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Some Generalizations of Semisimple Gamma Rings

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Abstract

In this paper we introduce and study the concepts of semisimple gamma modules , regular gamma modules and fully idempotent gamma modules as a generalization of semisimple Γ −ring. An R_Γ −module M is called fully R_Γ −idempotent (semisimple , regular) if $N = (N:_{R_\Gamma} M)\Gamma N$ for all R_Γ −submodule N of M (every R_Γ −submodule is a direct summand, for each $m \in M$, there exists $f \in Hom_{R_\Gamma}(M, R)$ and $\gamma \in \Gamma$ such that $m = f(m)\gamma m$. We study some properties and relationships between them.

Keywords: Semisimple Gamma Module, Multiplication Gamma Module, Duo Gamma Module, Fully Idempotent Gamma Module, Regular Gamma Module.

بعض التعميمات للمقاسات شبه البسيطة من نمط كاما

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الخلاصة

في هذا البحث قدمنا تعريف مفاهيم المقاسات شبه البسيطة من نمط كاما ، المقاسات المنتظمة من نمط كاما و المقاسات تامة اللانمو من نمط كاما كاعمام الى حلقة كاما شبه البسيطة . المقاس من نمط كاما يسمى تام اللانمو (شبه بسيط ، منتظم) اذا كان $N = (N:_{R_\Gamma} M)\Gamma N$ لكل مقاس شبه جزئي منه (كل مقاس جزئي منه مجموع مباشر، لكل عنصر m ينتمي له يوجد تشاكل من نمط كاما من M الى R و γ في Γ بحيث $m = f(m)\gamma m$. كما درسنا بعض الخواص لهذه المفاهيم والعلاقة فيما بينها.

1. Introduction

Let R and Γ be two additive abelian groups, R is called a Γ −ring (in the sense of Barnes), if there exists a mapping $\cdot : R \times \Gamma \times R \longrightarrow R$, written $\cdot (r, \gamma, s) \mapsto r\gamma s$ such that $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)c = a\alpha c + a\beta c, a\alpha(b + c) = a\alpha b + b\alpha c$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$ [1]. A subset A of Γ −ring R is said to be a right(left) ideal of R if A is an additive subgroup of R and $A\Gamma R \subseteq A(R\Gamma A \subseteq A)$, where $A\Gamma R = \{a\alpha r : a \in A,\ \alpha \in \Gamma,\ r \in R\}$. If A is both right and left ideal, we say that A is an ideal of R [1]. An element 1 in Γ −ring R is unity if there exists element $\gamma_\circ \in \Gamma$ such that $r = 1\gamma_\circ r = r\gamma_\circ 1$ for every $r \in R$, in this paper we denote $\gamma_\circ \in \Gamma$ to the element such that $1\gamma_\circ$ is the unity [2]. A Γ −ring can have more than one unity. A Γ −ring R is called commutative, if $a\gamma b = b\gamma a$ for any $a, b \in R$ and $\gamma \in \Gamma$ [2].

Let R be a Γ −ring and M be an additive abelian group. Then M together with a mapping $\cdot: R \times \Gamma \times M \rightarrow M$, $\cdot (r, \gamma, m) \mapsto r\gamma m$ such that $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$, $(r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$, $r(\gamma + \beta)m = r\gamma m + r\beta m$, $(r_1\gamma r_2)\beta m = r_1\gamma(r_2\beta m)$ where $r, r_1, r_2 \in R,\ \gamma, \beta \in \Gamma$ and $m, m_1, m_2 \in M$ is called a left R_Γ −module, similarly one can defined right R_Γ −module [1]. A

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left R_Γ −module M is unitary if there exist elements, say 1 in R and $\gamma_\circ \in \Gamma$ such that $1\gamma_\circ m = m$ for every $m \in M$ [1].

Let M be an R_Γ −module. A nonempty subset N of M is said to be an R_Γ −submodule of M (denoted by $N \leq M$) if N is a subgroup of M and $R\Gamma N \subseteq N$, where $R\Gamma N = \{r\alpha n : r \in R, \alpha \in \Gamma, n \in N\}$[1]. An R_Γ −module M is called simple if $R\Gamma M \neq 0$ and the only R_Γ −submodules of M are M and 0 [3]. A Γ −ring R is called simple if R is simple R_Γ −module. An R_Γ −submodule N of $R_\Gamma -$ module M is called essential (denote by $N \leq_e M$) if every nonzero R_Γ −submodule of M has nonzero intersection with N, equivalent to, for each nonzero element m in M there is $r_1, r_2, \dots, r_n \in R$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ such that $\sum_{i=1}^{n} r_i \gamma_i m (\neq 0) \in N$ [4]. If X is a nonempty subset of M, then the R_Γ −submodule of M generated by X denoted by $\langle X\rangle$ and $\langle X\rangle = \cap \{N \leq M : X \subseteq N\}$, X is called the generator of $\langle X\rangle$ and $\langle X\rangle$ is finitely generated if $|X| < \infty$, then $\langle X\rangle = \{\sum_{i=1}^{m} n_i x_i + \sum_{j=1}^{k} r_j \gamma_j x_j : k, m \in N, n_i \in Z, \gamma_j \in \Gamma, r_j \in R, x_i, x_j \in X\}$, In particular , if $X = \{x\}$, then $\langle X\rangle$ is called the cyclic R_Γ −submodule of M generated by x. If M is unitary, then $\langle x\rangle = \{\sum_{i=1}^{n} r_i \gamma_i x : n \in N, \gamma_i \in \Gamma, r_i \in R\}$ [1]. An $R_\Gamma -$ submodule N of M is a direct summand if there is an R_Γ −submodule K of M such that $M = N + K$ and $N \cap K = 0$, in this case M is written as $M = N \bigoplus K$ [5]. An $R_\Gamma -$ submodule N of M is closed in M if the only solution of the relation $N \leq_e K \leq M$ is $N = K$ [5].

Let M and N be two R_Γ −modules. A mapping $f : M \to N$ is called homomorphism of R_Γ −modules (simply R_Γ −homomorphism) if $f(x + y) = f(x) + f(y)$ and $f(r\gamma x) = r\gamma f(x)$ for each $x, y \in M, r \in R$ and $\gamma \in \Gamma$. An R_Γ −homomorphism is R_Γ −monomorphism if it is one-to-one and R_Γ −epimorphism if it is onto, the set of all R_Γ −homomorphisms from M into N denote by $Hom_{R_\Gamma}(M, N)$ in particular if $M = N$, $Hom_{R_\Gamma}(M, N)$ denote by $End_{R_\Gamma}(M)$. If M is R_Γ −module, then $End_{R_\Gamma}(M)$ is a Γ −ring with the mapping $\cdot : End_{R_\Gamma}(M) \times \Gamma \times End_{R_\Gamma}(M) \to End_{R_\Gamma}(M)$ denoted by $\cdot (f, \gamma, g) \mapsto f\gamma g$ where $f\,\gamma g(x) = g(f(1\gamma x))$, for $f, g \in End_{R_\Gamma}(M)$, $\gamma \in \Gamma$ and $x \in M$. If M is a left R_Γ −module, then M is a right $End_{R_\Gamma}(M)$ –module with the mapping $\cdot : M \times \Gamma \times End_{R_\Gamma}(M) \to M$ by $\cdot (x, \gamma, f) \mapsto x\gamma f$ where $x\gamma f = f(1\gamma x)$, for $f \in End_{R_\Gamma}(M)$, $\gamma \in \Gamma$ and $x \in M$ [1]. The set of rational numbers and the set of integers will be denoted by Q and Z. All modules in this paper are unitary left R_Γ −modules

2. Fully Idempotent Gamma Modules

In this section we introduce the concept of fully idempotent gamma modules and give some basic properties and characterizations of this concept.

Let N be an R_Γ −submodule of an R_Γ −module M. Then the residual of N in M denoted by $(N :_{R_\Gamma} M) = \{r \in R : r\Gamma M \subseteq N\}$, which is a left ideal of R [1]. An element r of a Γ −ring R is called idempotent if $r = r\gamma r$ for some $\gamma \in \Gamma$ [3]. The ideal I of a Γ −ring R is called idempotent if $I = I\Gamma I$ and R is called semisimple if every ideal of R is idempotent [6]. An element x of an R −module M is called idempotent if there exists $t \in (Rx :_R M)$ such that $x = tx$ [7]. A submodule N of an R −module M is called idempotent if $N = (N :_R M)N$ and M is called fully idempotent if every submodule of M is idempotent [8].

Remarks (2.1): Let N, K and L are R_Γ −submodules of an R_Γ −module M and $f \in End_{R_\Gamma}(M)$. Then

1. If $K \leq N$, then $(K :_{R_\Gamma} L) \subseteq (N :_{R_\Gamma} L)$.
2. If $K \leq N$, then $(L :_{R_\Gamma} N) \subseteq (L :_{R_\Gamma} K)$.
3. $(N :_{R_\Gamma} M) \cap (K :_{R_\Gamma} M) = (N \cap K :_{R_\Gamma} M)$.
4. $(N :_{R_\Gamma} M) \subseteq (N :_{R_\Gamma} f(M)) \cap (f(N) :_{R_\Gamma} f(M))$.
5. $(L :_{R_\Gamma} N + K) = (L :_{R_\Gamma} N) \cap (L :_{R_\Gamma} K)$.

Definition (2.2):

An R_Γ −submodule N of R_Γ −module M is called R_Γ −idempotent if $N = (N :_{R_\Gamma} M)\Gamma N$ and M is called fully R_Γ −idempotent if every R_Γ −submodule of M is R_Γ −idempotent. A Γ −ring R is called fully R_Γ −idempotent if it is fully R_Γ −idempotent R_Γ −module, that is R is semisimle Γ −ring.

Proposition(2.3): Let M be an R_Γ −module. Then M is fully R_Γ −idempotent if and only if every cyclic R_Γ −submodule is R_Γ −idempotent.

Proof:

Assume that $N \le M$, if $x \in N$, then $\langle x\rangle = (\langle x\rangle :_{R_\Gamma} M)\Gamma\langle x\rangle$, so $x \in (\langle x\rangle :_{R_\Gamma} M)\Gamma\langle x\rangle \subseteq (N:_{R_\Gamma} M)\Gamma\langle x\rangle \subseteq (N:_{R_\Gamma} M)\Gamma N$, hence $N = (N:_{R_\Gamma} M)\Gamma N$.

Proposition (2.4): Let M be an R_Γ −module. Then M is fully R_Γ −idempotent if and only if for any element $x \in M$, there exist $t_1, t_2, \dots, t_n \in (\langle x\rangle :_{R_\Gamma} M)$ and $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ such that $x = \sum_{i=1}^{n} t_i\gamma_i x$.

Proof:

Assume that $\langle x\rangle = (\langle x\rangle :_{R_\Gamma} M)\Gamma\langle x\rangle$, so $x = t\gamma\sum_{i=1}^{n} r_i\gamma_i x$ where $t \in (\langle x\rangle :_{R_\Gamma} M)$, $\gamma, \gamma_i \in \Gamma$ and $r_i \in R$, then $x = \sum_{i=1}^{n}(t\gamma r_i)\gamma_i x$. For each $i = 1, \dots, n$, $\beta \in \Gamma, m \in M$, $(t\gamma r_i)\beta m = t\gamma(r_i\beta m) \in t\Gamma M \subseteq \langle x\rangle$, so $t\gamma r_i \in (\langle x\rangle :_{R_\Gamma} M)$ for each $i = 1, \dots, n$. Conversely, for each $x \in M$, $x = \sum_{i=1}^{n} t_i\gamma_i x$ where $t_i \in (\langle x\rangle :_{R_\Gamma} M)$ and $\gamma_i \in \Gamma$, so $x \in (\langle x\rangle :_{R_\Gamma} M)\Gamma\langle x\rangle$, so $\langle x\rangle = (\langle x\rangle :_{R_\Gamma} M)\Gamma\langle x\rangle$, hence M is fully R_Γ −idempotent by proposition(2.3).

An R_Γ −module M is called multiplication if for each R_Γ −submodule N of M, then $N = I\Gamma M$ for some left ideal I of R. This is equivalent to saying that $N = (N:_{R_\Gamma} M)\Gamma M$ for every R_Γ −submodule N of M [2].

Proposition(2.5): Every cyclic R_Γ −module over commutative Γ −ring is multiplication.

Proof: Let N be an R_Γ −submodule of cyclic R_Γ −module M, then there is $x \in M$ such that $= \langle x\rangle$, if $n \in N$, then $n = \sum_{i=1}^{n} r_i\gamma_i x$ where $r_i \in R$ and $\gamma_i \in \Gamma$. Now for each $\beta \in \Gamma$ and $m \in M$, then $(\sum_{i=1}^{n} r_i\gamma_i 1)\beta m = (\sum_{i=1}^{n} r_i\gamma_i 1)\beta(\sum_{j=1}^{t} s_j\lambda_j x) = \sum_{i=1}^{n}\sum_{j=1}^{t} r_i\gamma_i 1\beta s_j\lambda_j x = \sum_{j=1}^{t}\sum_{i=1}^{n}(r_i\gamma_i 1)\beta(s_j\lambda_j 1)\gamma_\circ x = \sum_{j=1}^{t}\sum_{i=1}^{n}(s_j\lambda_j 1)\beta(r_i\gamma_i 1)\ \gamma_\circ x = \sum_{j=1}^{t} s_j\lambda_j 1\beta \sum_{i=1}^{n}(r_i\gamma_i 1)\gamma_\circ x = \sum_{j=1}^{t} s_j\lambda_j 1\beta\, n \in N$, so $\sum_{i=1}^{n}(r_i\gamma_i 1) \in (N:_{R_\Gamma} M)$, hence $N \subseteq (N:_{R_\Gamma} M)\gamma_\circ x \subseteq (N:_{R_\Gamma} M)\Gamma M$, thus $N = (N:_{R_\Gamma} M)\Gamma M$.

Proposition (2.6): Let M be an R_Γ −module, K and N be R_Γ −submodules of M. Then

1- If N is R_Γ −idempotent submodule of M, then N is multiplication , and hence every fully R_Γ −idempotent is multiplication.

2- If K and N are R_Γ −idempotent R_Γ −submodules of M, then so is $K + N$.

3- Let R be commutative Γ −ring. Then

(i) If I is idempotent ideal of R and N is R_Γ −idempotent M, then $I\Gamma N$ is R_Γ −idempotent submodule in M.

(ii) If K is R_Γ −idempotent in N and N is R_Γ −idempotent in M, then K is R_Γ −idempotent in M.

Proof:

1. $N = (N:_{R_\Gamma} M)\Gamma N \subseteq (N:_{R_\Gamma} M)\Gamma M \subseteq N$, so $N = (N:_{R_\Gamma} M)\Gamma M$.
2. $K + N = (K:_{R_\Gamma} M)\Gamma K + (N:_{R_\Gamma} M)\Gamma N \subseteq (K + N:_{R_\Gamma} M)\Gamma K + (K + N:_{R_\Gamma} M)\ \Gamma N = (K + N:_{R_\Gamma} M)\Gamma(K + N)$.
3. (i)
$(I\Gamma N:_{R_\Gamma} M)\Gamma(I\Gamma N) \subseteq I\Gamma N = (I\Gamma I)\Gamma(N:_{R_\Gamma} M)\Gamma N = I\Gamma(N:_{R_\Gamma} M)\Gamma I\Gamma N \subseteq (I\Gamma N:_{R_\Gamma} M)\Gamma(I\Gamma N)$, so $I\Gamma N = (I\Gamma N:_{R_\Gamma} M)\Gamma(I\Gamma N)$.

(ii) $(K:_{R_\Gamma} N)\Gamma N = (K:_{R_\Gamma} N)\Gamma(N:_{R_\Gamma} M)\Gamma N \subseteq (K:_{R_\Gamma} M)\Gamma N \subseteq (K:_{R_\Gamma} N)\Gamma N$, then $(K:_{R_\Gamma} N)\Gamma N = (K:_{R_\Gamma} M)\Gamma N$, also $K = (K:_{R_\Gamma} N)\Gamma K \subseteq (K:_{R_\Gamma} N)\Gamma N = (K:_{R_\Gamma} M)\Gamma N \subseteq (K:_{R_\Gamma} M)\Gamma M \subseteq K$, so $K = (K:_{R_\Gamma} M)\Gamma N$, thus $K = (K:_{R_\Gamma} N)\Gamma K = (K:_{R_\Gamma} N)\Gamma(K:_{R_\Gamma} M)\Gamma N = (K:_{R_\Gamma} M)\Gamma(K:_{R_\Gamma} N)\Gamma N = (K:_{R_\Gamma} M)\Gamma K$.

The following proposition shows that the concept of fully R_Γ −idempotent generalizes that of semisimple Γ −ring.

Proposition (2.7): If R is fully R_Γ −idempotent Γ −ring, then R is semisimple. The converse holds when R is commutative.

Proof:

Assume R is a fully R_Γ −idempotent Γ −ring and I is an ideal of R, then $I = (I:_{R_\Gamma} R)\Gamma I$. For each $t \in (I:_{R_\Gamma} R)$, then $t = t\gamma_\circ 1 \in t\Gamma R \subseteq I$, so $(I:_{R_\Gamma} R) \subseteq I$, thus $I = (I:_{R_\Gamma} R)\Gamma I \subseteq I\Gamma I \subseteq R\Gamma I \subseteq I$ and hence $I = I\Gamma I$. Conversely, let I be an ideal of R, it's enough to show that $I \subseteq (I:_{R_\Gamma} R)$, since $I\Gamma R \subseteq R\Gamma I \subseteq I$,then $I \subseteq (I:_{R_\Gamma} R)$, hence $I = I\Gamma I \subseteq (I:_{R_\Gamma} R)\Gamma I \subseteq I$, thus $I = (I:_{R_\Gamma} R)\Gamma I$.

Examples and Remarks (2.8):

1- Every idempotent element in $R-$module M is R_R-idempotent and every idempotent submodule N of M is idempotent $R_\Gamma-$submodule.

2- Every $R_\Gamma-$submodule of fully $R_\Gamma-$idempotent also fully $R_\Gamma-$idempotent. Let B is an $R_\Gamma-$submodule of M, for any $R_\Gamma-$submodule N of B, then $N=(N:_{R_\Gamma}M)\Gamma N$, by Remarks(2.1) $(N:_{R_\Gamma}M)\subseteq(N:_{R_\Gamma}B)$, so $N=(N:_{R_\Gamma}M)\Gamma N\subseteq(N:_{R_\Gamma}B)\Gamma N\subseteq R\Gamma N\subseteq N$, thus $N=(N:_{R_\Gamma}B)\Gamma N$.

3- Every simple $R_\Gamma-$module is fully $R_\Gamma-$idempotent.

4- Let $R=Z_2$, $\Gamma=Z$ and $M=Z_2\oplus Z_2$. Then M is not fully $R_\Gamma-$idempotent, since $Z_2\oplus(0)$ is not $R_\Gamma-$idempotent submodule. Note that M is not multiplication .

5- Let $R=\{(n\ \ n), n\in Q\}$ and $\Gamma=\{\binom{x}{y}, x,y\in Q\}$, then R is $\Gamma-$ring with $\cdot: R\times\Gamma\times R\to R$ by $(n\ \ n)\binom{x}{y}(m\ \ m)=\big((nx+ny)m\quad(nx+ny)m\big)$, since for any nonzero ideal I of R, take $0\neq(m\ \ m)\in I$ we can choose $x=y=\frac{1}{2m}$, then $(m\ \ m)=(m\ \ m)\binom{x}{y}(m\ \ m)\in I\Gamma I$, so $I=I\Gamma I$, hence R is semisimple and by proposition(2.7) R is fully $R_\Gamma-$idempotent.

1- Fully $R_\Gamma-$idempotent $R_\Gamma-$module over simple $\Gamma-$ring is simple. For each nonzero $R_\Gamma-$submodule N of M, then $N=(N:_R M)\Gamma N$, so $(N:_R M)=R$, hence $M=R\Gamma M=(N:_R M)\Gamma M\subseteq N$, thus $M=N$.

The product of two $R-$submodules N and K of an $R-$module M define as $NK=(N:_R M)(K:_R M)M$ [9].

Definition (2.9):

Let N and K are $R_\Gamma-$submodules of an $R_\Gamma-$module M. The product of N and K define by $=(N:_{R_\Gamma}M)\Gamma(K:_{R_\Gamma}M)\Gamma M$.

The following proposition gives a characterizations of fully $R_\Gamma-$idempotent $R_\Gamma-$modules.

Proposition (2.10):

Let M be $R_\Gamma-$module, then the following are equivalent:

1- M is fully $R_\Gamma-$idempotent.

2- $N=N^2$ for all $R_\Gamma-$submodule N of M.

3- $N\cap K=NK$ for all $R_\Gamma-$submodules N and K.

Proof:

(1)⇒(2) For each $R_\Gamma-$submodule N of M, $N=(N:_{R_\Gamma}M)\Gamma N$, then $N=(N:_{R_\Gamma}M)\Gamma(N:_{R_\Gamma}M)\Gamma N\subseteq(N:_{R_\Gamma}M)\Gamma(N:_{R_\Gamma}M)\Gamma M\subseteq(N:_{R_\Gamma}M)\Gamma N\subseteq N$, so $N=(N:_{R_\Gamma}M)\ \Gamma(N:_{R_\Gamma}M)\Gamma M=N^2$. (2) ⇒ (3) For each $R_\Gamma-$submodules N and K, then $N\cap K=(N\cap K)^2=(N\cap K:_{R_\Gamma}M)\Gamma(N\cap K:_{R_\Gamma}M)\Gamma M\subseteq(N:_{R_\Gamma}M)\Gamma(K:_{R_\Gamma}M)\Gamma M=NK$, so $N\cap K\subseteq NK$, since $NK=(N:_{R_\Gamma}M)\Gamma(K:_{R_\Gamma}M)\Gamma M\subseteq(N:_{R_\Gamma}M)\Gamma K\subseteq K$ also $NK=(N:_{R_\Gamma}M)\Gamma(K:_{R_\Gamma}M)\Gamma M\subseteq(N:_{R_\Gamma}M)\Gamma K\subseteq N$, then $NK\subseteq N\cap K$, thus $NK=N\cap K$. (3)⇒(1) $N=N\cap N=NN=N^2$.

We have proved that every fully $R_\Gamma-$idempotent $R_\Gamma-$module is multiplication, in the following corollary we discuss the converse.

Corollary(2.11): Every multiplication $\mathrm{R}_\Gamma-$module over semisimple $\Gamma-$ring is fully $\mathrm{R}_\Gamma-$idempotent.

Proof:

Let M be a multiplication $R_\Gamma-$module and $N\leq M$, then $N=(N:_{R_\Gamma}M)\Gamma M=(N:_{R_\Gamma}M)\Gamma(N:_{R_\Gamma}M)\Gamma M=N^2$, hence M is fully $R_\Gamma-$idempotent.

Let M and N be two $R_\Gamma-$modules. Then M is called $N-$injective if for any $R_\Gamma-$submodule A of N and $R_\Gamma-$homomorphism $f: A\to M$, there exists an $R_\Gamma-$homomorphism $g: N\to M$ such that $gi=f$ where i is the inclusion mapping. An $R_\Gamma-$module M is injective if it is $N-$injective for any $R_\Gamma-$module N. Every $R_\Gamma-$module M can be embedding in injective $R_\Gamma-$module which is called injective hull $E(M)$ [4]. An $R_\Gamma-$module M is called quasi-injective if and only if M is $M-$injective [5].

Proposition*(2.12):* Let M be an $R_\Gamma-$module with injective hull $E(M)$. If M is $R_\Gamma-$idempotent of $E(M)$, then M is quasi-injective.

Proof:

Assume that M is R_Γ −idempotent of $E(M)$, then $M = \left(M :_{R_\Gamma} E(M)\right)\Gamma M$, then for each $f \in End_{R_\Gamma}(E(M))$, $f(M) = f\left(\left(M :_{R_\Gamma} E(M)\right)\Gamma M\right) = \left(M :_{R_\Gamma} E(M)\right)\Gamma f(M) \subseteq \left(M :_{R_\Gamma} E(M)\right)\Gamma E(M) \subseteq M$, thus M is quasi-injective [5].

An R_Γ −module M is called duo if $f(N) \subseteq N$ for each R_Γ − submodule N of M and $f \in End_{R_\Gamma}(M)$. It is easy to see that every multiplication is duo.

Proposition(2.13): Let M be fully R_Γ − idempotent. Then M is duo.

Proof:

For each R_Γ − submodule N of M and $f \in End_{R_\Gamma}(M)$, then $N = \left(N :_{R_\Gamma} M\right)\Gamma N$. So $f(N) = f\left(\left(N :_{R_\Gamma} M\right)\Gamma N\right) = \left(N :_{R_\Gamma} M\right)\Gamma f(N) \subseteq N$.

The converse of Proposition(2.13) is not true in general for example Z_4 as Z_Z −module is multiplication and hence duo but not fully R_Γ −idempotent.

An R_Γ −submodule of quasi-injective need not be quasi-injective for example see Example(2.3) [4].

Corollary(2.14): Let M be fully R_Γ −idempotent. Then M is quasi-injective R_Γ −module if and only if every R_Γ −submodule of M is quasi-injective R_Γ −module.

Proof:

Assume that N is R_Γ −submodule of a quasi-injective R_Γ −module M, let K be R_Γ −submodule of N and let $f: K \to N$ be R_Γ −homomorphism, since M is quasi-injective, then there exists an R_Γ − homomorphism $g: M \to M$ such that $g i_N i_K = i_N f$ where i_N and i_K are inclusion maps, clear that g is extended of f and by Proposition(2.13) $g(N) \subseteq N$. The converse is obvious.

3. Semisimple Gamma Modules

In this section we extended the concept of semisimplicity from category of modules to the category of gamma modules.

Definition(3.1):

An R_Γ −module M is called semisimple if every R_Γ −submodule is a direct summand.

Examples(3.2):

1- $R = Z_6$ is Z_Z − ring with $\cdot: Z_6 \cdot_6 Z \cdot_6 Z_6 \to Z_6$ by $(n, k, m) \mapsto nkm$, the only ideals of Z_6 are 0 , Z_6 , $\langle 2\rangle$ and $\langle 3\rangle$, then Z_6 is semisimple.

2- Let $R = \{(a\ \ b)\,, a, b \in Q\}$(where Q is the ring of rational numbers) and $\Gamma = \{\begin{pmatrix} x \\ y \end{pmatrix}\,, x, y \in Q\}$. Then R is Γ −ring with $\cdot: R \times \Gamma \times R \to R$ by $(a\ \ b)\begin{pmatrix} x \\ y \end{pmatrix}(c\ \ d) = \left((ax + by)c \quad (ax + by)d\right)$. Take $J = \{(2n\ 2m), n, m \in Q\}$, then $(a\,b)\begin{pmatrix} k \\ t \end{pmatrix}(2n\ \ 2m) = (ax + by)(2n\ \ 2m) = (2(ax + by)n\ \ 2(ax + by)m) = (2n_1\ \ 2m_1) \in J$, so $R\Gamma J \subseteq J$, hence J is a left ideal of R, for any anther left ideal N of R, let $0 \neq (k\ \ t) \in N$, then $(2k\ \ 2t) \in J$ and since $R\Gamma N \subseteq N$, for $(1\,1) \in R$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \Gamma$ we have $(1\,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}(k\ t) = (2k\ \ 2t) \in N$, hence $N \cap J \neq 0$, so J can not be direct summand of R, thus R is not semisimple R_Γ −module. It is noted that R is semisimple Γ − ring, since if I is an ideal of R, then for each $(a\ \ b) \in I$ we can choose $\gamma = \begin{pmatrix} k \\ t \end{pmatrix} \in \Gamma$ such that if :

(i)- $a = 0\ and\ b = 0$ then $k = 0\ and\ t = 0$.

(ii)- $a \neq 0\ and\ b = 0$ then $k = \frac{1}{a}\ and\ t = 0$.

(iii)- $a = 0\ and\ b \neq 0$ then $k = 0\ and\ t = \frac{1}{b}$.

(iv)- $a \neq 0\ and\ b \neq 0$ then $k = \frac{1}{2a}\ and\ t = \frac{1}{2b}$. Then $(a\ \ b) = (ak + bt)(a\ \ b) = (a\ \ b)\begin{pmatrix} k \\ t \end{pmatrix}(a\ \ b) \in I\Gamma I$, so $I \subseteq I\Gamma I$, hence $I = I\Gamma I$, therefore R is semisimple Γ −ring.

3- Every simple R_Γ −module is semisimple.

4- Let $R' = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, a, b, c \in R$ ring of real numbers$\}$, $\Gamma = R$, then R' is Γ −ring with $\cdot: R' \times \Gamma \times R' \to R'$ by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} n \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} axn & ayn + bzn \\ 0 & czn \end{pmatrix}$. Take $L_1 = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, b \in R\right\}$, L_1 is a left ideal of R' and R' is not semisimple Γ −ring since $L_1 \Gamma L_1 = 0 \neq L_1$.

In the category of module it is known that a submodule is a direct summand if an only if there exists $f \in End(M)$ such that $N = f(M)$ and $f = f^2$.

Proposition(3.3): Let M be R_Γ −module, then $M = M\gamma f \oplus M\gamma_\circ(I - I\gamma f)$ for any $f \in End_{R_\Gamma}(M)$ such that $f = f\gamma f$ for some $\gamma \in \Gamma$.

Proof:

For each $x \in M$, then $x = x + f(1\gamma x) - f(1\gamma x) = f(1\gamma x) + I(x) - f(1\gamma x) = f(1\gamma x) + (I - I\gamma f)(1\gamma_\circ x) = x\gamma f + x\gamma_\circ(I - I\gamma f) \in M\gamma f + M\gamma_\circ(I - I\gamma f)$, so $M = M\gamma f + M\gamma_\circ(I - I\gamma f)$. Now if $y \in M\gamma f \cap M\gamma_\circ(I - I\gamma f)$, then $y = x\gamma f = t\gamma_\circ(I - I\gamma f)$ where $x, t \in M$, hence $y = f(1\gamma x) = (I - I\gamma f)(1\gamma_\circ t) = t - I\gamma f(t) = t - f(1\gamma t)$, so $1\gamma f(y) = 1\gamma f(t) - 1\gamma f(f(1\gamma t) = 1\gamma f(t) - f\big(1\gamma f(1\gamma t)\big) = 1\gamma f(t) - (f\gamma f)(1\gamma t) = 1\gamma f(t) - 1\gamma f(t) = 0$, hence $0 = f(1\gamma f(y)) = f(y)$, but $f(y) = f\big(f(1\gamma x)\big) = (f\gamma f)(x)$, therefore $y = 1\gamma f(x) = 0$, thus $M = M\gamma f \oplus M\gamma_\circ(I - I\gamma f)$.

Corollary(3.4): Let M be R_Γ −module, then $M = M\Gamma f \oplus M\Gamma(I - I\gamma f)$ for any $f \in End_{R_\Gamma}(M)$ such that $f = f\gamma f$ for some $\gamma \in \Gamma$.

Proof:

For any $y \in M\Gamma f \oplus M\Gamma(I - I\gamma f)$, $y = x\lambda f + t\beta(I - I\gamma f)$ where $x, t \in M$ and $\lambda, \beta \in \Gamma$, so $y = f(1\lambda x) + (I - I\gamma f)(1\beta t) = f\gamma f(1\lambda x) + (I - I\gamma f)(1\gamma_\circ(1\beta t)) = f(1\gamma f(1\lambda x)) + (1\beta t)\gamma_\circ(I - I\gamma f) = f(1\lambda x)\gamma f + (1\beta t)\gamma_\circ(I - I\gamma f) \in M\gamma f \oplus M\gamma_\circ(I - I\gamma f)$, hence $M\Gamma f \oplus M\Gamma(I - I\gamma f) \subseteq M\gamma f \oplus M\gamma_\circ(I - I\gamma f)$, thus $M\Gamma f \oplus M\Gamma(I - I\gamma f) = M\gamma f \oplus M\gamma_\circ(I - I\gamma f)$.

Corollary(3.5): Let N be an R_Γ −submodule of R_Γ −module M. Then N is a direct summand of M if and only if $N = M\gamma f$ where $f \in End_{R_\Gamma}(M)$ and $f = f\gamma f$ for some $\gamma \in \Gamma$.

Proof*:*

Assume N is a direct summand of M, then $M = N \oplus K$ for some R_Γ −submodule K of M, take $f: M \to M$ by $f(n + k) = n$ for any $n \in N$ and $k \in K$, then $f\gamma_\circ f(x) = f\big(1\gamma_\circ f(x)\big) = f(x)$ for any $x \in M$ and $N = f(M) = f(1\gamma_\circ M) = M\gamma_\circ f$.

Proposition(3.6): Every R_Γ −submodule of a semisimple R_Γ −module M is semisimple R_Γ −module.

Proof: For any R_Γ −submodule N of M, if $K \leq N$, then there exists an R_Γ −submodule K_1 such that $M = K \oplus K_1$, hence $N = N \cap M = N \cap (K \oplus K_1) = (N \cap K_1) \oplus K$.

Proposition(3.7): If R is semisimple R_Γ −module, then R is semisimple Γ −ring.

Proof: Let I be an ideal of R, then $R = I \oplus L$ for some ideal L of R, so $1 = e_1 + e_2$ for $e_1 \in I$, $e_2 \in L$, then for each $n \in I$,$n = n\gamma_\circ 1 = n\gamma_\circ(e_1 + e_2)$, thus $n\gamma_\circ e_2 = n - n\gamma_\circ e_1 \in I \cap L = 0$, hence $n = n\gamma_\circ e_1 \in I\gamma_\circ e_1 \subseteq I\gamma_\circ I$, so $I \subseteq I\Gamma I \subseteq I$, hence $I = I\Gamma I$, therefore R is semisimple Γ −ring.

The converse of Proposition(3.7) is not true in general, see Example(3.2)(2).

Proposition(3.8): Let M be a nonzero R_Γ −module. Then the following are equivalent:

1- M is semisimple R_Γ −module.
2- M is sum of simple R_Γ −submodules.
3- M has no proper essential R_Γ −submodules.

Proof:

(1) ⇒ (2) To show M has simple R_Γ −submodule , if $0 \neq N \leq M$, then for each $K \leq N$ we have K is a direct summand of N , so $M = K \oplus K_1$, hence $N = N \cap M = N \cap (K \oplus K_1) = (N \cap K_1) \oplus K$. Let $a(\neq 0) \in M$, take $\Omega = \{B \leq M : a \notin B\}$, then $\Omega \neq \phi$ since $0 \in \Omega$ by using Zorn's lemma there is maximal element B of Ω, $a \notin B$, hence B is a direct summand of M , then $M = B \oplus C$ for some R_Γ −submodule C of M. We claim C is simple , if not C has a proper R_Γ −submodule $D \neq 0$, so $C = D \oplus E$ for some R_Γ −submodule $E \neq 0$ since D is proper, hence $M = B \oplus D \oplus E$, by maximality of B , $a \in B \oplus D$ and $a \in B \oplus E$, so $a = b + d = b' + e$ for $b, b' \in B$, $d \in D$ and $e \in E$, then $d = e + (b' - b) \in D \cap (B \oplus E)$ and $e = d + (b - b') \in E \cap (D \oplus B)$, hence $d = e = 0$ and $b = b'$, so $a = b \in B$ which is a contradiction, thus C is simple R_Γ −submodule. Let $N_\circ$ is the sum of all simple R_Γ −submodule of M, then there is $L \leq M$ such that $M = N_\circ \oplus L$. If $L \neq 0$, then by proof L has a nonzero simple R_Γ −submodule T, then $T \subset L \cap N_\circ = 0$ which is a contradiction, hence $L = 0$ and $M = N_\circ$. (2) ⇒ (3) Assume that M has a proper R_Γ −submodule A, then there is $x \in M - A$ and by (2) M has simple R_Γ −submodule B' such that $x \in B'$, then $A \cap B' \leq B'$, so either $A \cap B' = B'$ a contradiction or $A \cap B' = 0$, thus A is not essential R_Γ −submodule of M. (3) ⇒ (1) Let $A \leq M$ and B complement of A, then $A \oplus B \leq_e M$ [5], so $M = A \oplus B$.

Proposition*(3.9):* Let R be a $\Gamma-$ ring. Then the following are equivalent:
1- R is semisimple $R_\Gamma-$module.
2- Every ideal of R is generated by an idempotent element.
3- R is sum of simple $R_\Gamma-$submodules.
4- Every $R_\Gamma-$module M has no proper essential $R_\Gamma-$submodules.
5- Every $R_\Gamma-$module is injective.
6- Every $R_\Gamma-$module is semisimple.

Proof:
(1)⇒(2) Let I be an ideal of R , By proof of Proposition(3.7), there exists $e_1 \in I$ such that $n = n\gamma_\circ e_1$ for each $n \in I$. In particular $e_1 = e_1\gamma_\circ e_1$ therefore e_1 is an idempotent and $n \in \langle e_1 \rangle$, thus $I \subseteq \langle e_1 \rangle$. (2)⇒(1) Let I be an ideal of R. Then there exists an idempotent element $e \in R$ such that $e = e\gamma e$ for some $\gamma \in \Gamma$ and $I = \langle e \rangle$. For each $r \in R, r = r\gamma e + r - r\gamma e = r\gamma e + r\gamma_\circ 1 - r\gamma e = r\gamma e + r\gamma_\circ 1 - (r\gamma_\circ 1)\gamma e$, hence $r = r\gamma e + r\gamma_\circ(1 - 1\gamma e)$, so $R \subseteq R\Gamma e + R\Gamma(1 - 1\gamma e)$. For each $x \in R\Gamma e$, $x = \sum_{i=1}^{n} r_i\gamma_i e = \sum_{i=1}^{n} r_i\gamma_i(e\,\gamma e) = (\sum_{i=1}^{n} r_i\gamma_i e)\gamma e = x\gamma e$. Now if $x \in R\Gamma(1 - 1\gamma e)$, then $x = \sum_{i=1}^{n} r_i\gamma_i\,(1 - 1\gamma e) = \sum_{i=1}^{n} r_i\gamma_i\,1 - \sum_{i=1}^{n} r_i\gamma_i 1\gamma e$, hence $x\gamma e = \sum_{i=1}^{n} r_i\gamma_i\,1\gamma e - \sum_{i=1}^{n} r_i\gamma_i 1\gamma e\,\gamma e = \sum_{i=1}^{n} r_i\gamma_i\,1\gamma e - \sum_{i=1}^{n} r_i\gamma_i\,1\gamma e = 0$, thus $R = R\Gamma e \oplus R\Gamma(1 - 1\gamma e)$. (1)⟺(3)⟺(4) by proposition(3.8). (5)⇒(6) By proposition(1.9) [4]. (6)⇒(1) Clear. (1)⇒(5) Let M be an $R_\Gamma-$module, for each ideal I of R and $R_\Gamma-$homomorphism $f: I \to M$, since R is semisimple $R_\Gamma-$module, then there exists an ideal J of R such that $R = I \oplus J$, define $g: R \to M$ by $g(r) = f(r)$ if $r \in I$ otherwise $g(r) = 0$ for each $r \in R$, then g is extension of f , so M is injective [4].

Semisimple $R_\Gamma-$modules and multiplications are different for example any semisimple $R_\Gamma-$module over simple $\Gamma-$ring is not multiplication. Since for any nonzero $R_\Gamma-$submodule N of M, if there exists an ideal I of R such that $N = I\Gamma M = I\Gamma(N \oplus K) = R\Gamma N + R\Gamma K = N + K \neq N$ for some $R_\Gamma-$submodule K of M which is a contradiction. In Particular ,$Z_2 \oplus Z_2$ as Z_Z- module is semisimple $R_\Gamma-$module but not multiplication. The Z_Z- module Z_4 is multiplication but not semisimple. Also semisimple $R_\Gamma-$module and fully $R_\Gamma-$idempotent are different for example $M = Z_2 \oplus Z_2$ as Z_Z-module is not fully $R_\Gamma-$idempotent since every fully $R_\Gamma-$idempotent is multiplication. For fully $R_\Gamma-$idempotent which is not semisimple see examples and remarks(2.8)(5) R is not semisimple by proposition(3.7).

Proposition (3.10): Let M be multiplication $\mathrm{R}_\Gamma-$module. If M is semisimple $\mathrm{R}_\Gamma-$module. Then M is fully $\mathrm{R}_\Gamma-$idempotent.

Proof:

For each $R_\Gamma-$submodule N of M, $M = N \oplus K$ for some $R_\Gamma-$submodule K of M, since M is multiplication, then $N = (N:_{R_\Gamma} M)\Gamma M = (N:_{R_\Gamma} M)\Gamma(N + K) = (N:_{R_\Gamma} M)\Gamma N + (N:_{R_\Gamma} M)\Gamma K$ but $(N:_{R_\Gamma} M)\Gamma K \subseteq N \cap K = 0$, so N is $R_\Gamma-$idempotent submodule.

Proposition (3.11): If M is semisimple $\mathrm{R}_\Gamma-$module, then M is quasi-injective.

Proof:

For each $R_\Gamma-$submodule N of M and $R_\Gamma-$homomorphism $f: N \to M$, since M is semisimple $R_\Gamma-$module, then $M = N \oplus K$ for some $K \leq M$. So for each $x \in M$, then $x = n + k$ where $n \in N$ and $k \in K$, define $g: M \to M$ by $g(x) = f(n)$ for each $x \in M$, clear that g is $R_\Gamma-$homomorphism and g is extended of f , so M is quasi-injective [5].

Lemma(3.12): Every $\Gamma-$ring R is $\mathrm{R}_\Gamma-$isomorphic to $\mathrm{End}_{\mathrm{R}_\Gamma}(\mathrm{R})$.

Proof:

Let R be a $\Gamma-$ring. For a fixed element r in R we can define $\lambda_r: R \to R$, by $\lambda_r(x) = x\gamma_\circ r$ for each $x \in R$, then λ_r is $R_\Gamma-$homomorphism, that is $\lambda_r \in End_{R_\Gamma}(R)$. Let $R^\ell = \{\lambda_r : r \in R\}$, then R^ℓ is abelian group with $(\lambda_r + \lambda_s) = \lambda_r(x) + \lambda_r(x)$ and R^ℓ is a $\Gamma-$ring with $\cdot : R^\ell \times \Gamma \times R^\ell \longrightarrow R^\ell$, by $\cdot(\lambda_r, \gamma, \lambda_s) \mapsto \lambda_r\gamma\lambda_s$ where $\lambda_r\gamma\lambda_s(x) = \lambda_s(1\gamma\lambda_r(x))$. For each $f \in End_{R_\Gamma}(R)$, $f(x) = f(x\gamma_\circ 1) = x\gamma_\circ f(1) = \lambda_{f(1)}(x)$, so $f = \lambda_{f(1)}$, hence $R^\ell = End_{R_\Gamma}(R)$. Define $\varphi: R \to R^\ell$ by $(r) = \lambda_r$, it is easy to show that φ is a $R_\Gamma-$ isomorphism see[4,example(2.12)], hence $R \cong R^\ell = End_{R_\Gamma}(R)$.

Lemma(3.12) show that if R is a commutative then $End_{R_\Gamma}(R)$ is commuatative. But this may not be true for an arbitrary $R_\Gamma-$module. For example consider V is a vector space over a field F of dimension 2, then V is an F_F-module. Let $f: V \to V$ by $f(v \quad u) = (u \quad v)$ and $g: V \to V$ by

$g(v\quad u)=(v\quad 0)$ be two R_Γ −homomorphisms, then for each $(v\quad u)\in V$ and $\gamma\in\Gamma$, $g\gamma f(v\quad u)=f\big(1\gamma g(v\quad u)\big)=1\gamma f(v\quad 0)=(0\quad 1\gamma v)$ and $f\gamma g(v\quad u)=g\big(1\gamma f(v\quad u)\big)=1\gamma g(u\quad v)=(1\gamma u\quad 0)$, so $f\gamma g\neq g\gamma f$.

Proposition(3.13): Let R be a commutative Γ −ring. If M is fully R_Γ −idempotent , then $End_{R_\Gamma}(M)$is commutative.

Proof: For each $f,g\in End_{R_\Gamma}(M)$, $\gamma\in\Gamma$ and $x\in M$, since $f(\langle x\rangle)\subseteq\langle x\rangle$ by Proposition(2.13), then $f(x)=\sum_{i=1}^{n}r_i\gamma_i x$ and $g(x)=\sum_{j=1}^{m}s_j\beta_j x$ where $r_i,s_j\in R$, $\gamma_j,\beta_j\in\Gamma$, then $(f\gamma g)(x)=g(1\gamma f(x)=1\gamma g(\sum_{i=1}^{n}r_i\gamma_i x)=1\gamma(\sum_{i=1}^{n}r_i\gamma_i g(x))=1\gamma(\sum_{i=1}^{n}r_i\gamma_i\sum_{j=1}^{m}s_j\beta_j x)=$ $1\gamma(\sum_{i=1}^{n}\sum_{j=1}^{m}r_i\gamma_i s_j\beta_j x)=1\gamma(\sum_{j=1}^{m}\sum_{i=1}^{n}r_i\gamma_i s_j\beta_j x)$, but $r_i\gamma_i s_j\beta_j x=(r_i\gamma_i 1)\gamma_\circ(s_j\beta_j 1)\gamma_\circ x=(s_j\beta_j 1)\gamma_\circ(r_i\gamma_i 1)\gamma_\circ x=s_j\beta_j r_i\gamma_i x$, so $(f\gamma g)(x)=1\gamma(\sum_{j=1}^{m}\sum_{i=1}^{n}s_j\beta_j r_i\gamma_i x)=1\gamma(\sum_{j=1}^{m}s_j\beta_j f(x))=1\gamma f(\sum_{j=1}^{m}s_j\beta_j x)=(g\gamma f)(x)$.

4. Regular Gamma Modules

In this section we extended the concept of regular gamma modules as a generalization of regular modules and semisimple gamma modules.

There are deferent of definitions of the regular Γ −ring. In [3] if R is Γ −ring , then $x\in R$ is called regular if there exists $s\in R$ such that $x=x\gamma s\gamma x$ for some $\gamma\in\Gamma$ and R is called regular if every element of R is regular. In [6] a Γ −ring R is called regular if for each $x\in R$ there exists $s\in R$ and $\gamma,\beta\in\Gamma$ such that $x=x\gamma s\beta x$. In [10] a Γ −ring R is called regular if for each $x\in R$ there exists $\gamma\in\Gamma$ such that $x=x\gamma x$. Note that if a Γ −ring is regular in the sense of [10] and [3] ,then R is regular in the sense of [6]. In this paper we take the definition of regular in the sense of [6]. A left module M is called regular if for any element $m\in M$ there exists $f\in Hom_R(M,R)$ such that $m=f(m)m$ [11].

Definition (4.1):

Let M be R_Γ −module. Then M is called regular if for each $m\in M$, there exists $f\in Hom_{R_\Gamma}(M,R)$ and $\gamma\in\Gamma$ such that $m=f(m)\gamma m$.

If R is a regular R_Γ −module, for each $x\in R$, there exists $f\in End_{R_\Gamma}(R)$ and $\gamma\in\Gamma$ such that $x=f(x)\gamma x=\lambda_r(x)\gamma x=x\gamma_\circ r\gamma x$ by lemma(3.12), so regular R_Γ −module is a generalization of regular Γ −ring.

An R_Γ −module M is called projective if for each R_Γ −epimorphism $\alpha: A\to B$ and $: M\to B$, there exists an R_Γ −homomorphism $\lambda: M\to A$ such that $\alpha\lambda=\beta$ [12].

Proposition (4.2): Let M be an R_Γ −module. Then M is regular if and only if every cyclic R_Γ −submodule of M is a projective direct summand.

Proof:

Assume $N=\langle x\rangle$ be a cyclic R_Γ −submodule of a regular R_Γ −module M , there exists $f\in Hom_{R_\Gamma}(M,R)$ such that $x=f(x)\gamma x$ for some $\gamma\in\Gamma$, define $W=\{m\in M: f(m)\gamma x=0\}$, clear W is R_Γ −submodule of M and for each $t\in M$, $t-f(t)\gamma x\in W$ since $f[t-f(t)\gamma x]\gamma x=[f(t)-f(f(t)\gamma x)]\gamma x=f(t)\gamma x-f(t)\gamma f(x)\gamma x=f(t)\gamma x-f(t)\gamma x=0$, hence $t=(t-f(t)\gamma x)+f(t)\gamma x\in W+\langle x\rangle$, so $M\subseteq W+\langle x\rangle$. Now if $\sum_{i=1}^{n}r_i\gamma_i x\in W\cap\langle x\rangle$, then $0=f(\sum_{i=1}^{n}r_i\gamma_i x)\gamma x=\sum_{i=1}^{n}r_i\gamma_i f(x)\gamma x=\sum_{i=1}^{n}r_i\gamma_i x$, thus $M=\langle x\rangle\oplus W$. Take $e=f(x)$, then $e\gamma e=f(x)\gamma f(x)=f(f(x)\gamma x)=f(x)=e$, define $\varphi: R\gamma_\circ e\to\langle x\rangle$ by $\varphi(r\gamma_\circ e)=(r\gamma_\circ e)\gamma x$, then φ is an R_Γ −isomorphism, hence $\langle x\rangle\cong R\gamma_\circ e$, so $\langle x\rangle$ is projective [12]. Conversely, for any $x\in M$, there exists an R_Γ −submodule N of M such that $M=\langle x\rangle\oplus N$. Define an R_Γ −homomorphism $f: R\to\langle x\rangle$ by $f(r)=r\gamma_\circ x$ for each $r\in R$. for each $\sum_{i=1}^{k}r_i\gamma_i x\in\langle x\rangle$, then $\sum_{i=1}^{k}r_i\gamma_i x=(\sum_{i=1}^{k}r_i\gamma_i 1)\gamma_\circ x=f(\sum_{i=1}^{k}r_i\gamma_i 1)$, so f is an R_Γ −epimorphism , since $\langle x\rangle$ is projective, then there exists $g:\langle x\rangle\to R$ such that $id_{\langle x\rangle}=fg$. Define an R_Γ −homomorphism $h: M\to R$ by $h\big(\sum_{i=1}^{k}r_i\gamma_i x+n\big)=g(\sum_{i=1}^{k}r_i\gamma_i x)$, then $x=id_{\langle x\rangle}(x)=f\big(g(x)\big)=g(x)\gamma_\circ x=h(x)\gamma_\circ x$, hence M is regular.

Examples (4.3):

1- Every R_Γ −submodule of regular R_Γ −module is regular.

2- In examples and remarks(2.8)(5) R is semisimple Γ −ring and fully R_Γ −idempotent, let J any principle ideal of R generated by the element $(m\ m)$, for any another ideal $L\neq 0$ of R, take $0\neq$

$(s\ \ s) \in L$, then $(s\ \ s)\begin{pmatrix}1\\1\end{pmatrix}(m\ \ m) = (sm+sm\quad sm+sm) \in J \cap L$, hence $J \cap L \neq 0$, so J can not be direct summand in R, thus R is not regular.

Proposition (4.4): If R is regular Γ −ring, then R is semisimple.

Proof:

For each ideal I of R, let $n \in I$, then $n = n\gamma s\beta n$ for some $s \in R$ and $\gamma, \beta \in \Gamma$, so $n \in I\gamma I \subseteq I\Gamma I$, hence $I = I\Gamma I$.

Proposition(4.5): Let M be duo regular R_Γ −module. Then M is fully R_Γ −idempotent.

Proof:

For each $x \in M$, there exists $f \in \mathrm{Hom}_{R_\Gamma}(M,R)$ and $\gamma \in \Gamma$ such that $x = f(x)\gamma x$. If $f(x)\beta m \in f(x)\Gamma M$ where $\beta \in \Gamma$ and $m \in M$, define $g: R \to M$ by $g(r) = r\beta m$, clear that g is an R_Γ −homomorphism, so $h = gf: M \to M$ is R_Γ −endomorphism and $f(x)\beta m = g\big(f(x)\big) = h(x)$, but $h(x) \in \langle x \rangle$ since M is duo, hence $f(x) \in \big(\langle x \rangle :_{R_\Gamma} M\big)$, thus M is fully R_Γ −idempotent by proposition(2.4).

Corollary (4.6): Let M be multiplication regular R_Γ −module. Then M is fully R_Γ −idempotent.

Proposition (4.7): Let M be an R_Γ −module. Then the following statements are equivalent:

1- M is regular.

2- For each R_Γ −module K, R_Γ −homomorphism $h: K \to M$ and $x \in h(K)$, there exists R_Γ −homomorphism $g: M \to K$ (g depends on x) such that $x = h(g(x))$.

3- For each R_Γ −homomorphism $h: R \to M$ and $\in h(R)$, there is R_Γ −homomorphism $g: M \to R$ such that $x = h(g(x))$.

Proof*:*

(1)⇒(2) Assume $h: K \to M$ is R_Γ −homomorphism and $x \in h(K)$, then there exists $q \in K$ such that $x = h(q)$, since M is regular, then there exists an R_Γ −homomorphism $f: M \to R$ such that $x = f(x)\gamma x$ for some $\gamma \in \Gamma$, define $g: M \to Q$ by $g(m) = f(m)\gamma q$, then g is an R_Γ −homomorphism and $h(g(x)) = h(f(x)\gamma q) = f(x)\gamma h(q) = f(x)\gamma x = x$. (2)⇒(3) Clear. (3)⇒(1) For each $x \in M$, define an R_Γ −homomorphism $h: R \to M$ by $h(r) = r\gamma_\circ x$, then there exists R_Γ −homomorphism $g: M \to R$ such that $x = h\big(g(x)\big) = g(x)\gamma_\circ x$.

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