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The Classical Continuous Optimal Control for Quaternary parabolic boundary value problem

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Abstract

The aim of this paper is to study the quaternary classical continuous optimal control for a quaternary linear parabolic boundary value problems(QLPBVPs). The existence and uniqueness theorem of the continuous quaternary state vector solution for the weak form of the QLPBVPs with given quaternary classical continuous control vector (QCCCV) is stated and proved via the Galerkin Method. In addition, the existence theorem of a quaternary classical continuous optimal control vector governinig by the QLPBVPs is stated and demonstrated. The Fréchet derivative for the cost function is derived. Finally, the necessary conditions for the optimality theorem of the proposed problem is stated and demonstrated.

Keywords: Quaternary Linear Parabolic Boundary Value Problems, Fréchet derivative , Necessary Conditions for the Optimality.

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السيطرة التقليدية الأمثلية المستمرة لمسائل القيم الحدودية الرباعية المكافئة

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قسم الرياضيات , كلية العلوم , الجامعة المستنصرية, بغداد , العراق

الخلاصة

هدفنا في هذا العمل هو دراسة السيطرة التقليدية الأمثلية المستمرة لمسائل القيم الحدودية الرباعية المكافئة، تم ذكر نص مبرهنة وجود ووحدانية الحل لمتجه الحالة للصيغة الضعيفة لرباعي مسائل القيم الحدودية عندما يكون متجه السيطرة التقليدية المستمرة معلوماً، بواسطة طريقة كاليركن. أيضاً، تم برهان مبرهنة وجود متجه رباعي لسيطرة أمثلية تقليدية مستمرة لهذه المسألة. تم ايجاد مشتقة فريشيه لدالة الهدف. أخيراً، تم ذكر نص وبرهان مبرهنة الشروط الضرورية لوجود متجه سيطرة مستمرة تقليدية لهذه المسألة المقترحة.

1. Introduction

Optimal control problems have a big role in many real life applications, for examples in economic[1], Medicine[2], Engineering[3], Robots[4] and many other areas in the sciences. These problems are usually ruled either by ordinary differential equations or by partial differential equations (PDEs), in particular, optimal control problems are studied for systems that are ruling by PDEs of elliptic or parabolic or hyperbolic type are achived and considered

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in [5 –10]. On the other hand, the authors in [11-13] disscsed and achieved the optimal control problems for systems ruling by triple PDEs of the previous three types.
In this work, we look at the optimal control problems that is formed by QLPBVPs. We also prove the existence and uniqueness theorem for the weak form of the continuous quaternary state vector solution by using the Galerkin Method when the quaternary classical continuous control vector is known. In addition, the existence theorem for a quaternary classical continuous optimal control vector is stated and demonstrated. The Fréchet derivative for the cost function is derived. We study the solution for quaternary adjoint boundary value problems related to the QLPBVPs. Finally, the necessary conditions for the optimality theorem of the proposed optimal control problem is stated and demosntrated.

2. Description of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded region with boundary $\Gamma = \partial\Omega$, $x = (x_1, x_2)$, $Q = I \times \Omega$, $I = [0, T]$, $\Gamma = \partial\Omega$, $\Sigma = \Gamma \times$ I. The quaternary classical continuous optimal control consists of the quaternary atate equations, which are given by the following QLPBVP:

$$y_{1t} - \Delta y_1 + y_1 - y_2 + y_3 + y_4 = f_1(x,t) + u_1 , \quad \text{in } Q \quad (1)$$
$$y_{2t} - \Delta y_2 + y_1 + y_2 - y_3 - y_4 = f_2(x,t) + u_2 , \quad \text{in } Q \quad (2)$$
$$y_{3t} - \Delta y_3 - y_1 + y_2 + y_3 + y_4 = f_3(x,t) + u_3 , \quad \text{in } Q \quad (3)$$
$$y_{4t} - \Delta y_4 - y_1 + y_2 - y_3 + y_4 = f_4(x,t) + u_4 , \quad \text{in } Q \quad (4)$$

with the following boundary conditions and initial conditions:

$$y_i(x,t) = 0 , \qquad \forall\, i = 1,2,3,4. \quad \text{on } \Sigma \quad (5)$$
$$y_i(x,0) = y_i^0(x) , \qquad \forall\, i = 1,2,3,4. \quad \text{on } \Omega \quad (6)$$

Where, $(f_1, f_2, f_3, f_4) = \big(f_1(x,t), f_2(x,t), f_3(x,t), f_4(x,t)\big) \in \big(L^2(Q)\big)^4$ is a given vector of functions for each $\vec{u} = (u_1, u_2, u_3, u_4) = \big(u_1(x,t), u_2(x,t), u_3(x,t), u_4(x,t)\big) \in \big(L^2(Q)\big)^4$ that is a admissible quaternary continuous classical control vector and $\vec{y} = (y_1, y_2, y_3, y_4) = \big(y_{u_1}, y_{u_2}, y_{u_3}, y_{u_4}\big) \in \big(\mathcal{H}^2(\Omega)\big)^4$ is the continuous quaternary state vector solution corresponding to the control vector $\vec{u}$.

The set of admissible quaternary continuous classical control vector (with $\vec{U}$ is a convex set) is defined by

$\overrightarrow{W_A} = \{\, \vec{u} = (u_1, u_2, u_3, u_4) \in \big(L^2(Q)\big)^4 \mid \vec{u} \in \vec{U} = U_1 \times U_2 \times U_3 \times U_4 \subset \mathbb{R}^4 \ a.e.\, in\, Q\}$,

The cost function is defined by

$$Min.\, G_0(\vec{u}) = \frac{1}{2}\left(\|y_1 - y_{1d}\|^2_{L^2(Q)} + \|y_2 - y_{2d}\|^2_{L^2(Q)} + \|y_3 - y_{3d}\|^2_{L^2(Q)} + \|y_4 - y_{4d}\|^2_{L^2(Q)}\right)$$
$$+ \frac{\beta}{2}\left(\|u_1\|^2_{L^2(Q)} + \|u_2\|^2_{L^2(Q)} + \|u_3\|^2_{L^2(Q)} + \|u_4\|^2_{L^2(Q)}\right), \beta > 0 \quad (7)$$

Let $\vec{V} = V_1 \times V_2 \times V_3 \times V_4 = \left\{\vec{v} :\ \vec{v} = (v_1, v_2, v_3, v_4) \in \big(\mathcal{H}^1(\Omega)\big)^4 , \vec{v} = 0 \ on\ \partial\Omega\right\}$,

The weak formulation of the QSEs:

The weak form of $(1) - (6)$ with $\vec{y} \in \big(\mathcal{H}_0^1(\Omega)\big)^4$ is given by

$$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1 + u_1, v_1), \quad (8.a)$$
$$(y_1^0, v_1) = (y_1(0), v_1), \qquad \forall v_1 \in V_1 \quad (8.b)$$
$$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2 + u_2, v_2), \quad (9.a)$$
$$(y_2^0, v_2) = (y_2(0), v_2), \qquad \forall v_2 \in V_2 \quad (9.b)$$
$$\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3 + u_3, v_3), \quad (10.a)$$
$$(y_3^0, v_3) = (y_3(0), v_3), \qquad \forall v_3 \in V_3 \quad (10.b)$$
$$\langle y_{4t}, v_4\rangle + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4 + u_4, v_4), \quad (11.a)$$
$$(y_4^0, v_4) = (y_4(0), v_4), \qquad \forall v_4 \in V_4. \quad (11.b)$$

The following assumption is important in our study to the quaternary classical continuous optimal control.

2. Assumption (A):

The function f_i, (for all $i = 1,2,3,4$) in the R.H.S. of (8.a), (9.a), (10.a), and (11.a) satisfy the following condition: $|f_i| \le \eta_i(x,t)$, where $(x,t) \in Q$ and $\eta_i \in L^2(Q,\mathbb{R})$.

3. The solution of the weak form

The next theorem gives the existence and uniqness for the the continuous quaternary state vector solution**.**

Theorem (3.1)

If the assumption (A) holds, then the weak form ((8) – (11)) has a unique solution $\vec{y} \in \left(L^2(I,V)\right)^4$ with $\vec{y}_t = (y_{1t}, y_{2t}, y_{3t}, y_{4t}) \in \left(L^2(I,V^*)\right)^4$ for each given QCCCV, $\vec{u} \in \left(L^2(Q)\right)^4$.

Proof: Let $\vec{V}_n = (V_{1n} \times V_{2n} \times V_{3n} \times V_{4n}) = (V_n \times V_n \times V_n \times V_n) \subset \vec{V}$ be the set of piecewise affine functions in Ω for any n such that $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n})$ with $v_{in} \in \mathrm{V_n} (\forall i = 1,2,3,4)$ and $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})$, let $\vec{y}_n$ be an approximation for $\vec{y}$ such that

$$y_{in} = \sum_{j=1}^{n} c_{ij}(t) v_{ij}(x) \qquad \forall i = 1, 2, 3, 4. \tag{12}$$

where $c_{ij}(t)$ is an unknown function of t for all $i = 1,2,3,4$, $j = 1,2, \dots \dots, n$.

Hence, ((8) – (11)) for all$v_i \in V_n$ become as follows:

$$\langle y_{1nt}, v_1 \rangle + (\nabla y_{1n}, \nabla v_1) + (y_{1n} - y_{2n} + y_{3n} + y_{4n}, v_1) = (f_1 + u_1, v_1) \tag{13.a}$$
$$(y_{1n}^0, v_1) = (y_1^0, v_1), \tag{13.b}$$
$$\langle y_{2nt}, v_2 \rangle + (\nabla y_{2n}, \nabla v_2) + (y_{1n} + y_{2n} - y_{3n} - y_{4n}, v_2) = (f_2 + u_2, v_2), \tag{14.a}$$
$$(y_{2n}^0, v_2) = (y_2^0, v_2), \tag{14.b}$$
$$\langle y_{3nt}, v_3 \rangle + (\nabla y_{3n}, \nabla v_3) + (-y_{1n} + y_{2n} + y_{3n} + y_{4n}, v_3) = (f_3 + u_3, v_3), \tag{15.a}$$
$$(y_{3n}^0, v_1) = (y_3^0, v_3), \tag{15.b}$$
$$\langle y_{4nt}, v_4 \rangle + (\nabla y_{4n}, \nabla v_4) + (-y_{1n} + y_{2n} - y_{3n} + y_{4n}, v_4) = (f_4 + u_4, v_4), \tag{16.a}$$
$$(y_{4n}^0, v_4) = (y_4^0, v_4), \tag{16.b}$$

Where, $y_{in}^0 = y_{in}^0(x) = y_{in}(x,0) \in V_n \subset V_i \subset L^2(\Omega)$ is the projection of y_i^0 with respect to the norm $\|.\|_{L^2(\Omega)}$, i.e. $\left\|y_{in}^0 - y_i^0\right\|_{L^2(\Omega)} \le \left\|y_i^0 - v_i\right\|_{L^2(\Omega)}$, for all $v_i \in V_n$, $i = 1,2,3,4$.

By substituting (12) for $i = 1, 2, 3, 4$ in (13) – (16) ,respectively, and then setting $v_i = v_{il}$, for all $i = 1, 2, 3, 4$ and $j = 1,2, \dots \dots, n$, one obtains that the equations from(13) to (16) are equivalent to the following linear system of ODEs of first order with initial conditions that has a unique solution:

$$A_1 \grave{C}_1(t) + B_1 C_1(t) - D C_2(t) + E C_3(t) + F C_4(t) = b_1, \tag{17.a}$$
$$A_1 C_1(0) = b_1^0 , \tag{17.b}$$
$$A_2 \grave{C}_2(t) + B_2 C_2(t) + K C_1(t) - M C_3(t) - N C_4(t) = b_2 , \tag{18.a}$$
$$A_2 C_2(0) = b_2^0 , \tag{18.b}$$
$$A_3 \grave{C}_3(t) + B_3 C_3(t) - P C_1(t) + Q C_2(t) + S C_4(t) = b_3 , \tag{19.a}$$
$$A_3 C_3(0) = b_3^0 , \tag{19.b}$$
$$A_4 \grave{C}_4(t) + B_4 C_4(t) - X C_1(t) + Y C_2(t) - Z C_3(t) = b_4 , \tag{20.a}$$
$$A_4 C_4(0) = b_4^0 , \tag{20.b}$$

Where $A_i = (a_{ilj})_{n\times n}$, $a_{ilj} = (v_{ij}, v_{il})$, $B_i = (b_{ilj})_{n\times n}$, $b_{ilj} = (\nabla v_{ij}, \nabla v_{il}) + (v_{ij}, v_{il})$, $D = (d_{lj})_{n\times n}$, $d_{lj} = (v_{2j}, v_{1l})$,$E = (e_{lj})_{n\times n}$,$e_{lj} = (v_{3j}, v_{1l})$,$F = (f_{lj})_{n\times n}$,$f_{lj} = (v_{4j}, v_{1l})$,$K = (k_{lj})_{n\times n}$, $k_{lj} = (v_{1j}, v_{2l})$, $= (m_{lj})_{n\times n}$,$m_{lj} = (v_{3j}, v_{2l})$,$N = (n_{lj})_{n\times n}$,$n_{lj} = (v_{4j}, v_{2l})$,$P = (p_{lj})_{n\times n}$,$p_{lj} = (v_{1j}, v_{3l})$,$Q = (q_{lj})_{n\times n}$,$q_{lj} = (v_{2j}, v_{3l})$, $S = (s_{lj})_{n\times n}$,$s_{lj} = (v_{4j}, v_{3l})$,$X = (x_{lj})_{n\times n}$, $x_{lj} = (v_{1j}, v_{4l})$, $Y = (y_{lj})_{n\times n}$, $y_{lj} = (v_{2j}, v_{4l})$,$Z = (z_{lj})_{n\times n}$,$z_{lj} = (v_{3j}, v_{4l})$ $b_i^0 = (b_{il}^0)$, $b_{il}^0 = (y_i^0, v_{il})$, $b_i = (b_{il})_{n\times 1}$, $b_{il} = (f_i + u_i, v_{il})$, $\grave{C}_i(t) = (\grave{c}_{il}(t))_{n\times 1}$ $C_i(t) = \left(c_{ij}(t)\right)_{n\times 1}$, $C_i(0) = \left(c_{ij}(0)\right)_{n\times 1}$, $\forall\, l = 1,2,\ldots\ldots,n$, $\forall i = 1,2,3,4$.

To prove the norm $\|\vec{y}_n^0\|_{L^2(\Omega)}$ is bounded, we have $y_1^0 = y_1^0(x) \in L^2(\Omega)$, then there is a sequence $\{v_{1n}^0\}$, $v_{1n}^0 \in V_n$, such that $v_{1n}^0 \longrightarrow y_1^0$ (strongly) in $L^2(\Omega)$, by applying the projection theorem [14] with (13.b), one has
$\|y_{1n}^0 - y_1^0\|_{L^2(\Omega)} \le \|y_1^0 - v_1\|_{L^2(\Omega)}$, $\forall\, v_1 \in V$, then for all $v_{1n}^0 \in V \subset V$, and for all n, we have
$\|y_{1n}^0 - y_1^0\|_{L^2(\Omega)} \le \|y_1^0 - v_{1n}^0\|_{L^2(\Omega)}$ this imples that $y_{1n}^0 \longrightarrow y_1^0$ (strongly) in $L^2(\Omega)$ with $\|y_{1n}^0\|_{L^2(\Omega)} \le b_1$.
Similarly, one has that $\|y_{2n}^0\|_{L^2(\Omega)} \le b_2$, $\|y_{3n}^0\|_{L^2(\Omega)} \le b_3$ and $\|y_{4n}^0\|_{L^2(\Omega)} \le b_4$,this leads to $\left\|\vec{\boldsymbol{y}}_{\boldsymbol{n}}^{\boldsymbol{0}}\right\|_{\boldsymbol{L}^2(\Omega)}$ is bounded.
Now to prove the norm $\|\vec{y}_n(t)\|_{L^\infty\left(I,L^2(\Omega)\right)}$ and $\|\vec{y}_n(t)\|_{L^2(Q)}$ are bounded.

Set $v_i = y_{in}$, $\forall\, i = 1,2,3,4$, in ((13) – (16)) respectively, if we integrate with respect to t from 0 to T, then collect the obtained four equations together to get
$\int_0^T \langle \vec{y}_{nt}\,, \vec{y}_n\rangle\, dt + \int_0^T [\|y_{1n}\|_{\mathcal{H}^1(\Omega)}^2 + \|y_{2n}\|_{\mathcal{H}^1(\Omega)}^2 + \|y_{3n}\|_{\mathcal{H}^1(\Omega)}^2 + \|y_{4n}\|_{\mathcal{H}^1(\Omega)}^2] dt =$
$\int_0^T [\,(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n}) + (f_4 + u_4, y_{4n})] dt$,
Or

$$\int_0^T \langle \vec{y}_{nt}\,, \vec{y}_n\rangle\, dt + \int_0^T \|\vec{y}_n\|_{\boldsymbol{\mathcal{H}}^1(\Omega)}^2\, dt = \int_0^T [\,(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n}) + (f_4 + u_4, y_{4n})] dt\,, \tag{21}$$

Since $\vec{y}_{nt} \in \left(L^2(I, V^*)\right)^4$ and $\vec{y}_n \in \left(L^2(I, V)\right)^4$, then lemma (1.2) in [15] can be used for the first term in L.H.S. of (21). On the other hand, since the second term is nonnegative,so we take $T = t \in [0, T]$. Finally, we use assumption (A) for the R.H.S. of (21),then it yields to
$\frac{1}{2}\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_{\boldsymbol{L}^2(\Omega)}^2\, dt \le \frac{1}{2}\int_0^t \int_\Omega [(\eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2) + (|u_1|^2 + |u_2|^2 + |u_3|^2 + |u_4|^4)]\, dxdt + \int_0^t \int_\Omega [(|y_{1n}|^2 + |y_{2n}|^2 + |y_{3n}|^2 + |y_{4n}|^2)]\, dxdt$
This implies to

$$\int_0^t \frac{d}{dt}\|\vec{y}_n(t)\|_{\boldsymbol{L}^2(\Omega)}^2\, dt \le \|\eta_1\|_{L^2(Q)}^2 + \|\eta_2\|_{L^2(Q)}^2 + \|\eta_3\|_{L^2(Q)}^2 + \|\eta_4\|_{L^2(Q)}^2 + \|u_1\|_{L^2(Q)}^2 + \|u_2\|_{L^2(Q)}^2 + \|u_3\|_{L^2(Q)}^2 + \|u_4\|_{L^2(Q)}^2 + 2\int_0^t \|y_{1n}\|_{L^2(\Omega)}^2 dt + 2\int_0^t \|y_{2n}\|_{L^2(\Omega)}^2 dt + 2\int_0^t \|y_{3n}\|_{L^2(\Omega)}^2 dt + 2\int_0^t \|y_{4n}\|_{L^2(\Omega)}^2 dt$$

Since $\|\eta_i\|_{L^2(Q)}^2 \le \grave{b}_i$, $\|u_i\|_{L^2(Q)}^2 \le c_i$, $\forall\, i = 1,2,3,4$, and $\|\vec{y}_n(0)\|_{\boldsymbol{L}^2(\Omega)}^2 \le b$. Then,we get
$\|\vec{y}_n\|_{\boldsymbol{L}^2(\Omega)}^2 \le c^* + 2\int_0^t \|\vec{y}_n\|_{\boldsymbol{L}^2(\Omega)}^2 dt$, $c^* = \grave{b}_1 + \grave{b}_2 + \grave{b}_3 + \grave{b}_4 + c_1 + c_2 + c_3 + c_4 + b$.
Using the Continuous Bellman Gronwall Inequality in [16], we obtain the following:
$\|\vec{y}_n(t)\|_{\boldsymbol{L}^2(\Omega)}^2 \le c^* e^{2T} = b^2(c)$, $\forall\, t \in [0, T]$. Then, $\|\vec{y}_n(t)\|_{\boldsymbol{L}^\infty\left(\boldsymbol{I}, \boldsymbol{L}^2(\Omega)\right)} \le b(c)$.
So that we get $\|\vec{y}_n(t)\|_{\boldsymbol{L}^2(\mathbf{Q})} = b_1(c)$.

To prove the norm $\|\vec{y}_n(t)\|^2_{L^2(I,V)}$ is bounded, we use lemma (1.2) in [15], for the first term in L.H.S. of (21), then we use the same above results which are obtained from the R.H.S. of (21), then set $t = T$ and $\|\vec{y}_n(0)\|^2_{L^2(\Omega)}$ is nonnegative. Hence, the equation (21) becomes:

$$\|\vec{y}_n(T)\|^2_{L^2(\Omega)} - \|\vec{y}_n(0)\|^2_{L^2(\Omega)} + 2\int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)} dt \le \|\eta_1\|^2_{L^2(Q)} + \|\eta_2\|^2_{L^2(Q)} + \|\eta_3\|^2_{L^2(Q)} + \|\eta_4\|^2_{L^2(Q)} + \|u_1\|^2_{L^2(Q)} + \|u_2\|^2_{L^2(Q)} + \|u_3\|^2_{L^2(Q)} + \|u_4\|^2_{L^2(Q)} + 2\|\vec{y}_n\|^2_{L^2(Q)}$$

Which gives:

$$\int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)} dt \le b_2^2(c), \text{ with } b_2^2(c) = \frac{\left(b+\grave{b}_1+\grave{b}_2+\grave{b}_3+\grave{b}_4+c_1+c_2+c_3+c_4+2b_1(c)\right)}{2},$$

i.e. $\|\vec{y}_n\|_{L^2(I,V)} \le b_2(c)$.

To disscuss the convergence of the solution, let $\{\vec{V}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{V}$ such that for all $\vec{v} = (v_1, v_2, v_3, v_4) \in \vec{V}$, there exists a sequence $\{\vec{v}_n\}$ with $\vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n}) \in \vec{V}_n$, for all n and $\vec{v}_n \longrightarrow \vec{v}$ stongly in $\vec{V}$, this gives $\vec{v}_n \longrightarrow \vec{v}$ (strongly) in $\left(L^2(\Omega)\right)^4$. Since problem (13) – (16) has a unique solution $\vec{y}_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n})$ for each n with $\vec{V}_n \subset \vec{V}$, hence corresponding to the sequence of subspaces $\{\vec{V}_n\}_{n=1}^{\infty}$, there exists a sequence of (approximation) problems like (13) – (16). Now by substituting $\vec{v} = \vec{v}_n = (v_{1n}, v_{2n}, v_{3n}, v_{4n})$ in these equations for $= 1,2,\ldots$, one gets the following:

$$\langle y_{1nt}, v_{1n}\rangle + (\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n}) - (y_{2n}, v_{1n}) + (y_{3n}, v_{1n}) + (y_{4n}, v_{1n}) = (f_1 + u_1, v_{1n}), \quad \forall v_{1n} \in V_n \tag{22.a}$$

$$(y_{1n}^0, v_{1n}) = (y_1^0, v_{1n}), \quad \forall v_{1n} \in V_n \tag{22.b}$$

$$\langle y_{2nt}, v_{2n}\rangle + (\nabla y_{2n}, \nabla v_{2n}) + (y_{1n}, v_{2n}) + (y_{2n}, v_{2n}) - (y_{3n}, v_{2n}) - (y_{4n}, v_{2n}) = (f_2 + u_2, v_{2n}), \quad \forall v_{2n} \in V_n \tag{23.a}$$

$$(y_{2n}^0, v_{2n}) = (y_2^0, v_{2n}), \quad \forall v_{2n} \in V_n \tag{23.b}$$

$$\langle y_{3nt}, v_{3n}\rangle + (\nabla y_{3n}, \nabla v_{3n}) - (y_{1n}, v_{3n}) + (y_{2n}, v_{3n}) + (y_{3n}, v_{3n}) + (y_{4n}, v_{3n}) = (f_3 + u_3, v_{3n}), \quad \forall v_{3n} \in V_n \tag{24.a}$$

$$(y_{3n}^0, v_{1n}) = (y_3^0, v_{3n}), \quad \forall v_{3n} \in V_n \tag{24.b}$$

$$\langle y_{4nt}, v_{4n}\rangle + (\nabla y_{4n}, \nabla v_{4n}) - (y_{1n}, v_{4n}) + (y_{2n}, v_{4n}) - (y_{3n}, v_{4n}) + (y_{4n}, v_{4n}) = (f_4 + u_4, v_{4n}), \quad \forall v_{4n} \in V_n \tag{25.a}$$

$$(y_{4n}^0, v_{4n}) = (y_4^0, v_{4n}). \quad \forall v_{4n} \in V_n \tag{25.b}$$

Then, problem ((22) – (25)) has a sequence of solutions $\{\vec{y}_n\}_{n=1}^{\infty}$, but from the previous steps, we got $\|\vec{y}_n\|_{L^2(Q)}$ and $\|\vec{y}_n\|_{L^2(I,V)}$ are bounded, then by Alaoglu's Theorem there is a subsequence of $\{\vec{y}_n\}_{n\in\mathbb{N}}$, for simplicity, say again $\{\vec{y}_n\}_{n\in\mathbb{N}}$ such that $\vec{y}_n \longrightarrow \vec{y}$ (weakly) in $\left(L^2(Q)\right)^4$ and in $\left(L^2(I, V)\right)^4$. Now, we multiply both sides of (22.a), (23.a), (24.a) and (25.a) by $\varphi_i \in C^1[0,T]$ for all $i = 1,2,3,4$, respectively, with $\varphi_i(T) = 0$ and $\varphi_i(0) \ne 0$ $for\ all$ $i = 1,2,3,4$, then integrating both sides (IBS) with respect to t from 0 to T, and then integrating by parts the first terms in L.H.S. of each obtained equation, one has:

$$-\int_0^T (y_{1n}, v_{1n})\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1n}, \nabla v_{1n}) + (y_{1n}, v_{1n})]\,\varphi_1(t)dt - \int_0^T (y_{2n}, v_{1n})\varphi_1(t)dt + \int_0^T (y_{3n}, v_{1n})\varphi_1(t)dt + \int_0^T (y_{4n}, v_{1n})\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_{1n})\varphi_1(t)dt + (y_{1n}^0, v_{1n})\varphi_1(0), \tag{26}$$

$-\int_0^T (y_{2n}, v_{2n})\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2n}, \nabla v_{2n}) + (y_{2n}, v_{2n})]\,\varphi_2(t)dt + \int_0^T (y_{1n}, v_{2n})\varphi_2(t)dt - \int_0^T (y_{3n}, v_{2n})\varphi_2(t)dt - \int_0^T (y_{4n}, v_{2n})\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_{2n})\varphi_2(t)dt + (y_{2n}^0, v_{2n})\varphi_2(0)$ (27)

$-\int_0^T (y_{3n}, v_{3n})\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3n}, \nabla v_{3n}) + (y_{3n}, v_{3n})]\,\varphi_3(t)dt - \int_0^T (y_{1n}, v_{3n})\varphi_3(t)dt + \int_0^T (y_{2n}, v_{3n})\varphi_3(t)dt + \int_0^T (y_{4n}, v_{3n})\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_{3n})\varphi_3(t)dt + (y_{3n}^0, v_{3n})\varphi_3(0)$ (28)

$-\int_0^T (y_{4n}, v_{4n})\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_{4n}, \nabla v_{4n}) + (y_{4n}, v_{4n})]\,\varphi_4(t)dt - \int_0^T (y_{1n}, v_{4n})\varphi_4(t)dt + \int_0^T (y_{2n}, v_{4n})\varphi_4(t)dt - \int_0^T (y_{3n}, v_{4n})\varphi_4(t)dt = \int_0^T (f_4 + u_4, v_{4n})\varphi_4(t)dt + (y_{4n}^0, v_{4n})\varphi_4(0)$ (29)

Since $\left.\begin{matrix} v_{in} \longrightarrow v_i \text{ ST in } L^2(\Omega) \\ v_{in} \longrightarrow v_i \text{ ST in } V \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} v_{in}\dot{\varphi}_i \longrightarrow v_i\dot{\varphi}_i \text{ ST in } L^2(Q) \\ v_{in}\varphi_i \longrightarrow v_i\varphi_i \text{ ST in } L^2(I,V) \end{matrix}\right.$

And since $y_{in} \longrightarrow y_i$ (weakly) in $L^2(Q)$ and in $L^2(I,V)$, $v_{in} \longrightarrow v_i$ (weakly) in $L^2(\Omega)$, $y_{in}^0 \longrightarrow y_i^0$ (weakly) in $L^2(\Omega)$ $\forall\, i = 1,2,3,4$, then ((26) – (29)) converge to:

$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1^0, v_1)\varphi_1(0)$ (30)

$-\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt + (y_2^0, v_2)\varphi_2(0)$ (31)

$-\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt + (y_3^0, v_3)\varphi_3(0)$ (32)

$-\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\varphi_4(t)dt + (y_4^0, v_4)\varphi_4(0)$ (33)

Now we have the following two cases:

<u>Case1:</u> Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, for all $i = 1,2,3,4$, substituting these values in (30) – (33) then using integrqtion by parts for the first terms in the L.H.S. of each one of the obtained equations, one gets that:

$\int_0^T \langle y_{1t}, v_1\rangle\varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt$ (34)

$\int_0^T \langle y_{2t}, v_2\rangle\varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt$ (35)

$\int_0^T \langle y_{3t}, v_3\rangle\varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt$ (36)

$\int_0^T \langle y_{4t}, v_4\rangle\varphi_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\varphi_4(t)dt$ (37)

Then ((34) – (37)) gives that $\vec{y}$ is a solution of the wf of the CQSVS ((8) – (11)).

<u>Case2:</u> Choose $\varphi_i \in C^1[0,T]$, s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) \neq 0$, $\forall\, i = 1,2,3,4$.

We integrate the first terms in the L.H.S. of (2.47) to obtain that:

$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0)$ (38)

Moreover, by subtracting (30) form (38), one gets that:

$(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0)$, , $\varphi_1(0) \neq 0$, this implies

$(y_1^0, v_1) = (y_1(0), v_1)$ that means, the initial condition (8.b) holds.
Similarly, one can get that that $(y_i^0, v_i) = (y_i(0), v_i)$, fro all all $i = 2, 3, 4$, that means the initial conditions (9.b),(10.b) and (11.b) hold.
The convergence for $\vec{y}_n$:

Substituting $v_i = y_{in}$, for all $i = 1, 2, 3, 4$, in (13.a), (14.a), (15.a) and (16.a) respectively, then collect the obtained equations together and, then integrating both sides of them from 0 to T. On the other hand, substituting $v_i = y_i$, $\forall i = 1, 2, 3, 4$, in (8.a), (9.a), (10.a) and (11.a) respectively ,then we add them together, and integrate both sides of the obtained equations from 0 to T to obtain that:

$$\int_0^T \langle \vec{y}_{nt}, \vec{y}_n \rangle\, dt + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n}) + (f_4 + u_4, y_{4n})]dt\,, \quad (39)$$

$$\int_0^T \langle \vec{y}_t, \vec{y} \rangle\, dt + \int_0^T \|\vec{y}\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3) + (f_4 + u_4, y_4)]dt\,, \quad (40)$$

Using Lemma (1.2) in [15] for the first terms in the L.H.S. of (39- 40), they become:

$$\frac{1}{2}\|\vec{y}_n(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}_n(0)\|^2_{L^2(\Omega)} + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n}) + (f_4 + u_4, y_{4n})]dt \quad (41)$$

$$\frac{1}{2}\|\vec{y}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}(0)\|^2_{L^2(\Omega)} + \int_0^T \|\vec{y}\|^2_{\mathcal{H}^1(\Omega)}\, dt = \int_0^T [\,(f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3) + (f_4 + u_4, y_4)]dt\,, \quad (42)$$

Since

$$\frac{1}{2}\|\vec{y}_n(T) - \vec{y}(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}_n(0) - \vec{y}(0)\|^2_{L^2(\Omega)} + \int_0^T (\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt = A_1 - B_1 - C_1. \quad (43)$$

Where

$$A_1 = \frac{1}{2}\|\vec{y}_n(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\vec{y}_n(0)\|^2_{L^2(\Omega)} + \int_0^T \|\vec{y}_n\|^2_{\mathcal{H}^1(\Omega)}\, dt,$$

$$B_1 = \frac{1}{2}\big(\vec{y}_n(T), \vec{y}(T)\big) - \frac{1}{2}\big(\vec{y}_n(0) - \vec{y}(0)\big) + \int_0^T \big(\vec{y}_n(t), \vec{y}(t)\big)dt,$$

$$C_1 = \frac{1}{2}\big(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) - \frac{1}{2}\big(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)\big) + \int_0^T \big(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t)\big)dt,$$

Since $\vec{y}_n^0 = \vec{y}_n(0) \longrightarrow \vec{y}^0 = \vec{y}(0)$ ST in $\big(L^2(\Omega)\big)^4$, (43.a)

$\vec{y}_n(T) \longrightarrow \vec{y}(T)$ ST in $\big(L^2(\Omega)\big)^4$. (43.b)

Then, $$\begin{cases} \big(\vec{y}(0), \vec{y}_n(0) - \vec{y}(0)\big) \longrightarrow 0 \\ \big(\vec{y}(T), \vec{y}_n(T) - \vec{y}(T)\big) \longrightarrow 0 \end{cases}, \quad (43.c)$$

And $$\begin{cases} \|\vec{y}_n(0) - \vec{y}(0)\|^2_{L^2(\Omega)} \longrightarrow 0 \\ \|\vec{y}_n(T) - \vec{y}(T)\|^2_{L^2(\Omega)} \longrightarrow 0 \end{cases}. \quad (43.d)$$

Since $\vec{y}_n \longrightarrow \vec{y}$ (weakly) in $\big(L^2(\mathrm{I}, \mathrm{V})\big)^4$, then

$$\int_0^T \big(\vec{y}(t), \vec{y}_n(t) - \vec{y}(t)\big)dt \longrightarrow 0\,, \quad (43.e)$$

again since $\vec{y}_n \longrightarrow \vec{y}$ (weakly) in $\big(L^2(Q)\big)^4$, then

$$\int_0^T [\,(f_1 + u_1, y_{1n}) + (f_2 + u_2, y_{2n}) + (f_3 + u_3, y_{3n}) + (f_4 + u_4, y_{4n})]dt \longrightarrow \int_0^T [\,(f_1 + u_1, y_1) + (f_2 + u_2, y_2) + (f_3 + u_3, y_3) + (f_4 + u_4, y_4)]dt \quad (43.f)$$

Now, when $n \longrightarrow \infty$ in both sides of (43), one has the following results:
1) The first two terms in the L.H.S. of (43) are tending to zero from (43.d).
2) From (41), we have

$$Eq.A_1 = \int_0^T [(f_1+u_1, y_{1n}) + (f_2+u_2, y_{2n}) + (f_3+u_3, y_{3n}) + (f_4+u_4, y_{4n})]dt \xrightarrow[(43.\text{f})]{\text{from}}$$
$$\int_0^T [(f_1+u_1, y_1) + (f_2+u_2, y_2) + (f_3+u_3, y_3) + (f_4+u_4, y_4)]dt.$$

(3) The equation $B_1 \longrightarrow$ L.H.S. of (43.b) $= \int_0^T [(f_1+u_1, y_1) + (f_2+u_2, y_2) + (f_3+u_3, y_3) + (f_4+u_4, y_4)]dt.$

(4) The first two terms in equation C_1 are tending to zero from (43.c) and the last term also tends to zero from (43.e), from these convergences and the results, (43) gives as $n \longrightarrow \infty$:

$$\int_0^T \|\vec{y}_n - \vec{y}\|^2_{\mathcal{H}^1(\Omega)} dt = \int_0^T (\vec{y}_n - \vec{y}, \vec{y}_n - \vec{y}) dt \longrightarrow 0.$$

This implies that $\vec{y}_n \longrightarrow \vec{y}$ (strongly) in $\left(L^2(I,V)\right)^4$.

Now ,we prove the uniqueness of the solution.

Let $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$ are two solutions of the weak form ((8.a)-(11.a)). We start with y_1 *and* $\bar{y}_1$ that means:

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1 + u_1, v_1), \forall v_1 \in V_1,$
$\langle \bar{y}_{1t}, v_1\rangle + (\nabla \bar{y}_1, \nabla v_1) + (\bar{y}_1, v_1) - (\bar{y}_2, v_1) + (\bar{y}_3, v_1) + (\bar{y}_4, v_1) = (f_1 + u_1, v_1), \forall v_1 \in V_1.$

By subtracting the second equation from the first one and substituting $v_1 = y_1 - \bar{y}_1$, we get

$$\langle (y_1 - \bar{y}_1)_t, y_1 - \bar{y}_1\rangle + (y_1 - \bar{y}_1, y_1 - \bar{y}_1) - (y_2 - \bar{y}_2, y_1 - \bar{y}_1) + (y_3 - \bar{y}_3, y_1 - \bar{y}_1) + (y_4 - \bar{y}_4, y_1 - \bar{y}_1) = 0, \quad (44)$$

We use the same previous way to obtain:

$$\langle (y_2 - \bar{y}_2)_t, y_2 - \bar{y}_2\rangle + (y_1 - \bar{y}_1, y_2 - \bar{y}_2) + (y_2 - \bar{y}_2, y_2 - \bar{y}_2) - (y_3 - \bar{y}_3, y_2 - \bar{y}_2) - (y_4 - \bar{y}_4, y_2 - \bar{y}_2) = 0, \quad (45)$$
$$\langle (y_3 - \bar{y}_3)_t, y_3 - \bar{y}_3\rangle - (y_1 - \bar{y}_1, y_3 - \bar{y}_3) + (y_2 - \bar{y}_2, y_3 - \bar{y}_3) + (y_3 - \bar{y}_3, y_3 - \bar{y}_3) + (y_4 - \bar{y}_4, y_3 - \bar{y}_3) = 0, \quad (46)$$
$$\langle (y_4 - \bar{y}_4)_t, y_4 - \bar{y}_4\rangle - (y_1 - \bar{y}_1, y_4 - \bar{y}_4) + (y_2 - \bar{y}_2, y_4 - \bar{y}_4) - (y_3 - \bar{y}_3, y_4 - \bar{y}_4) + (y_4 - \bar{y}_4, y_4 - \bar{y}_4) = 0, \quad (47)$$

Collecting ((44) – (47)), we use Lemma (1.2) in [15] to the frist term of the obtained equations, one obtains that:

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{y} - \vec{\bar{y}}\right\|^2_{L^2(\Omega)} + \left\|\vec{y} - \vec{\bar{y}}\right\|^2_{\mathcal{H}^1(\Omega)} = 0, \quad (48)$$

The second term of the L.H.S. of (48) is nonnegative, then we integrate both sides of (48) with respect to t from 0 to t to obtain for each $t \in I$ that:

$\int_0^t \frac{d}{dt}\left\|\vec{y} - \vec{\bar{y}}\right\|^2_{L^2(\Omega)} \le 0$, this gives $\left\|(\vec{y} - \vec{\bar{y}})(t)\right\|^2_{L^2(\Omega)} \le 0$, then $\left\|\vec{y} - \vec{\bar{y}}\right\|^2_{L^2(\Omega)} = 0$ (49).

Now,we integrate both sides of (48) from 0 to T, and use the given initial conditions and (49), we get

$2\int_0^T \left\|\vec{y} - \vec{\bar{y}}\right\|^2_{\mathcal{H}^1(\Omega)} dt = 0$. Hence, $\left\|\vec{y} - \vec{\bar{y}}\right\|^2_{L^2(\mathbf{I},\mathbf{V})} = 0$. Therefore, $\vec{y} = \vec{\bar{y}}$.

4. The existence of a quaternary classical continuous optimal control problem:

Theorem (4.1): In addition to assumption (A), we assume $\vec{u}$ and $\vec{u} + \overrightarrow{\delta u}$ bounded quaternary classical continuous optimal control vectos in $\left(L^2(Q)\right)^4$ and $\vec{y}$, $\vec{y} + \overrightarrow{\delta y}$ are their corresponding continuous quaternary state vector solution, respectively, then

$\|\delta\vec{y}\|_{L^2\left(I, L^2(\Omega)\right)} \le M\|\delta\vec{u}\|_{L^2(Q)}$, $M \in \mathbb{R}^+$
$\|\delta\vec{y}\|_{L^2(Q)} \le \bar{M}\|\delta\vec{u}\|_{L^2(Q)}$, $\bar{M} \in \mathbb{R}^+$
$\|\delta\vec{y}\|_{L^2(I,V)} \le \bar{M}_1\|\delta\vec{u}\|_{L^2(Q)}$, $\bar{M}_1 \in \mathbb{R}^+$.

Proof: Let $\vec{u}=(u_1,u_2,u_3,u_4)$ and $\vec{\bar{u}}=(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{u}_4)\in\left(L^2(Q)\right)^4$ be two given quaternary classical continuous optimal control vectos, then by Theorem (3.1) there exist $\vec{y}=\left(y_1=y_{u_1},y_2=y_{u_2},y_3=y_{u_3},y_4=y_{u_4}\right)$ and $\vec{\bar{y}}=(\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4)$ its their corresponding continuous quaternary state vector solutions and they satisfy equations ((8) – (11) that means

$$\langle\bar{y}_{1t},v_1\rangle+(\nabla\bar{y}_1,\nabla v_1)+(\bar{y}_1,v_1)-(\bar{y}_2,v_1)+(\bar{y}_3,v_1)+(\bar{y}_4,v_1)=(f_1+\bar{u}_1,v_1), \quad (50.a)$$
$$(\bar{y}_1(0),v_1)=(y_1^0,v_1) \quad (50.b)$$
$$\langle\bar{y}_{2t},v_2\rangle+(\nabla\bar{y}_2,\nabla v_2)+(\bar{y}_1,v_2)+(\bar{y}_2,v_2)-(\bar{y}_3,v_2)-(\bar{y}_4,v_2)=(f_2+\bar{u}_2,v_2) \quad (51.a)$$
$$(\bar{y}_2(0),v_2)=(y_2^0,v_2), \quad (51.b)$$
$$\langle\bar{y}_{3t},v_3\rangle+(\nabla\bar{y}_3,\nabla v_3)-(\bar{y}_1,v_3)+(\bar{y}_2,v_3)+(\bar{y}_3,v_3)+(\bar{y}_4,v_3)=(f_3+\bar{u}_3,v_3), \quad (52.a)$$
$$(\bar{y}_3(0),v_3)=(y_3^0,v_3), \quad (52.b)$$
$$\langle\bar{y}_{4t},v_4\rangle+(\nabla\bar{y}_4,\nabla v_4)-(\bar{y}_1,v_4)+(\bar{y}_2,v_4)-(\bar{y}_3,v_4)+(\bar{y}_4,v_4)=(f_4+\bar{u}_4,v_4), \quad (53.a)$$
$$(\bar{y}_4(0),v_4)=(y_4^0,v_4), \quad (53.b)$$

By subtracting (8.a-b) from (50.a-b), (9.a-b) from (51.a-b), (10.a-b) from (52.a-b) and (11.a-b) from (53.a-b) and setting $\delta y_i=\bar{y}_i-y_i$, $\delta u_i=\bar{u}_i-u_i$, $\forall i=1,2,3,4$, in the obtained equations to get:

$$\langle\delta y_{1t},v_1\rangle+(\nabla\delta y_1,\nabla v_1)+(\delta y_1,v_1)-(\delta y_2,v_1)+(\delta y_3,v_1)+(\delta y_4,v_1)=(\delta u_1,v_1), \quad (54.a)$$
$$(\delta y_1(0),v_1)=0, \quad (54.b)$$
$$\langle\delta y_{2t},v_2\rangle+(\nabla\delta y_2,\nabla v_2)+(\delta y_1,v_2)+(\delta y_2,v_2)-(\delta y_3,v_2)-(\delta y_4,v_2)=(\delta u_2,v_2), \quad (55.a)$$
$$(\delta y_2(0),v_2)=0, \quad (55.b)$$
$$\langle\delta y_{3t},v_3\rangle+(\nabla\delta y_3,\nabla v_3)-(\delta y_1,v_3)+(\delta y_2,v_3)+(\delta y_3,v_3)+(\delta y_4,v_3)=(\delta u_3,v_3), \quad (56.a)$$
$$(\delta y_3(0),v_3)=0, \quad (56.b)$$
$$\langle\delta y_{4t},v_4\rangle+(\nabla\delta y_4,\nabla v_4)-(\delta y_1,v_4)+(\delta y_2,v_4)-(\delta y_3,v_4)+(\delta y_4,v_4)=(\delta u_4,v_4), \quad (57.a)$$
$$(\delta y_4(0),v_4)=0, \quad (57.b)$$

By substituting $v_i=\delta y_i$, $\forall i=1,2,3,4$, in (54.a-b), (55.a-b), (56.a-b) and (57.a-b), respectiely,we add the obtained equations and use Lemma (1.2) in [15] for the first term in the L.H.S., they give:

$$\frac{1}{2}\frac{d}{dt}\|\delta\vec{y}\|^2_{\boldsymbol{L^2(\Omega)}}+\|\delta\vec{y}\|^2_{\boldsymbol{\mathcal{H}^1(\Omega)}}=(\delta u_1,\delta y_1)+(\delta u_2,\delta y_2)+(\delta u_3,\delta y_3)+(\delta u_4,\delta y_4)\ , \quad (58)$$

Since the second term of (58) is non negative, we integrate both sides with respective to t from 0 to t, it becomes:

$$\int_0^t\frac{d}{dt}\|\delta\vec{y}\|^2_{\boldsymbol{L^2(\Omega)}}\le\int_0^t\|\delta u_1\|^2_{L^2(\Omega)}dt+\int_0^t\|\delta y_1\|^2_{L^2(\Omega)}+\int_0^t\|\delta u_2\|^2_{L^2(\Omega)}dt+\int_0^t\|\delta y_2\|^2_{L^2(\Omega)}+$$
$$\int_0^t\|\delta u_3\|^2_{L^2(\Omega)}+\int_0^t\|\delta y_3\|^2_{L^2(\Omega)}+\int_0^t\|\delta u_4\|^2_{L^2(\Omega)}dt+\int_0^t\|\delta y_4\|^2_{L^2(\Omega)}$$
$$=\int_0^t\|\delta\vec{u}\|^2_{L^2(\Omega)}dt+\int_0^t\|\delta\vec{y}\|^2_{L^2(\Omega)}\ ,$$

Then ,

$$\|\delta\vec{y}\|^2_{\boldsymbol{L^2(\Omega)}}\le\|\delta\vec{u}\|^2_{\boldsymbol{L^2(Q)}}+\int_0^t\|\delta\vec{y}\|^2_{\boldsymbol{L^2(\Omega)}}dt$$

By using the continuous Bellman Gronwall Inequality in [16], one gets:

$\|\delta\vec{y}(t)\|^2_{\boldsymbol{L^2(\Omega)}}\le\|\delta\vec{u}\|^2_{\boldsymbol{L^2(Q)}}e^{\int_0^T dt}=M^2\|\delta\vec{u}\|^2_{\boldsymbol{L^2(Q)}}$, $M^2=e^T$, $t\in[0,T]$, this gives $\|\delta\vec{y}\|^2_{\boldsymbol{L^\infty\left(I,L^2(\Omega)\right)}}\le M\|\delta\vec{u}\|^2_{\boldsymbol{L^2(Q)}}$.

Now, since $\|\delta\vec{y}\|^2_{\boldsymbol{L^2(Q)}}\le T max_{t\in[0,T]}\|\delta\vec{y}(t)\|^2_{\boldsymbol{L^2(\Omega)}}\le TM^2\|\delta\vec{u}\|^2_{\boldsymbol{L^2(Q)}}$,

then $\|\delta\vec{y}\|^2_{\boldsymbol{L^2(Q)}}\le\bar{M}\|\delta\vec{u}\|^2_{\boldsymbol{L^2(Q)}}$, $\bar{M}^2=TM^2$, we

integrate both sides of (58) with respect to t from 0 to T, using the same manner which is used for the R.H.S., this gives:

$\|\delta\vec{y}(T)\|^2_{L^2(\Omega)} + 2\int_0^T \|\delta\vec{y}\|^2_{\mathcal{H}^1(\Omega)}dt \le \|\delta\vec{u}\|^2_{L^2(Q)} + \|\delta\vec{y}\|^2_{L^2(Q)} \le \bar{M}^2\|\delta\vec{u}\|^2_{L^2(Q)}$,so that
$\|\delta\vec{y}\|^2_{L^2(I,V)} \le \bar{M}_1^2\|\delta\vec{u}\|^2_{L^2(Q)}$, where $\bar{M}_1^2 = (1+\bar{M}^2)/2$ .Therfore,
$\|\delta\vec{y}\|^2_{L^2(I,V)} \le \bar{M}\|\delta\vec{u}\|^2_{L^2(Q)}$.

Theorem (4.2): With assumption (A), the operator $\vec{u} \longrightarrow \vec{y}_{\vec{u}}$ is continuous from $\left(L^2(Q)\right)^4$ into $(L^\infty\left(I, L^2(\Omega)\right)^4$ or into $\left(L^2(I,V)\right)^4$ or into $\left(L^2(Q)\right)^4$.

Proof: Let $\vec{\bar{u}}$ and $\vec{u}$ be two quaternary classical continuous optimal controls, and $\vec{\bar{y}}$,$\vec{y}$ are their corresponding continuous quaternary state vector solutions, $\delta\vec{u} = \vec{\bar{u}} - \vec{u}$ and $\delta\vec{y} = \vec{\bar{y}} - \vec{y}$, then using the first result in Theorem (4.1), one gets that:
$\vec{\bar{y}} \xrightarrow[L^2(I,L^2(\Omega)]{} \vec{y}$ if $\vec{\bar{u}} \xrightarrow[L^2(Q)]{} \vec{u}$, i.e. The operator $\vec{u} \longmapsto \vec{y}_{\vec{u}}$ is Lipschitz continuous from L^2(Q) into $L^\infty\left(I, L^2(\Omega)\right)$. Similarlly, this operator is also Lipschitz continuous from $L^2(Q)$ into $L^2(Q)$ and into $L^2(I,V)$.

Lemma (4.1) [8]: The norm $\|.\|_{L^2(\Omega)}$ is weakly lower semi continuous.

Lemma (4.2): The cost function (7) is weakly lower semicontinuous.

Proof: From lemma (4.1), the norm $\|\vec{u}\|_{L^2(Q)}$ is weakly lower semicontinuous, on the other hand when $\vec{u}_k \longrightarrow \vec{u}$ (weakly) in $\boldsymbol{L^2(Q)}$, then (by theorem (2.3.2)) $\vec{y}_k \longrightarrow \vec{y} = \vec{y}_{\vec{u}}$ (weakly) in $\boldsymbol{L^2(Q)}$, which gives $\|\vec{y} - \vec{y}_d\|_{L^2(Q)}$ is weakly lower semicontinuous (by Lemma (4.1)), hence $G_0(\vec{u})$ is weakly lower semicontinuous.

Lemma (4.3) [8]: The norm $\|.\|^2_{L^2(Q)}$ is strictly convex.

Theorem (4.3): If the cost function $G_0(\vec{u})$ in (7), is coercive, then there exists a quaternary classical continuous optimal control vector.

Proof: Since $G_0(\vec{u}) \ge 0$ and it is coercive and we seek about the minimum, then there is a minimizing sequence $\{\vec{u}_k\} = \{(u_{1k}, u_{2k}, u_{3k}, u_{4k})\} \in \vec{W}_A$, $\forall\, k$ s.t. $\lim_{n\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\bar{u}}_k \in \vec{W}_A} G_0(\vec{\bar{u}})$, then there is a constant, $C > 0$ s.t. $\|\vec{u}_k\|_{L^2(Q)} \le C$, and then by Alaoglu's Theorem there exists a subsequence of $\{\vec{u}_k\}$, for simplicity say again $\{\vec{u}_k\}$ s.t. $\vec{u}_k \longrightarrow \vec{u}$ (weakly) in $\left(L^2(Q)\right)^4$ as $k \longrightarrow \infty$.
Now, by Theorem (3.1) corresponding to the sequence of control $\{\vec{u}_k\}$ there is a sequence of unique solution $\{\vec{y}_k\}$, but the norm $\|\vec{y}_k\|_{L^\infty\left(I, L^2(\Omega)\right)}$, $\|\vec{y}_k\|_{L^2(Q)}$ and $\|\vec{y}_k\|_{L^2(I,V)}$ are bounded, then also by Alaoglu's Theorem there exists a subsequence for simplicity say again $\{\vec{y}_k\}$of of $\{\vec{y}_k\}$, such $\vec{y}_k \longrightarrow \vec{y}$ (weakly) in $\left(L^\infty\left(I, L^2(\Omega)\right)\right)^4$, in $\left(L^2(Q)\right)^4$ and in $\left(L^2(I,V)\right)^4$.
Now, consider (13.a), (14.a), (15.a) and (16.a) and rewrite them as:
$\langle y_{1kt}, v_1\rangle = -(\nabla y_{1k}, \nabla v_1) - (y_{1k}, v_1) + (y_{2k}, v_1) - (y_{3k}, v_1) - (y_{4k}, v_1) + (f_1 + u_{1k}, v_1)$,
$\langle y_{2kt}, v_2\rangle = -(\nabla y_{2k}, \nabla v_2) - (y_{1k}, v_2) - (y_{2k}, v_2) + (y_{3k}, v_2) + (y_{4k}, v_2) + (f_2 + u_{2k}, v_2)$,
$\langle y_{3kt}, v_3\rangle = -(\nabla y_{3k}, \nabla v_3) + (y_{1k}, v_3) - (y_{2k}, v_3) - (y_{3k}, v_3) - (y_{4k}, v_3) + (f_3 + u_{3k}, v_3)$,
$\langle y_{4kt}, v_4\rangle = -(\nabla y_{4k}, \nabla v_4) + (y_{1k}, v_4) - (y_{2k}, v_4) + (y_{3k}, v_4) - (y_{4k}, v_4) + (f_4 + u_{4k}, v_4)$,
By adding these equations and integrating both sides from 0 to T, then taking the absolute value, using Cauchy Schwartz Inequality. Finally, by using assumption (A), we get
$\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right| \le$
$\|\nabla y_{1k}\|_{L^2(Q)}\|\nabla v_1\|_{L^2(Q)} + \|y_{1k}\|_{L^2(Q)}\|v_1\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)}\|v_1\|_{L^2(Q)} +$
$\|y_{3k}\|_{L^2(Q)}\|v_1\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)}\|v_1\|_{L^2(Q)} + \|\nabla y_{2k}\|_{L^2(Q)}\|\nabla v_2\|_{L^2(Q)} +$

$\|y_{1k}\|_{L^2(Q)}\|v_2\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)}\|v_2\|_{L^2(Q)} + \|y_{3k}\|_{L^2(Q)}\|v_2\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)}\|v_2\|_{L^2(Q)} +$
$\|\nabla y_{3k}\|_{L^2(Q)}\|\nabla v_3\|_{L^2(Q)} + \|y_{1k}\|_{L^2(Q)}\|v_3\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)}\|v_3\|_{L^2(Q)} +$
$\|y_{3k}\|_{L^2(Q)}\|v_3\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)}\|v_3\|_{L^2(Q)} + \|\nabla y_{4k}\|_{L^2(Q)}\|\nabla v_4\|_{L^2(Q)} +$
$\|y_{1k}\|_{L^2(Q)}\|v_4\|_{L^2(Q)} + \|y_{2k}\|_{L^2(Q)}\|v_4\|_{L^2(Q)} + \|y_{3k}\|_{L^2(Q)}\|v_4\|_{L^2(Q)} + \|y_{4k}\|_{L^2(Q)}\|v_4\|_{L^2(Q)} +$
$\|\eta_1\|_{L^2(Q)}\|v_1\|_{L^2(Q)} + \|\eta_2\|_{L^2(Q)}\|v_2\|_{L^2(Q)} + \|\eta_3\|_{L^2(Q)}\|v_3\|_{L^2(Q)} + \|\eta_4\|_{L^2(Q)}\|v_4\|_{L^2(Q)} +$
$\|u_1\|_{L^2(Q)}\|v_1\|_{L^2(Q)} + \|u_2\|_{L^2(Q)}\|v_2\|_{L^2(Q)} + \|u_3\|_{L^2(Q)}\|v_3\|_{L^2(Q)} + \|u_4\|_{L^2(Q)}\|v_4\|_{L^2(Q)}$,
Since for each $i = 1,2,3,4$, the following are satisfied:
$\|\nabla y_{ik}\|_{L^2(Q)} \le \|\nabla \vec{y}_k\|_{\boldsymbol{L^2(Q)}} \le \|\nabla \vec{y}_k\|_{\boldsymbol{L^2(I,V)}}$, $\|v_i\|_{L^2(Q)} \le \|\vec{v}\|_{\boldsymbol{L^2(Q)}} \le \|\vec{v}\|_{\boldsymbol{L^2(I,V)}}$, $\|\nabla v_i\|_{L^2(Q)} \le \|\nabla \vec{v}\|_{\boldsymbol{L^2(Q)}} \le \|\vec{v}\|_{\boldsymbol{L^2(I,V)}}$, $\|y_{ik}\|_{L^2(Q)} \le \|\vec{y}_k\|_{\boldsymbol{L^2(Q)}} \le \|\vec{y}_k\|_{\boldsymbol{L^2(I,V)}}$, $\|u_{ik}\|_{L^2(Q)} \le \|\vec{u}_k\|_{\boldsymbol{L^2(Q)}} \le C$, $\|\eta_i\|_{L^2(Q)} \le \grave{b}_i$.
Then, the above inequality becomes:

$$\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right| \le 20\|\vec{y}_k\|_{\boldsymbol{L^2(I,V)}}\|\vec{v}\|_{\boldsymbol{L^2(I,V)}} + \left(\grave{b}_1 + \grave{b}_2 + \grave{b}_3 + \grave{b}_4 + 4C\right)\|\vec{v}\|_{\boldsymbol{L^2(I,V)}}$$

Or

$$\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right| \le \left(20 b_2(c) + \grave{b}(c)\right)\|\vec{v}\|_{\boldsymbol{L^2(I,V)}} ,$$

where $\|\vec{y}_k\|_{\boldsymbol{L^2(I,V)}} \le b_2(c)$ and $\grave{b}(c) = \grave{b}_1 + \grave{b}_2 + \grave{b}_3 + \grave{b}_4 + 4C$,
this implies to
$\frac{\left|\int_0^T \langle \vec{y}_{kt}, \vec{v}\rangle dt\right|}{\|\vec{v}\|_{\boldsymbol{L^2(I,V)}}} \le b_3(c)$,with $b_3(c) = 20 b_2(c) + \grave{b}(c)$, thus $\|\vec{y}_{kt}\|_{\boldsymbol{L^2(I,V^*)}} \le b_3(c)$, $\forall\, \vec{y}_{kt} \in \vec{V}^*$.
Since for each k, $\vec{y}_k$ is a solution of the weak form ((8) – (11)), then we get

$$\langle y_{1kt}, v_1\rangle + (\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1) - (y_{2k}, v_1) + (y_{3k}, v_1) + (y_{4k}, v_1) = (f_1 + u_{1k}, v_1), \quad (59)$$

$$\langle y_{2kt}, v_2\rangle + (\nabla y_{2k}, \nabla v_2) + (y_{1k}, v_2) + (y_{2k}, v_2) - (y_{3k}, v_2) - (y_{4k}, v_2) = (f_2 + u_{2k}, v_2), \quad (60)$$

$$\langle y_{3kt}, v_3\rangle + (\nabla y_{3k}, \nabla v_3) - (y_{1k}, v_3) + (y_{2k}, v_3) + (y_{3k}, v_3) + (y_{4k}, v_3) = (f_3 + u_{3k}, v_3), \quad (61)$$

$$\langle y_{4kt}, v_4\rangle + (\nabla y_{4k}, \nabla v_4) - (y_{1k}, v_4) + (y_{2k}, v_4) - (y_{3k}, v_4) + (y_{4k}, v_4) = (f_4 + u_{4k}, v_4), \quad (62)$$

Let $\varphi_i \in C^1[0,T]$ such that $\varphi_i(T) = 0$ and $\varphi_i(0) \neq 0$, for all $i = 1,2,3,4$, we rewrite the first erms in L.H.S. of (59) – (62), then we multiply both sides of each one by φ_i for all $i = 1, 2, 3, 4$, respectively, and integrate both sides with respect to t from 0 to T. We also integrate the first terms in the L.H.S. of each equations to obtain that:

$$-\int_0^T (y_{1k}, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_{1k}, \nabla v_1) + (y_{1k}, v_1)]\,\varphi_1(t)dt - \int_0^T (y_{2k}, v_1)\varphi_1(t)dt + \int_0^T (y_{3k}, v_1)\varphi_1(t)dt + \int_0^T (y_{4k}, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_{1k}, v_1)\varphi_1(t)dt + (y_{1k}(0), v_1)\varphi_1(0), \quad (63)$$

$$-\int_0^T (y_{2k}, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_{2k}, \nabla v_2) + (y_{2k}, v_2)]\,\varphi_2(t)dt + \int_0^T (y_{1k}, v_2)\varphi_2(t)dt - \int_0^T (y_{3k}, v_2)\varphi_2(t)dt - \int_0^T (y_{4k}, v_2)\varphi_2(t)dt = \int_0^T (f_2 + u_{2k}, v_2)\varphi_2(t)dt + (y_{2k}(0), v_2)\varphi_2(0), \quad (64)$$

$$-\int_0^T (y_{3k}, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_{3k}, \nabla v_3) + (y_{3k}, v_3)]\,\varphi_3(t)dt - \int_0^T (y_{1k}, v_3)\varphi_3(t)dt + \int_0^T (y_{2k}, v_3)\varphi_3(t)dt + \int_0^T (y_{4k}, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_{3k}, v_3)\varphi_3(t)dt + (y_{3k}(0), v_3)\varphi_3(0), \quad (65)$$

$$-\int_0^T (y_{4k}, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_{4k}, \nabla v_4) + (y_{4k}, v_4)]\,\varphi_4(t)dt - \int_0^T (y_{1k}, v_4)\varphi_4(t)dt + \int_0^T (y_{2k}, v_4)\varphi_4(t)dt - \int_0^T (y_{3k}, v_4)\varphi_4(t)dt = \int_0^T (f_4 + u_{4k}, v_4)\varphi_4(t)dt + (y_{4k}(0), v_4)\varphi_4(0), \quad (66)$$

Since $\vec{y}_k \longrightarrow \vec{y}$ (weakly) in $\left(L^2(Q)\right)^4$ and in $\left(L^2(I,V)\right)^4$, $\vec{u}_k \longrightarrow \vec{u}$ (weakly) in $\left(L^2(Q)\right)^4$, $y_{ik}(0) \longrightarrow y_i(0)$ (weakly) in $L^2(\Omega)$, $\forall\, i = 1,2,3,4$, then as $k \longrightarrow \infty$, equations ((63) – (66)), converge to:

$$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1^0(0), v_1)\varphi_1(0) \quad (67)$$

$$-\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt + (y_2^0(0), v_2)\varphi_2(0) \quad (68)$$

$$-\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt + (y_3^0(0), v_3)\varphi_3(0) \quad (69)$$

$$-\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\varphi_4(t)dt + (y_4^0(0), v_4)\varphi_4(0), \quad (70)$$

Now, the following two cases are considered:

Case1: Choose $\varphi_i \in D[0,T]$, i.e. $\varphi_i(0) = \varphi_i(T) = 0$, $\forall\, i = 1,2,3,4$. Now,we integrate the first terms in the L.H.S. of (67) – (70), one has $\forall\, v_i \in V$, $\forall\, \varphi_i \in D[0,T]$ that:

$$\int_0^T \langle y_{1t}, v_1\rangle\varphi_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt, \quad (71)$$

$$\int_0^T \langle y_{2t}, v_2\rangle\varphi_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt, \quad (72)$$

$$\int_0^T \langle y_{3t}, v_3\rangle\varphi_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt, \quad (73)$$

$$\int_0^T \langle y_{4t}, v_4\rangle\varphi_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\varphi_4(t)dt, \quad (74)$$

Which they give for all$v_i \in V$, a.e. on I:

$\langle y_{1t}, v_1\rangle + (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (y_3, v_1) + (y_4, v_1) = (f_1 + u_1, v_1)$,
$\langle y_{2t}, v_2\rangle + (\nabla y_2, \nabla v_2) + (y_1, v_2) + (y_2, v_2) - (y_3, v_2) - (y_4, v_2) = (f_2 + u_2, v_2)$, $\langle y_{3t}, v_3\rangle + (\nabla y_3, \nabla v_3) - (y_1, v_3) + (y_2, v_3) + (y_3, v_3) + (y_4, v_3) = (f_3 + u_3, v_3)$,
$\langle y_{4t}, v_4\rangle + (\nabla y_4, \nabla v_4) - (y_1, v_4) + (y_2, v_4) - (y_3, v_4) + (y_4, v_4) = (f_4 + u_4, v_4)$,
i.e. $\vec{y}$ is satisfied the weak form of the continuous quaternary state vector solutions

Case2: Choose $\varphi_i \in C^1[I]$, s.t. $\varphi_i(T) = 0$ and $\varphi_i(0) \neq 0$, $\forall\, i = 1,2,3,4$.

We integrate the first terms in the L.H.S. of ((71) – (74)), to obtain:

$$-\int_0^T (y_1, v_1)\dot{\varphi}_1(t)dt + \int_0^T [(\nabla y_1, \nabla v_1) + (y_1, v_1)]\,\varphi_1(t)dt - \int_0^T (y_2, v_1)\varphi_1(t)dt + \int_0^T (y_3, v_1)\varphi_1(t)dt + \int_0^T (y_4, v_1)\varphi_1(t)dt = \int_0^T (f_1 + u_1, v_1)\varphi_1(t)dt + (y_1(0), v_1)\varphi_1(0), \quad (75)$$

$$-\int_0^T (y_2, v_2)\dot{\varphi}_2(t)dt + \int_0^T [(\nabla y_2, \nabla v_2) + (y_2, v_2)]\,\varphi_2(t)dt + \int_0^T (y_1, v_2)\varphi_2(t)dt - \int_0^T (y_3, v_2)\varphi_2(t)dt - \int_0^T (y_4, v_2)\varphi_2(t)dt = \int_0^T (f_2 + u_2, v_2)\varphi_2(t)dt + (y_2(0), v_2)\varphi_2(0), \quad (76)$$

$$-\int_0^T (y_3, v_3)\dot{\varphi}_3(t)dt + \int_0^T [(\nabla y_3, \nabla v_3) + (y_3, v_3)]\,\varphi_3(t)dt - \int_0^T (y_1, v_3)\varphi_3(t)dt + \int_0^T (y_2, v_3)\varphi_3(t)dt + \int_0^T (y_4, v_3)\varphi_3(t)dt = \int_0^T (f_3 + u_3, v_3)\varphi_3(t)dt + (y_3(0), v_3)\varphi_3(0), \quad (77)$$

$$-\int_0^T (y_4, v_4)\dot{\varphi}_4(t)dt + \int_0^T [(\nabla y_4, \nabla v_4) + (y_4, v_4)]\,\varphi_4(t)dt - \int_0^T (y_1, v_4)\varphi_4(t)dt + \int_0^T (y_2, v_4)\varphi_4(t)dt - \int_0^T (y_3, v_4)\varphi_4(t)dt = \int_0^T (f_4 + u_4, v_4)\varphi_4(t)dt + (y_4(0), v_4)\varphi_4(0), \quad (78)$$

By subtracting ((75) – (78)) from ((67) – (70)), respectively, one has that:

$(y_1^0, v_1)\varphi_1(0) = (y_1(0), v_1)\varphi_1(0)$, $\varphi_1(0) \neq 0$, $\forall \varphi_1 \in C^1[0,T]$,
$(y_2^0, v_2)\varphi_2(0) = (y_2(0), v_2)\varphi_2(0)$, $\varphi_2(0) \neq 0$, $\forall \varphi_2 \in C^1[0,T]$,
$(y_3^0, v_3)\varphi_3(0) = (y_3(0), v_3)\varphi_3(0)$, $\varphi_3(0) \neq 0$, $\forall \varphi_3 \in C^1[0,T]$,
$(y_4^0, v_4)\varphi_4(0) = (y_4(0), v_4)\varphi_4(0)$, $\varphi_4(0) \neq 0$, $\forall \varphi_4 \in C^1[0,T]$,
So we get $y_i^0 = y_i(0) = y_i^0(x)$ for all $i = 1,2,3,4$, then $\vec{y}$ is a solution of the weak form of the quaternary state seuations.

Since $G_0(\vec{u})$ is W.L.S.C. from Lemma (4.2), and since $\vec{u} \longrightarrow \vec{\bar{u}}$ (weakly) in $\left(L^2(\Omega)\right)^4$, then $G_0(\vec{u}) \leq \lim_{k\to\infty} \inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}) = \lim_{k\to\infty} G_0(\vec{u}_k) = \inf_{\vec{\bar{u}} \in \vec{W}_A} G_0(\vec{\bar{u}})$,this implies $G_0(\vec{u}) \leq \inf_{\vec{\bar{u}} \in \vec{W}_A} G_0(\vec{\bar{u}})$ and $G_0(\vec{u}) = \min_{\vec{\bar{u}} \in \vec{W}_A} G_0(\vec{\bar{u}})$. Therfore, $\vec{u}$ is a the quaternary classical continuous optimal control.

5. The necessary conditions for the optimality theorem :

In order to state, the necessary conditions for the optimality theorem for the quaternary classical continuous optimal control the Fréchet derivativeof the cost function (7) is derived.

Theorem (5.1): Consider the cost function (7), the adjoint $(z_1, z_2, z_3, z_4) = (z_{u_1}, z_{u_2}, z_{u_3}, z_{u_4})$ equations corresponding to the state $(y_1, y_2, y_3, y_4) = (y_{u_1}, y_{u_2}, y_{u_3}, y_{u_4})$ equations ((1) – (6)) are given by:

$$-z_{1t} - \Delta z_1 + z_1 + z_2 - z_3 - z_4 = (y_1 - y_{1d}), \quad \text{in } Q \quad (79)$$
$$-z_{2t} - \Delta z_2 + z_2 - z_1 + z_3 + z_4 = (y_2 - y_{2d}), \quad \text{in } Q \quad (80)$$
$$-z_{3t} - \Delta z_3 + z_3 + z_1 - z_2 - z_4 = (y_3 - y_{3d}), \quad \text{in } Q \quad (81)$$
$$-z_{4t} - \Delta z_4 + z_4 + z_1 - z_2 + z_3 = (y_4 - y_{4d}), \quad \text{in } Q \quad (82)$$

With the following boundary conditions and initial conditions :

$$z_i(x,t) = 0, \quad \forall i = 1,2,3,4. \quad \text{on } \Sigma \quad (83)$$
$$z_i(x,T) = 0, \quad \forall i = 1,2,3,4. \quad \text{on } \Omega \quad (84)$$

Then, the Fréchet derivative of G_0 is given by
$\left(\grave{G}_0(\vec{u}), \delta\vec{u}\right) = (\vec{z} + \beta\vec{u}, \delta\vec{u})$ with $\vec{u} \in \overrightarrow{W_A}$.

Proof: For $v_i = V_i, \forall i = 1,2,3,4$. The weak form of ((79) – (84)) is :

$$-\langle z_{1t}, v_1\rangle + (\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) - (z_3, v_1) - (z_4, v_1) = (y_1 - y_{1d}), \quad (85)$$
$$-\langle z_{2t}, v_2\rangle + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) + (z_3, v_2) + (z_4, v_2) = (y_2 - y_{2d}), \quad (86)$$
$$-\langle z_{3t}, v_3\rangle + (\nabla z_3, \nabla v_3) + (z_3, v_3) + (z_1, v_3) - (z_2, v_3) - (z_4, v_3) = (y_3 - y_{3d}), \quad (87)$$
$$-\langle z_{4t}, v_4\rangle + (\nabla z_4, \nabla v_4) + (z_4, v_4) + (z_1, v_4) - (z_2, v_4) + (z_3, v_4) = (y_4 - y_{4d}), \quad (88)$$

The weak form (85) – (88) has a unique solution and this can be proved from using the same manner which is utilized to proof of Theorem (3.1). Now , we substitute $v_i = z_i, for\ all\ i = 1,2,3,4$, in (54.a), (55.a), (56.a) and (57.a) ,respectively, they become:

$$\langle \delta y_{1t}, z_1\rangle + (\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) + (\delta y_3, z_1) + (\delta y_4, z_1) = (\delta u_1, z_1), \quad (89)$$
$$\langle \delta y_{2t}, z_2\rangle + (\nabla\delta y_2, \nabla z_2) + (\delta y_1, z_2) + (\delta y_2, z_2) - (\delta y_3, z_2) - (\delta y_4, z_2) = (\delta u_2, z_2), \quad (90)$$
$$\langle \delta y_{3t}, z_3\rangle + (\nabla\delta y_3, \nabla z_3) - (\delta y_1, z_3) + (\delta y_2, z_3) + (\delta y_3, z_3) + (\delta y_4, z_3) = (\delta u_3, z_3), \quad (91)$$
$$\langle \delta y_{4t}, z_4\rangle + (\nabla\delta y_4, \nabla z_4) - (\delta y_1, z_4) + (\delta y_2, z_4) - (\delta y_3, z_4) + (\delta y_4, z_4) = (\delta u_4, z_4), \quad (92)$$

Also, substituting $v_i = \delta y_i$,for all $i = 1,2,3,4$, in ((85)- (88)), respectively, to get:

$$-\langle z_{1t}, \delta y_1\rangle + (\nabla z_1, \nabla\delta y_1) + (z_1, \delta y_1) + (z_2, \delta y_1) - (z_3, \delta y_1) - (z_4, \delta y_1) = (y_1 - y_{1d}, \delta y_1), \quad (93)$$
$$-\langle z_{2t}, \delta y_2\rangle + (\nabla z_2, \nabla\delta y_2) + (z_2, \delta y_2) - (z_1, \delta y_2) + (z_3, \delta y_2) + (z_4, \delta y_2) = (y_2 - y_{2d}, \delta y_2), \quad (94)$$
$$-\langle z_{3t}, \delta y_3\rangle + (\nabla z_3, \nabla\delta y_3) + (z_3, \delta y_3) + (z_1, \delta y_3) - (z_2, \delta y_3) - (z_4, \delta y_3) = (y_3 - y_{3d}, \delta y_3), \quad (95)$$
$$-\langle z_{4t}, \delta y_4\rangle + (\nabla z_4, \nabla\delta y_4) + (z_4, \delta y_4) + (z_1, \delta y_4) - (z_2, \delta y_4) + (z_3, \delta y_4) = (y_4 - y_{4d}, \delta y_4). \quad (96)$$

Now,we integrate both sides of equations (85)- (92) with respect to t from 0 to T, and we also integrate the first terms of the L.H.S. of each obtained equations from ((89)- (92)), then by subtracting the resulting equations from ((93)−(96)) and then collecting all the obtained equations together, it gives:

$$\langle (y_1+\delta y_1)_t, v_1\rangle + (\nabla(y_1+\delta y_1), \nabla v_1) + (y_1+\delta y_1, v_1) - (y_2+\delta y_2, v_1) + (y_3+\delta y_3, v_1) + (y_4+\delta y_4, v_1) = (f_1+u_1+\delta u_1, v_1), \quad (97)$$

$$\langle (y_2+\delta y_2)_t, v_2\rangle + (\nabla(y_2+\delta y_2), \nabla v_2) + (y_1+\delta y_1, v_2) + (y_2+\delta y_2, v_2) - (y_3+\delta y_3, v_2) - (y_4+\delta y_4, v_2) = (f_2+u_2+\delta u_2, v_2), \quad (98)$$

$$\langle (y_3+\delta y_3)_t, v_3\rangle + (\nabla(y_3+\delta y_3), \nabla v_3) - (y_1+\delta y_1, v_3) + (y_2+\delta y_2, v_3) + (y_3+\delta y_3, v_3) + (y_4+\delta y_4, v_3) = (f_3+u_3+\delta u_3, v_3), \quad (99)$$

$$\langle (y_4+\delta y_4)_t, v_4\rangle + (\nabla(y_4+\delta y_4), \nabla v_4) - (y_1+\delta y_1, v_4) + (y_2+\delta y_2, v_4) - (y_3+\delta y_3, v_4) + (y_4+\delta y_4, v_4) = (f_4+u_4+\delta u_4, v_4), \quad (100)$$

Which means that $\vec{y}+\overrightarrow{\delta y}$ is the corresponding solution for the $\vec{u}+\overrightarrow{\delta u}$.

Now, from the cost function (7) one can get that:

$$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = (\delta u_1, z_1) + (\delta u_2, z_2) + (\delta u_3, z_3) + (\delta u_4, z_4) + (\beta u_1, \delta u_1) + (\beta u_2, \delta u_2) + (\beta u_3, \delta u_3) + (\beta u_4, \delta u_4) + \frac{1}{2}\|\delta\vec{y}\|^2_{L^2(Q)} + \frac{\beta}{2}\|\delta\vec{u}\|^2_{L^2(Q)},$$

Or

$$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = (\delta\vec{u}, \vec{z}) + (\beta\vec{u}, \delta\vec{u}) + \frac{1}{2}\|\delta\vec{y}\|^2_{L^2(Q)} + \frac{\beta}{2}\|\delta\vec{u}\|^2_{L^2(Q)}$$

From the first result of Theorem (4.1)

$\|\delta\vec{y}\|_{L^2(Q)} \le \bar{M}\|\delta\vec{u}\|_{L^2(Q)}$, $\bar{M} > 0$, then

$\frac{1}{2}\|\delta\vec{y}\|^2_{L^2(Q)} = \varepsilon_1(\delta\vec{u})\|\delta\vec{u}\|_{L^2(Q)}$, with $\varepsilon_1(\delta\vec{u}) = \frac{1}{2}\bar{M}^2\|\delta\vec{u}\|_{L^2(Q)}$,

and $\frac{\beta}{2}\|\delta\vec{u}\|^2_{L^2(Q)} = \varepsilon_2(\delta\vec{u})\|\delta\vec{u}\|_{L^2(Q)}$, Where $\varepsilon_i(\delta\vec{u}) \longrightarrow 0$ as $\|\delta\vec{u}\|_{L^2(Q)} \longrightarrow 0$.

Then, one gets that:

$\frac{1}{2}\|\delta\vec{y}\|^2_{L^2(Q)} + \frac{\beta}{2}\|\delta\vec{u}\|^2_{L^2(Q)} = \varepsilon(\delta\vec{u})\|\delta\vec{u}\|_{L^2(Q)}$,

where$\varepsilon(\delta\vec{u}) = \varepsilon_1(\delta\vec{u}) + \varepsilon_2(\delta\vec{u}) \longrightarrow 0$, as $\|\delta\vec{u}\|_{L^2(Q)} \longrightarrow 0$.

Then, it yields

$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = (\vec{z}+\beta\vec{u}, \delta\vec{u}) + \varepsilon(\delta\vec{u})\|\delta\vec{u}\|_{L^2(Q)}$.

From the definition of the Fréchet derivative of G_0, one gets that:

$G_0(\vec{u}+\delta\vec{u}) - G_0(\vec{u}) = \left(\grave{G}_0(\vec{u}), \delta\vec{u}\right) + \varepsilon(\delta\vec{u})\|\delta\vec{u}\|_{L^2(Q)}$, with $\vec{u} \longrightarrow 0$, as $\|\delta\vec{u}\|_{L^2(Q)} \longrightarrow 0$.

Thus: $\left(\grave{G}_0(\vec{u}), \delta\vec{u}\right) = (\vec{z}+\beta\vec{u}, \delta\vec{u})$.

Theorem (5.2):

The necessary condition for the quaternary classical continuous optimal control of the above problem is $\grave{G}_0(\vec{u}) = \vec{z}+\beta\vec{u} = 0$ with $\vec{y} = \vec{y}_{\vec{u}}$ and $\vec{z} = \vec{z}_{\vec{u}}$.

Proof: If $\vec{u}$ is an quaternary classical continuous optimal control of the problem, then

$G_0(\vec{\vec{u}}) = \min_{\vec{\vec{u}}\in\vec{W}_A} G_0(\vec{u})$, $\forall\, \vec{u} \in \left(L^2(\Omega)\right)^4$ that means

$\grave{G}_0(\vec{\vec{u}}) = 0$ or $\vec{z}+\beta\vec{\vec{u}} = 0$, i.e.

The necessary conditions for the optimality is $\left(\grave{G}_0(\vec{\vec{u}}), \delta\vec{u}\right) \ge 0$

, this gives $\left(\vec{z}+\beta\vec{\vec{u}}, \delta\vec{u}\right) \ge 0$ with $\vec{u} = \vec{w} - \vec{\vec{u}} \Longrightarrow \left(\vec{z}+\beta\vec{\vec{u}}, \vec{w}\right) \ge \left(\vec{z}+\beta\vec{\vec{u}}, \vec{\vec{u}}\right)$, $\forall\, \vec{w} \in \left(L^2(\Omega)\right)^4$

6. Conclusion

In this paper, the existence theorem for the continuous quaternary state vector solution of the weak form for the QLPBVPs is stated and proved using the Galerkin method successfully. The existence of a quaternary classical continuous optimal control ruling by the considered

QLPBVPs is stated and proved. The existence and uniqueness of the solution of the adjiont eautions associated with the QLSPDEs is studied. Furthermore the derivation of the Fréchet derivative of the cost function is derived. Finally, the necessary conditions for the optimality of the proposed quaternary classical continuous optimal control problem is stated and is proved.

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