \hat{n} = 24$

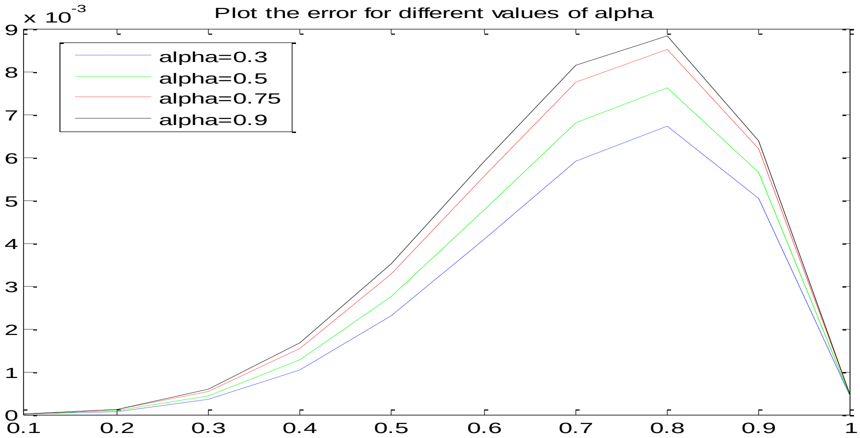

Figure 2- Plot the error for different values of alpha

Table 1-The absolute errors for some different values of α and x, t

(x,t)	$\boldsymbol{\alpha = 0.3}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.75}$	$\boldsymbol{\alpha = 0.9}$
(0.1,0.1)	**5.2065e-006**	**7.0789e-006**	**8.1245e-006**	**7.3206e-006**
(0.2,0.2)	**7.4259e-005**	**9.6464e-005**	**1.1836e-004**	**1.2244e-004**
(0.3,0.3)	**3.5494e-004**	**4.4501e-004**	**5.4504e-004**	**5.8270e-004**
(0.4,0.4)	**1.0498e-003**	**1.2808e-003**	**1.5450e-003**	**1.6619e-003**
(0.5,0.5)	**2.3137e-003**	**2.7605e-003**	**3.2678e-003**	**3.5004e-003**
(0.6,0.6)	**4.0896e-003**	**4.7865e-003**	**5.5550e-003**	**5.9000e-003**
(0.7,0.7)	**5.9111e-003**	**6.8004e-003**	**7.7381e-003**	**8.1332e-003**
(0.8,0.8)	**6.7225e-003**	**7.6181e-003**	**8.5068e-003**	**8.8416e-003**
(0.9,0.9)	**5.0516e-003**	**5.6551e-003**	**6.2069e-003**	**6.3780e-003**
(1.0,1.0)	**4.4541e-004**	**4.4541e-004**	**4.4784e-004**	**4.4834e-004**

Example 2[32]: We consider the following non-homogeneous advection dispersion equation of time fractional order

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial x} + f(x,t), \tag{42}$$
$$0 < x < 1,\ 0 < t \le 1,\ 0 < \alpha \le 1$$

Where $f(x,t) = \frac{10x^2(1-x)t^{1-\alpha}}{\Gamma(2-\alpha)} - 10(t+1)\left[\frac{2}{\Gamma(1)} - \frac{6x}{\Gamma(2)}\right] + 10(t+1)\left[\frac{2x}{\Gamma(2)} - \frac{6x^2}{\Gamma(3)}\right]$,

With initial condition

$$u(x,0) = 10x^2(1-x), \qquad 0 \le x \le 1 \tag{43}$$

and boundary conditions

$$\begin{cases} u(0,t) = 0 \\ u(1,t) = 0, \quad 0 < t \le 1 \end{cases} \tag{44}$$

The exact solution is $u(x,t) = 10(t+1)x^2(1-x)$.

We Apply the present method for solving Eq. (42), The diagram of the comparision between the exact and approximate solution for $\hat{m} = \hat{n} = 12$ $(s_1 = s_2 = 1, M = N = 12)$, and $\alpha = 0.7$ is presented in Figure-3. Also in Table- 2 we compares the Numerical result for the present methods that we have been obtained when $\alpha = 0.5$ with result given in [32] for different values of x and $t = 1$ for $\hat{m} = \hat{n} = 24$ $(s_1 = s_2 = 2, M = N = 12)$.

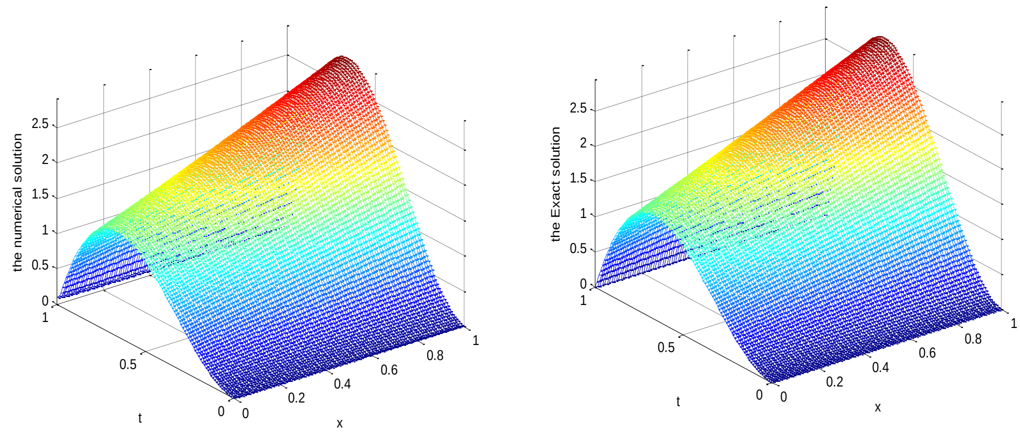

Figure 3- The comparison between the exact and approximate solution for $\hat{m} = \hat{n} = 24$ and $\alpha = 0.7$

Table 2- The absolute errors for $\alpha = 0.5$ and different values of x $with$ $t = 1$

(x,t)	Exact solution	Present method Numerical solution	Method [32] Numerical solution
(0.1,1.0)	0.1800000	0.17775401	0.17760929
(0.2,1.0)	0.6400000	0.63545182	0.63439865
(0.3,1.0)	1.2600000	1.25138745	1.25076281
(0.4,1.0)	1.9200000	1.90954789	1.90718308
(0.5,1.0)	2.5000000	2.48700110	2.48418437
(0.6,1.0)	2.8800000	2.87001534	2.86234006
(0.7,1.0)	2.9400000	2.92456444	2.92227732
(0.8,1.0)	2.5600000	2.55014586	2.54468333
(0.9,1.0)	1.6200000	1.61125478	1.61031324

6. Conclusions

In this work, the Chebyshev wavelets –Galerkin methods was successfully extended to solve the advection dispersion equation of time fractional order. The obtained results revealed that the proposed method is accurate and efficient in comparison with the result of [31] and [32]. The solution obtained by this method is in excellent agreement with the exact one.

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