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Negative Coefficients Subclass of Multivalent Functions

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Abstract

In this paper, we define a new subclass L℘ of multivalent functions defined by the generalized integral operator with negative coefficients in the open unit disk U. We also give and study some interesting properties such as coefficient estimates, subordination theorems and integral means inequalities by using the famous Littlewood's subordination theorem. Finally, we conclude a type of inequalities that is upper bound and lower bound for topology multivalent functions of all analytic functions.

Keywords: Multivalent function, Generalized integral operator, Integral means.

المعاملات السالبة كصنف جزئي من الدوال متعددة التكافؤ

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الخلاصة

في هذا البحث تم تعريف صفوف جزئية جديدة للدوال متعددة التكافؤ L℘ بواسطة مؤثر التكامل العام مع المعاملات السالبة في قرص الوحدة U. سوف نقدم بعض المبرهنات للتبعية التفاضلية بالإضافة الى الخواص التكاملية مثل تقديرات المعاملات والمتراجحات وكذلك تم استخدام المبرهنة الشهيرة لتل وود للتبعية التفاضلية وكذلك استنتجنا أنواع من المتراجحات ذات القيد الأعلى والاسفل.

1- Introduction:

Let L_1 be the class of functions of the form:

$$f(\omega) = \omega^{\wp+\hbar} + \sum_{k=2}^{\infty} a_k\ \omega^{k+\wp+\hbar}, (0 \le \hbar < 1, \wp \in \mathbb{N} = \{1,2,\dots\}) \quad (1)$$

which are analytic and multivalent in the open unit disk $U = \{\omega \in \mathbb{C}: |\omega| < 1\}$ [1].
Let L_2 be the subclass of L_1 of functions of the form:

$$f(\omega) = \omega^{\wp+\hbar} - \sum_{k=2}^{\infty} a_k\ \omega^{k+\wp+\hbar}, \qquad (a_k \ge 0, \omega \in U). \quad (2)$$

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Observe that the authors introduced and studied some classes of analytic topology functions such as the form (1) in [1]. See also ([2], [3])

$$\left|\frac{(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}-(f(\omega))''}{\frac{-\mu}{\wp+\hbar}\left(f(\omega)\right)''+2\mu(\wp+\hbar-1)z^{\wp+\hbar-2}}\right|<\gamma, \tag{3}$$

where $\delta>0, c>-(\wp+\hbar), 0<\mu<1, 0<\gamma\leq\wp+\hbar, 0\leq\hbar<1$.

2- Coefficient Estimates:

Theorem 1*: Let* $f\in L_2$ *be a function that is defined by (2), then*
$f\in L\wp$ if and only if

$$\sum_{k=2}^{\infty}a_k\frac{\left((k+\wp+\hbar)(k+\wp+\hbar-1)\right)(\mu+1)}{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}\leq 1, \tag{4}$$

where $\delta>0, c>-(\wp+\hbar), 0\leq\hbar<1$ *and* $0<\gamma\leq\wp+\hbar$.

Proof: Suppose that the inequality (4) is verified and let $|\omega|=1$, we obtain

$$\left|(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}-(f(\omega))''\right|-\gamma\left|\frac{-\mu}{\wp+\hbar}\left(f(\omega)\right)''+2\mu(\wp+\hbar-1)\omega^{\wp+\hbar-2}\right|$$

$$=\left|\sum_{k=2}^{\infty}(k+\wp+\hbar)(k+\wp+\hbar-1)a_k\omega^{k+\wp+\hbar-2}\right|\quad-\gamma\left|\mu(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}+\sum_{k=2}^{\infty}\mu(k+\wp+\hbar)(k+\wp+\hbar-1)a_k\omega^{k+\wp+\hbar-2}\right| \tag{5}$$

$$\leq\sum_{k=2}^{\infty}(k+\wp+\hbar)(k+\wp+\hbar-1)a_k-\gamma\mu(\wp+\hbar)(\wp+\hbar-1)$$

$$+\sum_{k=2}^{\infty}\mu(k+\wp+\hbar)(k+\wp+\hbar-1)\,a_k$$

By the hypothesis, we obtain.

$$\leq\sum_{k=2}^{\infty}\left((k+\wp+\hbar)(k+\wp+\hbar-1)\right)(\mu+1)a_k\leq\gamma\mu(\wp+\hbar)(\wp+\hbar-1).$$

Therefore, by maximum modulus principle, we have $f\in L_2$.

Conversely, let $f\in L\wp$,then

$$\left|\frac{(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}-(f(\omega))''}{\frac{-\mu}{\wp+\hbar}\left(f(\omega)\right)''+2\mu(\wp+\hbar-1)\omega^{\wp+\hbar-2}}\right|<\gamma, (\omega\in U).$$

that is

$$\frac{\left|\sum_{k=2}^{\infty}(k+\wp+\hbar)(k+\wp+\hbar-1)a_k\omega^{k+\wp+\hbar-2}\right|}{\left|\mu(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}+\sum_{k=2}^{\infty}\mu(k+\wp+\hbar)(k+\wp+\hbar-1)\,a_k\omega^{k+\wp+\hbar-2}\right|}<\gamma, \tag{6}$$

Since$|\operatorname{Re}(\omega)|\leq|f(\omega)|$ for all ω, we have

$$\left|Re\left\{\frac{\sum_{k=2}^{\infty}(k+\wp+\hbar)(k+\wp+\hbar-1)a_k\omega^{k+\wp+\hbar-2}}{\mu(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}+\sum_{k=2}^{\infty}\mu(k+\wp+\hbar)(k+\wp+\hbar-1)a_k\omega^{k+\wp+\hbar-2}}\right\}\right|<\gamma, \tag{7}$$

$$\frac{\sum_{k=2}^{\infty}(k+\wp+\hbar)(k+\wp+\hbar-1)a_k}{\mu(\wp+\hbar)(\wp+\hbar-1)\omega^{\wp+\hbar-2}+\sum_{k=2}^{\infty}\mu(k+\wp+\hbar)(k+\wp+\hbar-1)a_k\omega^{k+\wp+\hbar-2}}\leq\gamma$$

which gives (4).

Finally, the result is directly obtained with external function f which is given by:

$$f(\omega)=\omega^{\wp+\hbar}-\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\omega^{k+\wp+\hbar}, k\geq 2. \qquad (8)$$

Corollary 1: let $f\in L(\wp,\hbar,\delta,\gamma)$, then

$$a_k\leq\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}, \qquad k\geq 2. \qquad (9)$$

The equality (9) is satisfied for the function f that is given by (8).

3. Integral means inequality for the class $L\wp$

In 1925, Littlewood [4] proved the following subordination theorem, see also [5]and [6].

Definition 1 [6], [7] : Let f and g be analytic which are in the unit disk U. Then we say that f is subordinate to g, denoted by $f\prec g$ *or* $f(\omega)\prec g(\omega)$, if there exists a Schwarz function w with $w(0)=0$, $|w(\omega)|<1$ $(\omega\in U)$ which is analytic in U, such that

$f(\omega)=g(w(\omega))$ $(\omega\in U)$. In particular, if the function f is univalent in U, we have the following equivalence relationship holds [10]

$f(\omega)\prec g(\omega)$ $(\omega\in U)$ if and only if $f(0)=g(0)$ and $f(U)\subset g(U)$.

Theorem 2: (Littlewood [4], [8]) If f and g are two analytic functions in U such that $f\prec g$, then for $\alpha>0$ and $\omega=r\,e^{i\vartheta}$ and $0<r<1$.

$$\int_0^{2\pi}|f(\omega)|^{\alpha}\,d\vartheta\leq\int_0^{2\pi}|(g(\omega))|^{\alpha}\,d\vartheta.$$

This theorem is needed to prove the following theorem.

Theorem 3: Let $F\in L\wp$ and suppose that f_k is defined by

$$f_k(z)=\omega^{\wp+\hbar}-\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\omega^{k+\wp+\hbar}, \qquad k\geq 2.$$

If we have the following analytic function w which is given by

$$\{w(\omega)\}^{k-\wp-\hbar}=\frac{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}\sum_{k=2}^{\infty}a_k\ \omega^k.$$

This implies that for $\omega=r\,e^{i\vartheta}$ and $0<r<1$.

$$\int_0^{2\pi}\left|f\left(r\,e^{i\vartheta}\right)\right|^{\alpha}d\vartheta\leq\int_0^{2\pi}\left|f_{k+p+\hbar}\left(r\,e^{i\vartheta}\right)\right|^{\alpha}d\vartheta, \qquad \alpha>0.$$

Proof: We have to show that

$$\int_0^{2\pi}\left|1-\sum_{k=2}^{\infty}a_k\ \omega^k\right|^{\alpha}d\vartheta\leq\int_0^{2\pi}\left|1-\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\omega^k\right|^{\alpha}d\vartheta.$$

It would be enough to show that by using Littlewood 's subordination theorem,

$$1-\sum_{k=2}^{\infty}a_k\ \omega^k\prec 1-\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\omega^k.$$

by setting

$$1-\sum_{k=2}^{\infty}a_k\ \omega^k=1-\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\{w(\omega)\}^k.$$

We have

$$\{w(\omega)\}^{k-\wp-\hbar}=\frac{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}\sum_{k=2}^{\infty}a_k\ \omega^k,$$

Which easily yields that *w (0) = 0.* Moreover, by using (4), we obtain

$$|\{w(\omega)\}|^{k-\wp-\hbar} = \left|\frac{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}\sum_{k=2}^{\infty} a_k\ \omega^k\right|$$
$$\leq \frac{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}\sum_{k=2}^{\infty} a_k\ |\omega|^{k-\wp-\hbar} \leq |\omega| < 1.$$

Theorem 4: *Let* $\alpha > 0.$ *if* $f \in L(\wp,\hbar,\delta,\gamma)$ *and*

$$f_k(\omega) = \omega^{\wp+\hbar} - \frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\omega^{k+\wp+\hbar}, k \geq 2.$$

Then for $\omega = r\, e^{i\vartheta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|f'(\mathrm{r}\, e^{i\vartheta})\right|^{\alpha} d\vartheta \leq \int_0^{2\pi} \left|f'_{k+p+\eta}(\mathrm{r}\, e^{i\vartheta})\right|^{\alpha} d\vartheta,$$

Proof: It is enough to show that

$$1 - \sum_{k=2}^{\infty}\left(\frac{k+\wp+\hbar}{\wp+\hbar}\right) a_k\ \omega^k$$
$$\prec 1 - \frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}\left(\frac{k+\wp+\hbar}{\wp+\hbar}\right)\omega^k.$$

This follows because

$$|\{w(\omega)\}|^{k-\wp-\hbar} = \left|\sum_{k=2}^{\infty}\frac{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)}{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)} a_k\ \omega^k\right|$$
$$\leq |\omega|^{k-\wp-\hbar}\sum_{k=2}^{\infty}\frac{\big((k+\wp+\hbar)(k+\wp+\hbar-1)\big)(\mu+1)}{\mu\gamma(\wp+\hbar)(\wp+\hbar-1)} a_k \leq\ |\omega|^{k-\wp-\hbar} \leq |\omega|.$$

Now, we discuss for the integral means inequalities for $f \in L\wp$ and g that are defined by

$$g(\omega) = \omega^{\wp+\hbar} - \mathrm{b}_{\mathrm{i}+\wp+\hbar}\omega^{\mathrm{i}+\wp+\hbar} - \mathrm{b}_{2\mathrm{i}}\omega^{2\mathrm{i}}, b_i \geq 0, i \geq 2. \tag{10}$$

Theorem 5: Let $f \in L\wp$ and g is given by (10). If f satisfies

$$\sum_{k=2}^{\infty} a_k \leq b_{2i-\wp-\hbar} + b_i, \qquad (b_i < b_{2i-\wp-\hbar}) \tag{11}$$

and there exists a function w which is an analytic such that

$$\mathrm{b}_{2\mathrm{i}}(\mathrm{w}(\omega))^{2\mathrm{i}-\wp-\hbar} + b_i\big(w(\omega)\big)^i - \sum_{k=2}^{\infty} a_k\ \omega^k\ = 0.$$

Then for $\alpha > 0$ and $\omega = \mathrm{r}\, e^{i\vartheta}$ with $0 < r < 1$

$$\int_0^{2\pi} |f(\omega)|^{\alpha}\, d\vartheta \leq \int_0^{2\pi} |(\mathrm{g}(\omega))|^{\alpha}\, d\vartheta.$$

Proof: By putting $\omega = \mathrm{r}\ \mathrm{e}^{\mathrm{i}\vartheta}$ and $0 < r < 1$, we see that

$$\int_0^{2\pi} |f(\omega)|^\alpha \, d\vartheta = \int_0^{2\pi} \left| \omega^{\wp+\hbar} - \sum_{k=2}^{\infty} a_k \ \omega^{k+\wp+\hbar} \right|^\alpha d\vartheta$$

$$= r^{(\wp+\hbar)\alpha} \int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} a_k \ \omega^k \right|^\alpha d\vartheta$$

And

$$\int_0^{2\pi} |g(\omega)|^\alpha \, d\vartheta = \int_0^{2\pi} \left| \omega^{\wp+\hbar} - b_{i+\wp+\hbar} \omega^{i+\wp+\hbar} - b_{2i+\wp+\hbar-\wp-\hbar} \, \omega^{2i+\wp+\hbar-\wp-\hbar} \right|^\alpha d\vartheta$$

$$= r^{(\wp+\hbar)\alpha} \int_0^{2\pi} \left| 1 - b_{i+\wp+\hbar} \omega^i - b_{2i+\wp+\hbar-p-\eta} \omega^{2i-\wp-\hbar} \right|^\alpha d\vartheta.$$

By applying Theorem (2), we have to show that

$$1 - \sum_{k=2}^{\infty} a_k \ a^k \prec 1 - b_{i+\wp+\hbar} \omega^i - b_{2i} \omega^{2i-\wp-\hbar} \, .$$

Now we define the function w in this way

$$1 - \sum_{k=2}^{\infty} a_k \ a^k = 1 - b_{i+\wp+\hbar} (w(\omega))^i - b_{2i} (w(\omega))^{2i-\wp-\hbar}$$

or by

$$b_{2i} (w(\omega))^{2i-\wp-\hbar} + b_i \big(w(\omega)\big)^i - \sum_{k=2}^{\infty} a_k \ \omega^k = 0. \tag{12}$$

Since for $\omega = 0$

$$b_{2i} (w(\omega))^{2i-\wp-\hbar} + b_i \big(w(\omega)\big)^i = 0,$$

Then there exists a function w which is analytic in U such that $w(0) = 0$.

After, we show that the analytic function w satisfies $|w(\omega)| < 1$ $(\omega \in U)$ for the condition (11). From (11), we know that

$$\left| b_{2i} (w(\omega))^{2i-\wp-\hbar} + b_i \big(w(\omega)\big)^i \right| = \left|\sum_{k=2}^{\infty} a_k \ \omega^k \right| < \sum_{k=2}^{\infty} a_k \ \text{ for } \ \omega \in U.$$

Therefore, $b_{2i} |w(\omega)|^{2i-\wp-\hbar} + b_i |w(\omega)|^i - \sum_{k=2}^{\infty} a_k < 0.$ (13)

Letting $t = | w(\omega)|^i$ $(t \geq 0)$ in (14) and G(t) be a function can be defined as follows

$$G(t) = b_{2i-p-\hbar} t^{i-\wp-\hbar} + b_i t - \sum_{k=2}^{\infty} a_k.$$

If $G(1) \geq 0$, we obtain that $t < 1$ for $G(t) < 0$. Indeed, we have

$$G(1) = b_{2i-\wp-\hbar} + b_i - \sum_{k=2}^{\infty} a_k \geq 0,$$

that is $\sum_{k=2}^{\infty} a_k \leq b_{2i-\wp-\hbar} + b_i$.

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