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The group action on a projective plane over finite field of order sixteen

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Abstract

The goal of this paper is to construct an arcs of size five and six with stabilizer groups of type alternating group of degree five $\boldsymbol{A_5}$ and degree six $\boldsymbol{A_6}$. Also construct an arc of degree five and size 15 with its stabilizer group, and then study the effect of $\boldsymbol{A_5}$ and $\boldsymbol{A_6}$ on the points of projective plane. Also, find a pentastigm which has the points on a line. Partitions on projective plane of order sixteen into subplanes and arcs have been described.

Keywords: Projective Plane, Arcs.

تأثير الزمر على المستوى الاسقاطي للحقل المنتهي من الرتبة السادسة عشر

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الخلاصة

الهدف من هذا البحث هو تشكيل اقواس من الحجم خمسة وستة مع الزمر المثبته من النوع الزمر المتناوبه من الدرجه خمسه A_5 والدرجة سته A_6 ودراسة تأثير A_5 ، A_6 على نقاط المستوي الاسقاطي كذلك تشكيل قوس من الدرجه الخامسة وحجم 15 مع زمرته المثبته. كذلك أيجاد pentastigm ذات نقاط على الخط. تم وصف تجزئات على المستوي الاسقاطي من الرتبه السادسة عشر الى مستويات جزئية واقواس.

1. Introduction

The projective plane $PG(2,q)$ is a 2-dimensional projective space over F_q. In a plane, each point **P** is joined to the remaining points by a pencil which consists of $q+1$ lines each of these lines contains **P** and q other points [1]. Hence the plane contains $q(q+1)+1 = q^2+q+1 = \theta(2,q)$ points [1] and by duality a plane contains q^2+q+1 lines. The integer q is called the order of the plane. Throughout, $\Upsilon = \{U_0 = \mathbf{P}(1,0,0), U_1 = \mathbf{P}(0,1,0), U_2 = \mathbf{P}(0,0,1), U = \mathbf{P}(1,1,1)\}$ denotes the standard frame in $PG(2,q)$ [1] . The projective plane of order sixteen, $PG(2,16)$ contains

- 273 points and lines ;
- 17 points on each line ;
- 17 lines passing through each point.

Associated to any topic in mathematics is its history. Arcs were first introduced by[2]. Further development began with [3] showed that every $(q+1)$-arc in $PG(2,q)$ is a conic. In [3] found important applications of curves over finite fields to coding theory. As to geometry over a finite fields, it has been thoroughly studied in the major treaties [1], for more informations see [4-9].

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2. Definitions and Basic Properties

Definition 2.1[1]: An $(n; r)$ arc K or arc of degree r in $PG(k, q)$ with $n \geq r+1$ is a set of n points with property that every hyperplane meets K in at most r points of K and there is some hyperplane meeting K in exactly r points. An $(n; 2)$-arc is also called an n-*arc*.

Definition 2.2[1]: An $(n; r)$-arc is complete if it is maximal with respect to inclusion; that is, it is not contained in an $(n+1; r)$ -arc.

Definition 2.3[1]: A line of $PG(k, q), k > 1$, is an i-*secant* of an $(n; r)$- arc K if $\left| \ell \cap K \right| = i$. A 2-secant is called a bisecant, a 1-secant is a unisecant (tangent) and a 0-secant is an external line.

Let c_i be the number of points of $PG(k, q) \setminus K$ with index exactly i . So, the parameters c_0 is the number of points through which no bisecant of K passes and c_3 is the number of points where three bisecants meet.

Theorem 2.4[1]: (The Fundamental Theorem of Projective Geometry)

If { $\mathbf{P_0}, \dots, \mathbf{P_{n+1}}$} and {$\mathbf{\acute{P}_0}, \dots, \mathbf{\acute{P}_{n+1}}$} are both subsets of $PG(n, q)$ of cardinality $n+2$ such that no $n+1$ points chosen from the same set lie in a hyperplane, then there exists a unique projectivity $\mathfrak{I}$ such that $\mathbf{\acute{P}_i} = \mathbf{P_i}\mathfrak{I}$ for $i = 0, 1, \dots, n+1$.

Definition 2.5[1]: Let Ω_1 and Ω_2 be two projective spaces of dimension n . A projectivity $\mathfrak{I}: \Omega_1 \to \Omega_2$ is a bijection given by a non-singular $(n+1) \times (n+1)$ matrix A such that $\mathbf{P}(X') = \mathbf{P}(X)$ if and only if t $tX' = XA$, where $t \in F_q \backslash \{0\}$. Two projective spaces X_1 and X_2 are projectively equivalent if there is a projectivity between them.

Remark 2.6[1]:(1) An $(n+1; r)$arc is constructing from an $(n; r)$-arc, K by adding one point of index zero to K.

(2) Two arcs $K$and K' are projectively equivalent if there is a projectivity between them.

(3) With parameters c_0, K is complete if and only if $c_0 = 0$.

Some groups that occur in this paper are listed below. For more details see [5].

$\boldsymbol{Z_n}$ = cyclic group of order n;

$\boldsymbol{Z_2} \times \boldsymbol{Z_2}$= the direct product of two copies of the cyclic group of order 2;

$\boldsymbol{A_n}$ = alternating group of degree n;

$\boldsymbol{D_n}$ = dihedral group of order $2n = \langle r, s \mid r^n = s^2 = (rs^2) = 1 \rangle$;

3. Construction of F_{16}

To construct of the field of order sixteen F_{16} in exponential and polynomial form, take $G(X) = X^4 + X + 1$, which is irreducible over F_2, since the elements of F_2 are may be taken as 0,1 with $1 + 1 = 0$. We show first that G has no linear factor, for

$G(0) = 1$, $G(1) = 1$.

There remains the possibility that (in $F[X]$ the ring of polynomials over F_2)

$X^4 + X + 1 = (X^2 + aX + b)(X^2 + cX + d)$.

Equating coefficients gives

$c + a = 0$, $\quad d + ac + b = 0$,

$ad + bc = 1$, $\quad bd = 1$.

Hence, $b = d = 1$, we deduce that $ac = 0$, and so either $a = 0$ or $c = 0$, and we have a contradiction.

We have shown that G is irreducible over F_2. Let ω be the element $X + \langle X^4 + X + 1 \rangle$ in the field $F_{16} = F_2[X] / \langle X^4 + X + 1 \rangle$, where $\langle X^4 + X + 1 \rangle$ the principle ideal generated by $X^4 + X + 1$, with $G(\omega) = \omega^4 + \omega + 1 = 0$, let $O^{\times}$ be the order of element with respect to multiplication and O^{+} be the order of element with respect to addition, the elements of F_{16} are given in Table-1, as follows:

Table 1-The elements of F_{16}.

Exponentials	Polynomials	Vectors	$O^{\times}$	O^{+}
0	0	(0,0,0,0)	-	1
1	1	(1,0,0,0)	1	2
ω	ω	(0,1,0,0)	15	2
ω^2	ω^2	(0,0,1,0)	15	2
ω^3	ω^3	(0,0,0,1)	5	2
ω^4	$\omega+1$	(1,1,0,0)	15	2
ω^5	$\omega^2+\omega$	(0,1,1,0)	3	2
ω^6	$\omega^3+\omega^2$	(0,0,1,1)	5	2
ω^7	$\omega^3+\omega+1$	(1,1,0,1)	15	2
ω^8	ω^2+1	(1,0,1,0)	15	2
ω^9	$\omega^3+\omega$	(0,1,0,1)	5	2
ω^{10}	$\omega^2+\omega+1$	(1,1,1,0)	3	2
ω^{11}	$\omega^3+\omega^2+\omega$	(0,1,1,1)	15	2
ω^{12}	$\omega^3+\omega^2+\omega+1$	(1,1,1,1)	5	2
ω^{13}	$\omega^3+\omega^2+1$	(1,0,1,1)	15	2
ω^{14}	ω^3+1	(1,0,0,1)	15	2

From Table- 1, we deduce that

1. $(F_{16}\setminus\{0\},\times)\simeq(\boldsymbol{Z_{15}},\times)$
2. $(F_{16},+)\simeq(\boldsymbol{Z_2}\times\boldsymbol{Z_2}\times\boldsymbol{Z_2}\times\boldsymbol{Z_2},+)$

4. Construction of $\boldsymbol{PG(2,q)}$

The polynomial of degree three $F(X)=X^3+X+\omega^7$ is irreducible and primitive over F_{16}, since, $F(t)\neq 0$ for all t in F_{16}. Let ϖ is a primitive root over F_{4096}, where $F_{4096}=\{0,1,\varpi,\varpi^2,\dots,\varpi^{4094};\varpi^{4095}=1\}$ take $F(\varpi)=\varpi^3+\varpi+\omega^7=0$, this implies $\varpi=\varpi^3+\omega^7$, and $\varpi^{4095}=(\varpi^3-\omega^7)^{4095}$ by mathematical programming language GAP, we have $\varpi^{4095}=1$, hence F is a primitive over F_{16}.

The points of $PG(2,16)$ are generated by a nonsingular matrix

$$T=C(F)=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \omega^7 & 1 & 0\end{pmatrix}$$

Such that, $\mathbf{P}_i=\mathbf{P}(1,0,0)\,T^i, i=0,1,\dots,272$. They are given as follows:

$\mathbf{P_0}=\mathbf{P}(1,0,0)$, $\mathbf{P_1}=\mathbf{P}(0,1,0)$, ..., $\mathbf{P_{253}}=\mathbf{P}(1,1,1)$,
$\mathbf{P_{254}}=\mathbf{P}(\omega^7,0,1)$, $\mathbf{P_{255}}=\mathbf{P}(\omega^{13},1,0)$, ..., $\mathbf{P_{272}}=\mathbf{P}(1,0,1)$.

Let $\ell_1=v(Z)$; that is, ℓ_1 is the line passing through points $\mathbf{P}(x,y,z)$ with third coordinate equal to zero. Then ℓ_1 forms the following difference set, with $\mathbf{P}_i=i, i=0,\dots,272$.

0 1 3 7 15 31 63 90 116 127 136 181 194 204 233 238 255

The points $\mathbf{P}_i=i$ and the lines ℓ_i of $PG(2,16)$ can be represented by the following array.

$\ell_1=\{0,1,3,7,15,31,63,90,116,127,136,181,194,204,233,238,255\}$;

$\ell_2=\{1,2,4,8,16,32,64,91,117,128,137,182,195,205,234,239,256\}$;

⋮

$\ell_{273}=\{272,0,2,6,14,30,62,89,115,126,135,180,193,203,232,237,254\}$.

A vector representation of the points in $PG(2,16)$ by three coordinates over F_{16} is given in Table- 2, as follows:

Table 2- Type of elements of $PG(2,16)$.

Type of elements	No. of elements
$\mathbf{P}(x, y, 1)$ $\mathbf{P}(x, 1,0)$ $\mathbf{P}(1,0,0)$	256 16 1
	$\theta(2,16)$

The action of $\langle T^{39}\rangle$ on $PG(2,16)$ is given as following:

$$0 \xrightarrow{T^{39}} 39 \xrightarrow{T^{39}} 78 \xrightarrow{T^{39}} 117 \xrightarrow{T^{39}} 156 \xrightarrow{T^{39}} 195 \xrightarrow{T^{39}} 234 \xrightarrow{T^{39}} 0.$$

By adding $1,\dots,38$ to previous orbit modulo 273, we have thirty nine disjoint orbits and geometrically, every one of them constructs a projective subplane of order two $PG(2,2)$.

The action of $\langle T^{21}\rangle$ on $PG(2,16)$ is given as following:

$$0 \xrightarrow{T^{21}} 21 \xrightarrow{T^{21}} 42 \xrightarrow{T^{21}} 63 \xrightarrow{T^{21}} 84 \xrightarrow{T^{21}} 105 \xrightarrow{T^{21}} \dots \xrightarrow{T^{21}} 189 \xrightarrow{T^{21}} 210 \xrightarrow{T^{21}} 231 \xrightarrow{T^{21}} 252 \xrightarrow{T^{21}} 0.$$

By adding $1,\dots,20$ to previous orbit modulo 273, we have twenty one disjoint orbits and geometrically, every one of them constructs a complete arc of degree two.

The action of $\langle T^{13}\rangle$ on $PG(2,16)$ is given as following :

$$0 \xrightarrow{T^{13}} 13 \xrightarrow{T^{13}} 26 \xrightarrow{T^{13}} 39 \xrightarrow{T^{13}} 52 \xrightarrow{T^{13}} 65 \xrightarrow{T^{13}} \dots \xrightarrow{T^{13}} 221 \xrightarrow{T^{13}} 234 \xrightarrow{T^{13}} 247 \xrightarrow{T^{13}} 260 \xrightarrow{T^{13}} 0.$$

By adding $1,\dots,12$ to previous orbit modulo 273, we have thirteen disjoint orbits and geometrically, every one of them constructs a projective subplane of order four $PG(2,4)$.

5. The Unique 4-Arc in $PG(2,16)$

The Fundamental Theorem of Projective Geometry is applied to the projective plane, the frame Υ is projectively the unique 4-arc in $PG(2,16)$.The frame points in $PG(2,16)$ are the points 0,1,2,253 in numeral form. The stabilizer group of Υ is $\boldsymbol{S_4}$, which can be found by transforming Υ to its 24 permutations. The two matrices marked by g_1, g_2 are generators of $\boldsymbol{S_4}$.

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ g_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Remark 5.1.

1. The values of the constants c_i for any 4-arc are
$c_0 = 182,\ \ c_1 = 84,\ \ c_2 = 3$
2. The three diagonal points of the frame are
$U_0 U_1 \cap U_2 U = \mathbf{P}(1,1,0)$;
$U_0 U \cap U_1 U_2 = \mathbf{P}(0,1,1)$;
$U_0 U_2 \cap U_1 U = \mathbf{P}(1,0,1)$.
These points are collinear.
3. The diagonal points are exactly the three points of index two. The set of diagonal points is fixed by $\boldsymbol{S_4}$, the stabilizer group of the frame.

6. An 5-Arcs in $PG(2,16)$

The number of points on the sides of a tetrastigm is $l(4,16) = 91$. Hence the number of points not on the sides of tetrastigm is $l^*(4,16) = 273 - 91 = 182$. The projective group $\boldsymbol{S_4}$ of the standard frame Υ splits the 182 points not on the bisecants of Υ into 10 disjoint orbits as follows:

1) {9,10,249,141,262,19,260,191,150,65,170,100,26,43,158,187,72,99, 87,120,41,93 ,165,212}.
2) {11,109,220,229,70,131,259,33,22,71,36,92,243,189,106,25,79,28, 38,73,113, 59 ,46,112} .
3) {12,145,185,207,103,82,176,173,217,266,124,188,183,49,257,228, 235,247,53, 47,161 ,154,213,122}.
4) {17,206,211,241,69,85,50,160,268,146,78,184,60,265,105,230,81, 108,75,248, 221 ,244,208,48}.
5) {18,159,223,250,95,216,77,130,164,240,56,197,162,49,142,218, 246,210,219, 245 ,98,133,149,198}.
6) {24,40,104,107,172,74,177,34,123,39,132,37,178,215,175,190,129, 111,267, 222 ,84,153,242,80}.
7) {42,52,251,155,269,66,157,196,163,169,171,186,148,55,209,121, 174,261,225, 76 270,54,86,166}.
8) {44,57,226,143,67,263}.
9) {96,167,139,252,118,199}.
10) {101,151}.

Hence, ten 5-arcs are constructed by adding one point from each orbit to Υ. They are listed with their stabilizer groups in Table- 3 as following:

Table 3- 5-arcs in PG (2,16).

No.	The 5-arc	Stabilizer
1	{0,1,2,253,9}	$\boldsymbol{I}$
2	{0,1,2,253,11}	$\boldsymbol{I}$
3	{0,1,2,253,12}	$\boldsymbol{Z_2 \times Z_2}$
4	{0,1,2,253,17}	$\boldsymbol{I}$
5	{0,1,2,253,18}	$\boldsymbol{I}$
6	{0,1,2,253,24}	$\boldsymbol{Z_2 \times Z_2}$
7	{0,1,2,253,42}	$\boldsymbol{I}$
8	{0,1,2,253,44}	$\boldsymbol{Z_2 \times Z_2}$
9	{0,1,2,253,96}	$\boldsymbol{Z_2 \times Z_2}$
10	{0,1,2,253,101}	$\boldsymbol{A_5}$

In Table- 4 all equivalent 5-arcs with their matrix transformation are listed

Table 4- Equivalent 5-arcs in $PG\ (2,16)$.

No.	Equivalent 5-arcs	Matrix transformation
1	$1 \to 2$	$\begin{pmatrix} 0 & \omega^7 & 0 \\ \omega^4 & \omega^9 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
2	$1 \to 4$	$\begin{pmatrix} \omega^2 & \omega^{10} & 1 \\ \omega^2 & \omega^2 & \omega^2 \\ 0 & 0 & \omega^8 \end{pmatrix}$
3	$1 \to 5$	$\begin{pmatrix} 0 & \omega^{13} & 0 \\ 0 & 0 & 1 \\ \omega^{12} & \omega^{13} & 1 \end{pmatrix}$
4	$1 \to 7$	$\begin{pmatrix} 0 & \omega^{11} & 0 \\ 0 & 0 & \omega^3 \\ \omega^{14} & \omega^{14} & \omega^{14} \end{pmatrix}$
5	$3 \to 9$	$\begin{pmatrix} 0 & 0 & \omega^2 \\ \omega^8 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
6	$6 \to 8$	$\begin{pmatrix} \omega^{14} & 0 & 0 \\ \omega^{14} & \omega^3 & 1 \\ 0 & \omega^3 & 0 \end{pmatrix}$

This gives the following conclusion

Theorem 6.1. In $PG\ (2,16)$, there are precisely four projectively distinct 5-arcs, as summarized in Table- 5, as follows:

Table 5- Inequivalent 5-arcs in $PG(2,16)$.

Symbol	The 5-arc	Stabilizer
A_1	{0,1,2,253,9}	$\boldsymbol{I}$
A_2	{0,1,2,253,12}	$\boldsymbol{Z_2 \times Z_2}$
A_3	{0,1,2,253,24}	$\boldsymbol{Z_2 \times Z_2}$
A_4	{0,1,2,253,101}	$\boldsymbol{A_5}$

Remark 6.2.

1. The values of the constants c_i for any 5-arc are
   $c_0 = 133$, $c_1 = 120$, $c_2 = 15$
2. The 5-arcs A_2 and A_3 have the same constants c_i and isomorphic stabilizer groups but they are inequivalent.

3. Because of the one-to-one correspondence between the projective line $PG\ (1,16)$ and a conic, let

$$C^* = v(Y^2 - XZ) = \{\mathbf{P}(t^2, t, 1); t \in F_{16} \cup \{\infty = \mathbf{P}(1,0,0)\}\}$$

be a conic. Then the four pentads δ_i as given in [9] correspond to inequivalent four 5-arcs δ_i^*on the conic C^*. Each 5-arc $\delta_i^*, i = 1, \ldots, 4$ is equivalent to one of A_j, $j = 1, \ldots, 4$. These equivalences and the matrix transformations are given in Table- 6, as follows:

Table 6- Transforming δ_i^* to A_j.

$\delta_i^* \cong A_j$	Matrix transformation
$\delta_1^* = \{0,2,253,190,207\} \cong A_3$	$\begin{pmatrix} \omega^5 & 0 & 0 \\ \omega^2 & \omega & 1 \\ \omega & \omega & \omega \end{pmatrix}$
$\delta_2^* = \{0,2,253,190,215\} \cong A_1$	$\begin{pmatrix} \omega & 1 & \omega^{14} \\ \omega^{10} & \omega^7 & \omega^4 \\ \omega^{12} & 0 & 0 \end{pmatrix}$
$\delta_3^* = \{0,2,253,190,176\} \cong A_2$	$\begin{pmatrix} \omega & 0 & 0 \\ \omega^4 & \omega^3 & \omega^2 \\ \omega^{10} & \omega^{10} & \omega^{10} \end{pmatrix}$
$\delta_4^* = \{0,2,253,101,151\} \cong A_4$	$\begin{pmatrix} 1 & 0 & 0 \\ \omega^{10} & \omega^5 & 1 \\ 0 & 0 & \omega^{10} \end{pmatrix}$

7. Collinearities of the Diagonal Points of Pentastigm in $PG(2,16)$

Let $\mathbf{P_0} = U_0$, $\mathbf{P_1} = U_1$, $\mathbf{P_2} = U_2$, $\mathbf{P_3} = U$, $\mathbf{P_4} = \mathbf{P}(a_0, a_1, a_2)$ be the vertices of a pentastigm δ. Since the vertices of δ form a 5-arc then $\mathbf{P_4}$ cannot be collinear with any pair of other vertices, so $a_0 a_1 a_2 (a_0 - a_1)(a_0 - a_2)(a_1 - a_2) \neq 0$.

Lemma7.1. [1] The condition that five diagonal points of a pentastigm δ are collinear in $PG(2,q)$ is that $x^2 = x + 1$ has a solution in F_q.

Since in F_{16} the equation $x^2 = x + 1$ has two solutions ω^5, ω^{10} so there is a pentastigm with five collinear diagonal points in $PG(2,16)$.

The pentastigm δ which has the 5-arc $A_4 = \{U_0, U_1, U_2, U, \mathbf{P}(\omega^{10}, \omega^5, 1)\}$ as vertices has five diagonal points which are collinear as shown below.

The fifteen diagonal points of A_4 in coordinate form are

01.23 = $\mathbf{P}(1,1,0)$,
01.24 = $\mathbf{P}(\omega^5,1,0)$,
01.34 = $\mathbf{P}(\omega^{10},1,0)$,
02.13 = $\mathbf{P}(1,0,1)$,
02.14 = $\mathbf{P}(\omega^{10},0,1)$,
02.34 = $\mathbf{P}(\omega^5,0,1)$,
03.12 = $\mathbf{P}(0,1,1)$,
03.14 = $\mathbf{P}(\omega^{10},1,1)$,
03.24 = $\mathbf{P}(\omega^5,1,1)$,
04.12 = $\mathbf{P}(0,\omega^5,1)$,
04.13 = $\mathbf{P}(1,\omega^5,1)$,
04.23 = $\mathbf{P}(\omega^5,\omega^5,1)$,
12.34 = $\mathbf{P}(0,\omega^{10},1)$,
13.24 = $\mathbf{P}(1,\omega^{10},1)$,
14.23 = $\mathbf{P}(\omega^{10},\omega^{10},1)$.

Amongst these, the five diagonal points

04.12 = $\mathbf{P}(0,\omega^5,1)$,
13.24 = $\mathbf{P}(1,\omega^{10},1)$,
01.23 = $\mathbf{P}(1,1,0)$,
02.34 = $\mathbf{P}(\omega^5,0,1)$,
03.14 = $\mathbf{P}(\omega^{10},1,1)$.

lie on the line $v(\omega^5 X + \omega^5 Y + \omega^{10} Z)$.

Amongst these, the five diagonal points

04.12 = $\mathbf{P}(0,\omega^5,1)$,
14.23 = $\mathbf{P}(\omega^{10},\omega^{10},1)$,
03.24 = $\mathbf{P}(\omega^5,1,1)$,
01.34 = $\mathbf{P}(\omega^{10},1,0)$,

$02.13 = \mathbf{P}(1,0,1)$.
lie on the line $v(\omega^3 X + \omega^{13}\mathrm{Y} + \omega^3 Z)$.
Amongst these, the five diagonal points
$02.14 = \mathbf{P}(\omega^{10}, 0,1)$,
$13.24 = \mathbf{P}(1, \omega^{10}, 1)$,
$03.12 = \mathbf{P}(0,1,1)$,
$01.34 = \mathbf{P}(\omega^{10}, 1,0)$,
$04.23 = \mathbf{P}(\omega^5, \omega^5, 1)$.
lie on the line $v(\omega^4 X + \omega^{14}\mathrm{Y} + \omega^{14} Z)$.
Amongst these, the five diagonal points
$01.24 = \mathbf{P}(\omega^5, 1,0)$,
$14.23 = \mathbf{P}(\omega^{10}, \omega^{10}, 1)$,
$03.12 = \mathbf{P}(0,1,1)$,
$04.13 = \mathbf{P}(1, \omega^5, 1)$,
$02.34 = \mathbf{P}(\omega^5, 0,1)$.
lie on the line $v(\omega^{11} X + \omega\mathrm{Y} + \omega Z)$.
Amongst these, the five diagonal points
$02.14 = \mathbf{P}(\omega^{10}, 0,1)$,
$03.24 = \mathbf{P}(\omega^5, 1,1)$,
$01.23 = \mathbf{P}(1,1,0)$,
$04.13 = \mathbf{P}(1, \omega^5, 1)$,
$12.34 = \mathbf{P}(0, \omega^{10}, 1)$.
lie on the line $v(\omega X + \omega^6\mathrm{Y} + \omega Z)$.

Remark 7.2:

(1) The ten sides of the pentastigm δ are separated into five pairs such that no pair meets at vertex. Also the point $\mathbf{P_{101}} = (\omega^{10}, \omega^5, 1)$ satisfies the equations
(1) $a_2 = a_0 + a_1$, $a_0^2 - a_1^2 = a_0 a_1$,
(2) $a_2 = a_0 + a_1$, $a_0^2 - a_1^2 = -a_0 a_1$,
(3) $a_0 = a_1 + a_2$, $a_1^2 - a_2^2 = -a_1 a_2$,
(4) $a_1 = a_0 + a_2$, $a_0^2 - a_2^2 = -a_0 a_2$,
(5) $a_0 = a_1 + a_2$, $a_1^2 - a_2^2 = a_1 a_2$,
(6) $a_1 = a_0 + a_2$, $a_0^2 - a_2^2 = a_0 a_2$.
which are the conditions for the above collinearities.

(2) The fifteen diagonals points of A_4 are exactly the fifteen points of index two.

8. The Group Action of $\boldsymbol{A_5}$ on the Pentad A_4

The group

$$\boldsymbol{A_5} = \langle g, h; g^2 = h^3 = (gh)^5 = I \rangle$$

$$g = \begin{pmatrix} \omega^{10} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega^5 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is the stabilizer group of the 5-arc $A_4 = \{0,1,2,253,101\}$.

The Group $\boldsymbol{A_5}$ acts transitively on A_4 as given below:

$0 \stackrel{h}{\longmapsto} 1$,
$0 \stackrel{h^2}{\longmapsto} 253$,
$0 \stackrel{h^2 g}{\longmapsto} 101$,
$0 \stackrel{hg}{\longmapsto} 2$.

The group $\boldsymbol{A_5}$ has a subgroup of type $\boldsymbol{D_5}$ generated by α_1, α_2, where

$$\alpha_1 = \begin{pmatrix} \omega^{10} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \omega^5 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \omega^{10} & \omega^5 & 0 \\ 1 & 1 & 1 \\ 0 & \omega^{10} & 0 \end{pmatrix}$$

$\boldsymbol{D_5} = \langle \alpha_1, \alpha_2; {\alpha_1}^2 = {\alpha_2}^5 = I, \alpha_2\alpha_1 = \alpha_1{\alpha_2}^{-1}\rangle$.

Each of the five projectivities $\alpha_1, \alpha_1\alpha_2, \alpha_1{\alpha_2}^2, \alpha_1{\alpha_2}^3, \alpha_1{\alpha_2}^4$ fixed 13 points amongst the 133 points of index zero by transforming each point to itself. Each of these 13 points lies on a line which is a unisecant to A_4 and a bisecant of the conic

$C_{A_4} = v(XY + \omega^{10}XZ + \omega^5 YZ)$.

These lines are

$\ell_{36} = v(\omega^3 \mathrm{Y} + \omega^{13}Z)$;

$\ell_{244} = v(X + \omega^5 Z)$;

$\ell_{137} = v(X + \mathrm{Y} + Z)$;

$\ell_{16} = v(X + \omega^5 \mathrm{Y} + \omega^{10}Z)$;

$\ell_{149} = v(\omega^6 X + \omega \mathrm{Y})$.

In Table- 7, each row contains the projectivity f that fixes the set of 13 points which lies on the line ℓ_i .

Table 7- Projectivities fixing 13 points.

f	Set of 13 point lies on ℓ_i fixed by f	ℓ_i
α_1	{17,36,38,42,50,66,98,151,162,171,216,229,268}	ℓ_{36}
$\alpha_1\alpha_2$	{33,60,86,106,151,164,174,208,225,243,244,246,250}	ℓ_{244}
$\alpha_1{\alpha_2}^2$	{44,57,67,96,118,139,143,151,167,199,226,252,263}	ℓ_{137}
$\alpha_1{\alpha_2}^3$	{18,22,46,78,105,131,142,151,196,209,219,248,270}	ℓ_{16}
$\alpha_1{\alpha_2}^4$	{11,56,69,79,108,113,130,148,149,151,155,163,211}	ℓ_{149}

The five lines $\ell_{36}, \ell_{244}, \ell_{137}, \ell_{16}, \ell_{149}$ are concurrent at a point $\mathbf{P}(\omega^5, \omega^{10}, 1) = 151$ which is fixed by $\boldsymbol{A_5}$ as well.

9. Properties of the 6-Arc $B = \{0,1.2,253,101,151\}$

Let

$K = \{ U_0, U_1, U_2, U, \mathbf{P}(a, b, 1), \mathbf{P}(c, d, 1)\}$

$= \{ \mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}, \mathbf{P_4}, \mathbf{P_5}, \mathbf{P}_6\}$

be a 6-arc. A point of index three is called a Brianchon point or B-point for short. write $ij.kl.mn = \mathbf{P}_i\mathbf{P}_j \cap \mathbf{P}_k\mathbf{P}_l \cap \mathbf{P}_m\mathbf{P}_n$ for a B-point. There are fifteen ways of choosing three bisecants no two of which intersect on K. The stabilizer group of B is $\boldsymbol{A_6}$.

The fifteen B-points of $\mathrm{B} = \{0,1,2,253,101,151\}$ are

1. 12.34.56 = 136;
2. 12.35.46 = 15;
3. 12.36.45 = 238;
4. 13.24.56 = 272;
5. 13.25.46 = 30;
6. 13.26.45 = 203;
7. 14.23.56 = 137;
8. 14.25.36 = 264;
9. 14.26.35 = 97 ;
10. 15.23.46 = 16;
11. 15.24.36 = 179;
12. 15.26.34 = 258;
13. 16.23.45 = 239;
14. 16.24.35 = 125;
15. 16.25.34 = 35.

The set $K_{15} = \{ 15,16,30,35,97,125,136,137,179,203,238,239,258,264,272\}$ of B-points of B is an arc of degree 5. The number of points not on the 5-secants is not equal to zero, this implies that K_{15} is an incomplete. The stabilizer group $G_{k_{15}}$ is $\boldsymbol{A_6}$.

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