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Solving of Chromatic Number, Chromatic Polynomial and Chromaticity for a Kind of 6-Bridge Graph Using Maplesoft

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Abstract

Maplesoft is a technical computation forms which is a heart of problem solving in mathematics especially in graph theory. Maplesoft has established itself as the computer algebra system for researchers. Maplesoft has more mathematical algorithms which is covering a wide range of applications. A new family $\theta(3,3,3,4,5,5)$ of 6-bridge graph still not completely solved for chromatic number, chromatic polynomial and chromaticity. In this paper we apply maplesoft on a kind of 6-bridge graph (G) to obtain chromatic number, chromatic polynomial and chromaticity. The computations are shown that graph G contents 3 different colours for all vertices, 112410 different ways to colour a graph such that any two adjacent vertices have different colour by using 3 different colour, graph G has isomorphic graph which has same chromatic polynomial of graph G. The odd number of vertices located in one of these bridges made chromatic number 3. The chromatic number was the important factor that made the number of way 112410. A bijection function α created isomorphic graph H to graph G and the chromatic polynomial of H was $P(H,\lambda) = P(G,\lambda)$.

Keywords: Chromatic Number, Chromatic Polynomial, Chromaticity, 6-Bridge Graph, Maplesoft.

طريقة حل عدد التلوين ومتعددة حدود التلوين واللونية لنوع من بيانات الجسور الستة باستخدام برنامج Maplesoft

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الخلاصة

برنامج Maplesoft هو أحد اشكال الحساب الفني بحيث يمثل جوهر حل المشكلة في الرياضيات خصوصا في نظرية البيانات. هذا البرنامج أسس نفسه كنظام جبري للباحثين. كما انه يحتوي على العديد من الخوارزميات الرياضية التي تغطي نطاق واسع من التطبيقات. احدى العوائل الجديدة لبيان الجسور الستة $\theta(3,3,3,4,5,5)$ لاتزال لم تحل من ناحية عدد التلوين ومتعددة حدود التلوين واللونية. في هذه الورقة البحثية سوف نقوم بإيجاد عدد التلوين ومتعددة حدود التلوين واللونية لهذا النوع من بيانات الجسور الستة باستخدام برنامج Maplesoft. تُظهر الحسابات ان هذا النوع من البيانات يحتوي على ثلاثة ألوان مختلفة لجميع

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الرؤوس، كما يحتوي على 112410طريقة مختلفة لتلوين رؤوس البيان بحيث ان أي رأسين متجاورين مختلفين في الون باستخدام ثلاثة ألوان مختلفة، هذا البيان له بيان متماثل بحيث ان البيان المتماثل له نفس متعددة حدود التلوين. العدد الفردي للرؤوس الموجودة في أحد الجسور لذلك البيان يجعل عدد الألوان ثلاثة. ان عدد التلوين هو العامل المهم الذي جعل عدد الطرق لتلوين ذلك البيان هي 112410. ان الدالة المتقابلة α هي التي أنشأت البيان المتماثل والذي بدورة امتلك متعددة حدود تلوين مساوية لمتعددة حدود تلوين البيان الأصلي.

1. Introduction

At the core of maple, symbolic computation engine is second to none in terms of scalability and performance. Indeed, symbolic was the core focus when maple was first conceived at university of waterloo in 1980 and to this day maple continues to be the benchmark software for symbolic computing [1]. Maplesoft combines symbolic manipulation, numerical mathematics, outstanding graphics and sophisticated programming language. Maplesoft has established itself as the computer algebra system of choice for many computer users including commercial, government scientists, engineers, mathematics, science, engineering teachers, researchers, student enrolled in mathematics science and engineering courses. However, due to its special syntax required to make maple perform in the way intended [2]. Some kinds of 6-bridge graphs still not completely solved for chromatic number, chromatic polynomial and chromaticity. For each integer $k \geq 2$, let θ_k be the multigraph with two vertices and k edges. Any subdivision of θ_k is called multi-bridge graph or k-bridge graph, denoted by $\theta(a_1, a_2, \ldots, a_k)$ where $a_1, a_2, \ldots, a_k \in \mathbb{N}$ and $a_1 \leq a_2 \leq \cdots, \leq a_k$ to be the graph obtained by replacing the edges of θ_k by paths of length $a_1, a_2, \ldots, a_k$ respectively [3]. $\chi(G)$ is a chromatic number of graph G which is the minimum number of colours in a proper λ-colouring of G [4]. $P(G, \lambda)$ is a chromatic polynomial of graph G which is the number of distinct λ-colourings of G [5]. The study of chromaticity of a graph G consist of basic problems which is "G chromatically unique and if it is not unique, then what can be said about it". Two graphs G and H are said to be chromatically equivalent or, simply, χ-equivalent, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$ [3]. A graph G is said to be chromatically unique or in short χ-unique if $H \cong G$ this means (H is isomorphic to G) for any graph H such that $H \sim G$ [6]. This paper obtains chromatic number, chromatic polynomial and chromaticity for some kinds of 6-bridge graphs which is represent as new families $\theta(3,3,3,4,5,5)$. Moreover, the methods will be different from the previous ways to obtain chromatic number, chromatic polynomial and chromaticity using maplesoft.

2. Methodology

Maplesoft is a key material which used in this study. To describe a key procedure and technique for maplesoft to compute chromatic number, chromatic polynomial and chromaticity are given as the following:

2.1 Computing Chromatic Number of $\theta(3, 3, 3, 4, 5, 5)$

To compute chromatic number, we will give it by the following steps of commands.

i. > with(GraphTheory)

ii. > G:=Graph (19, {{1, 2}, {2, 3}, {3, 19}, {19, 5}, {5, 4}, {4, 1}, {1, 6}, {6, 7}, {7, 19}, {19, 10}, {10, 9}, {9, 8}, {8, 1}, {1, 11}, {11, 12}, {12, 13}, {13, 14}, {14, 19}, {19, 18}, {18, 17}, {17, 16}, {16, 15}, {15, 1}}) (to determine a kind of graph which is an undirected unweighted graph with 19 vertices and 23 edges)

iii. > DrawGraph(G, style=planar)

iv. > ChromaticNumber(G, 'col')

v. > col

To illustrate the method of chromatic number function, it is by create a graph G. Then determine a kind of a graph G which is an undirected unweighted graph consists of order pair, the first part is the number of vertices 19 and the second part is a group of sets such that each set is an edge in a graph G. After that choosing the type to draw a graph G. Finally computing chromatic number of graph G and assigned the list of colour classes of an optimal proper colouring of vertices which is [1, 3, 5, 7, 9, 12, 14, 16, 18], [2, 4, 6, 8, 10, 11, 13, 15, 17] and [19].

2.2Computing Chromatic Polynomial of $\boldsymbol{\theta(3,3,3,4,5,5)}$

To compute chromatic polynomial, we will give it by the following steps of commands.

i. > with(GraphTheory)

ii. > G:=Graph (19, {{1, 2}, {2, 3}, {3, 19}, {19, 5}, {5, 4}, {4, 1}, {1, 6}, {6, 7}, {7, 19}, {19, 10}, {10, 9}, {9, 8}, {8, 1}, {1, 11}, {11, 12}, {12, 13}, {13, 14}, {14, 19}, {19, 18}, {18, 17}, {17, 16}, {16, 15}, {15, 1}})

iii. > DrawGraph(G, style=planar)

iv. > P:=ChromaticPolynomial(G, ‘λ’)

$$P := \lambda(-1+\lambda)(\lambda-2)(\lambda^2-2\lambda+2)(\lambda^2-3\lambda+3)(\lambda^{12}-15\lambda^{11}+105\lambda^{10}-456\lambda^9 + 1378\lambda^8-3077\lambda^7+5248\lambda^6-6938\lambda^5+7093\lambda^4-5484\lambda^3 + 3047\lambda^2-1094\lambda+193) \quad (1)$$

v. > asympt(P, λ)

$$\lambda^{19}-23\,\lambda^{18}+253\,\lambda^{17}-1771\,\lambda^{16}+8855\,\lambda^{15}-33646\,\lambda^{14}+100890\,\lambda^{13} - 244642\,\lambda^{12}+487380\,\lambda^{11}-805408\,\lambda^{10}+1108755\,\lambda^9 - 1270509\,\lambda^8+1204205\,\lambda^7-931899\,\lambda^6+576026\,\lambda^5 - 274413\,\lambda^4+94864\lambda^3-21234\lambda^2+2316\,\lambda \quad (2)$$

vi.> eval((2), [λ=3])

To clarify the use of chromatic polynomial function, it is by create a graph G. Then determine a kind of a graph G which is an undirected unweighted graph consists of order pair, the first part is the number of vertices 19 and the second part is a group of sets such that each set is an edge in a graph G. After that choosing the type to draw a graph G. Finally determined the polynomial (Equation 2) in which will be substituting 3 colour to obtain chromatic polynomial.

2.3 Computing Chromaticity of $\boldsymbol{\theta(3,3,3,4,5,5)}$

To illustrate the chromaticity, we need to show that G is chromatically unique this means G is χ-uniqe if $H \cong G$ for any graph H such that $P(H,\lambda)=P(G,\lambda)$.

If we want to determine whether or not these graphs are isomorphic. In order to determine the isomorphism, we check the following

i. Number of vertices are equal

ii. Number of edges are equal

iii. Same structure similarities or differences this means (degree of vertices)

iv. Is there any bijection function $\alpha: V(G) \to V(H)$ which preserves adjacency and nonadjacency this means $\big(uv \in E(G) \Leftrightarrow \alpha(u)\alpha(v) \in E(G)\big)$.

Define a function $\alpha: V(G) \to V(H)$ such that

$1 \longrightarrow a$	$2 \longrightarrow g$	$3 \longrightarrow f$	$4 \longrightarrow m$	$5 \longrightarrow l$	$6 \longrightarrow s$
$7 \longrightarrow r$	$8 \longrightarrow b$	$9 \longrightarrow c$	$10 \longrightarrow d$	$11 \longrightarrow h$	$12 \longrightarrow i$
$13 \longrightarrow j$	$14 \longrightarrow k$	$15 \longrightarrow n$	$16 \longrightarrow o$	$17 \longrightarrow p$	$18 \longrightarrow q$
$19 \longrightarrow e$					

then

$\alpha(12) = ag$ $(12) \in E(G) \Leftrightarrow (ag) \in E(H)$

$\alpha(23) = gf$ $(23) \in E(G) \Leftrightarrow (gf) \in E(H)$

$\alpha(3\ 19) = fe$ $(3\ 19) \in E(G) \Leftrightarrow (fe) \in E(H)$

$\alpha(19\ 5) = el$ $(19\ 5) \in E(G) \Leftrightarrow (el) \in E(H)$

$\alpha(54) = lm$ $(54) \in E(G) \Leftrightarrow (lm) \in E(H)$

$\alpha(41) = ma$ $(41) \in E(G) \Leftrightarrow (ma) \in E(H)$

$\alpha(16) = as$ $(16) \in E(G) \Leftrightarrow (as) \in E(H)$

$\alpha(67) = sr$ $(67) \in E(G) \Leftrightarrow (sr) \in E(H)$

$\alpha(7\ 19) = re$ $(7\ 19) \in E(G) \Leftrightarrow (re) \in E(H)$

$\alpha(19\ 10) = ed$ $(19\ 10) \in E(G) \Leftrightarrow (ed) \in E(H)$

$\alpha(10\ 9) = dc$ $(10\ 9) \in E(G) \Leftrightarrow (dc) \in E(H)$

$\alpha(98) = cb$ $(98) \in E(G) \Leftrightarrow (cb) \in E(H)$

$\alpha(81) = ba$ $(81) \in E(G) \Leftrightarrow (ba) \in E(H)$
$\alpha(1\ 11) = ah$ $(1\ 11) \in E(G) \Leftrightarrow (ah) \in E(H)$
$\alpha(11\ 12) = hi$ $(11\ 12) \in E(G) \Leftrightarrow (hi) \in E(H)$
$\alpha(12\ 13) = ij$ $(12\ 13) \in E(G) \Leftrightarrow (ij) \in E(H)$
$\alpha(13\ 14) = jk$ $(13\ 14) \in E(G) \Leftrightarrow (jk) \in E(H)$
$\alpha(14\ 19) = ke$ $(14\ 19) \in E(G) \Leftrightarrow (ke) \in E(H)$
$\alpha(19\ 18) = eq$ $(19\ 18) \in E(G) \Leftrightarrow (eq) \in E(H)$
$\alpha(18\ 17) = qp$ $(18\ 17) \in E(G) \Leftrightarrow (qp) \in E(H)$
$\alpha(17\ 16) = po$ $(17\ 16) \in E(G) \Leftrightarrow (po) \in E(H)$
$\alpha(16\ 15) = on$ $(16\ 15) \in E(G) \Leftrightarrow (on) \in E(H)$
$\alpha(15\ 1) = na$ $(15\ 1) \in E(G) \Leftrightarrow (na) \in E(H)$

thus α is a bijective since it is a map from a set $V(G)$ to a set $V(H)$ that is (one-to-one and onto). Therefore, the graph H is isomorphic to G.

To make sure, we will give it by the following steps of commands.
i.> with(GraphTheory)
ii.> G:=Graph (19, {{1, 2}, {2, 3}, {3, 19}, {19, 5}, {5, 4}, {4, 1}, {1, 6}, {6, 7}, {7, 19}, {19, 10}, {10, 9}, {9, 8}, {8, 1}, {1, 11}, {11, 12}, {12, 13}, {13, 14}, {14, 19}, {19, 18}, {18, 17}, {17, 16}, {16, 15}, {15, 1}})
iii.> H:=Graph (s, {{a, b}, {b, c}, {c, d}, {d, e}, {e, f}, {f, g}, {g, a}, {a, h}, {h, i}, {i, j}, {j, k}, {k, e}, {e, l}, {l, m}, {m, a}, {a, n}, {n, o}, {o, p}, {p, q}, {q, e}, {e, r}, {r, s}, {s, a}})
iv.> DrawGraph([G, H])
v.> IsIsomorphic(G, H, α)
vi.> α

To compute chromatic polynomial, we will give it by the following steps of commands.
i.> with(GraphTheory)
ii.> H:=Graph (s, {{a, b}, {b, c}, {c, d}, {d, e}, {e, f}, {f, g}, {g, a}, {a, h}, {h, i}, {i, j}, {j, k}, {k, e}, {e, l}, {l, m}, {m, a}, {a, n}, {n, o}, {o, p}, {p, q}, {q, e}, {e, r}, {r, s}, {s, a}})
iii.> DrawGraph(H, style=spring)
iv.> P:=ChromaticPolynomial(H, ‘λ’)

$$\begin{aligned} P \coloneqq \lambda(-1+\lambda)(\lambda-2)(\lambda^2-2\lambda+2)(\lambda^2-3\lambda+3)(\lambda^{12}-15\lambda^{11}+105\lambda^{10}-456\lambda^9 \\ +1378\lambda^8-3077\lambda^7+5248\lambda^6-6938\lambda^5+7093\lambda^4-5484\lambda^3 \\ +3047\lambda^2-1094\lambda+193) \end{aligned} \tag{3}$$

v. > asympt(P, λ)

$$\begin{aligned} \lambda^{19}-23\,\lambda^{18}+253\,\lambda^{17}-1771\,\lambda^{16}+8855\,\lambda^{15}-33646\,\lambda^{14}+100890\,\lambda^{13} \\ -244642\,\lambda^{12}+487380\,\lambda^{11}-805408\,\lambda^{10}+1108755\,\lambda^9 \\ -1270509\,\lambda^8+1204205\,\lambda^7-931899\,\lambda^6+576026\,\lambda^5 \\ -274413\,\lambda^4+94864\lambda^3-21234\lambda^2+2316\,\lambda \end{aligned} \tag{4}$$

vi. > eval((4), [λ=3])

To show that the use of isomorphic and chromatic polynomial functions, it is by create two graphs G and H. Then determine a kind of a graph G which is an undirected unweighted graph consists of order pair, the first part is the number of vertices 19 and the second part is a group of sets such that each set is an edge in a graph G. Similarly, for graph H. After that choosing the type to draw two graphs G and H. Finally finding out whether graphs are isomorphic or not and making a mapping from G to H. To clarify the use of chromatic polynomial function, it is by create a graph H. Then determine a kind of a graph H which is an undirected unweighted graph consists of order pair, the first part is the number of vertices 19 and the second part is a group of sets such that each set is an edge in a graph H. After that choosing the type to draw a graph H. Finally determined the polynomial (Equation 4) in which will be substituting 3 colour to obtain chromatic polynomial.

3- Results

Table 1 -Computing chromatic number

6-bridge graph	$\chi(G)$	Colour (1)	Colour 2	Colour 3
$\theta(3,3,3,4,5,5)$	3	[1, 3, 5, 7, 9, 12, 14, 16, 18]	[2, 4, 6, 8, 10, 11, 13, 15, 17]	[19]

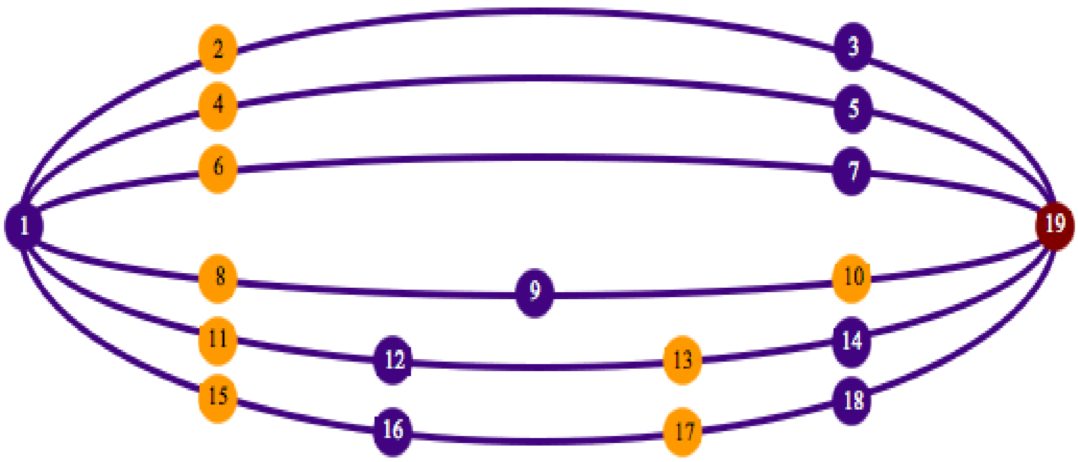

Figure 1-Graph G contains 19 vertices and 23 edges.

Table 2 -Computing chromatic polynomial

6-bridge graph	$P(G,\lambda)$	$P(G,3)$
$\theta(3,3,3,4,5,5)$	$\lambda^{19} - 23\,\lambda^{18} + 253\,\lambda^{17} - 1771\,\lambda^{16} + 8855\,\lambda^{15} - 33646\,\lambda^{14} + 100890\,\lambda^{13} - 244642\,\lambda^{12} + 487380\,\lambda^{11} - 805408\,\lambda^{10} + 1108755\,\lambda^{9} - 1270509\,\lambda^{8} + 1204205\,\lambda^{7} - 931899\,\lambda^{6} + 576026\,\lambda^{5} - 274413\,\lambda^{4} + 94864\lambda^{3} - 21234\lambda^{2} + 2316\,\lambda$	112410

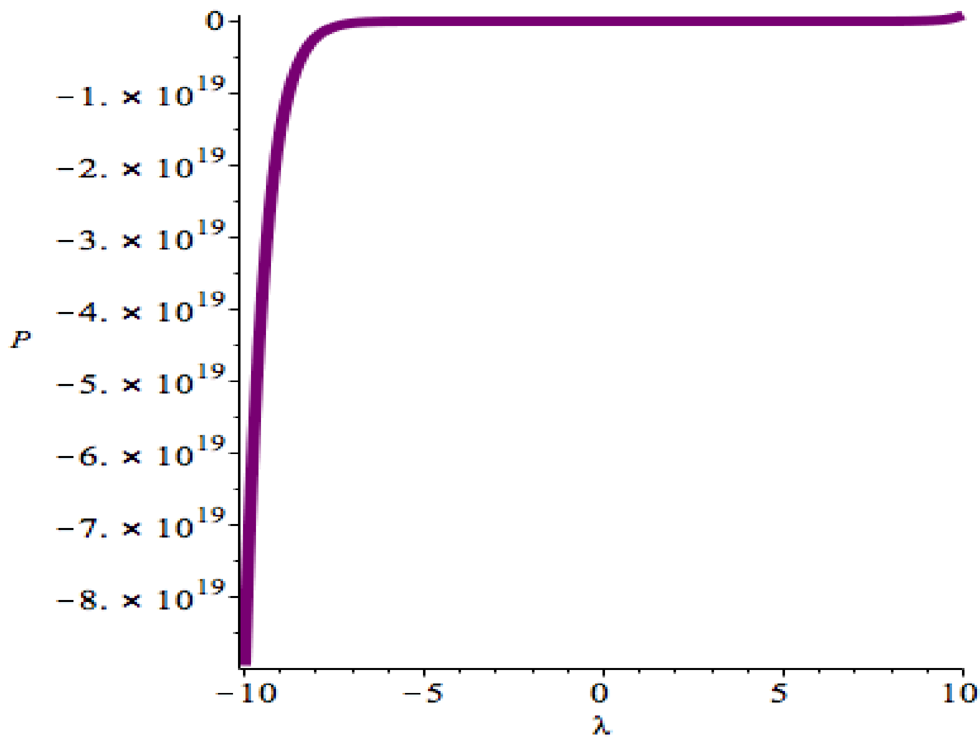

Figure 2-2-D Plot of G

Table 3 -studying the chromaticity

6-bridge graph	**Graph**	$\boldsymbol{\alpha: V(G) \to V(H)}$	**Relation**	$\boldsymbol{P(H,\lambda)}$	$\boldsymbol{P(G,3)}$
$\theta(3,3,3,4,5,5)$	H	$[1 = a, 2 = g, 3 = f, 4 = m, 5 = l, 6 = s, 7 = r, 8 = b, 9 = c, 10 = d, 11 = h, 12 = i, 13 = j, 14 = k, 15 = n, 16 = o, 17 = p, 18 = q, 19 = e]$	$H \cong G$	$\lambda^{19} - 23\lambda^{18} + 253\lambda^{17} - 1771\lambda^{16} + 8855\lambda^{15} - 33646\lambda^{14} + 100890\lambda^{13} - 244642\lambda^{12} + 487380\lambda^{11} - 805408\lambda^{10} + 1108755\lambda^{9} - 1270509\lambda^{8} + 1204205\lambda^{7} - 931899\lambda^{6} + 576026\lambda^{5} - 274413\lambda^{4} + 94864\lambda^{3} - 21234\lambda^{2} + 2316\lambda$	112410

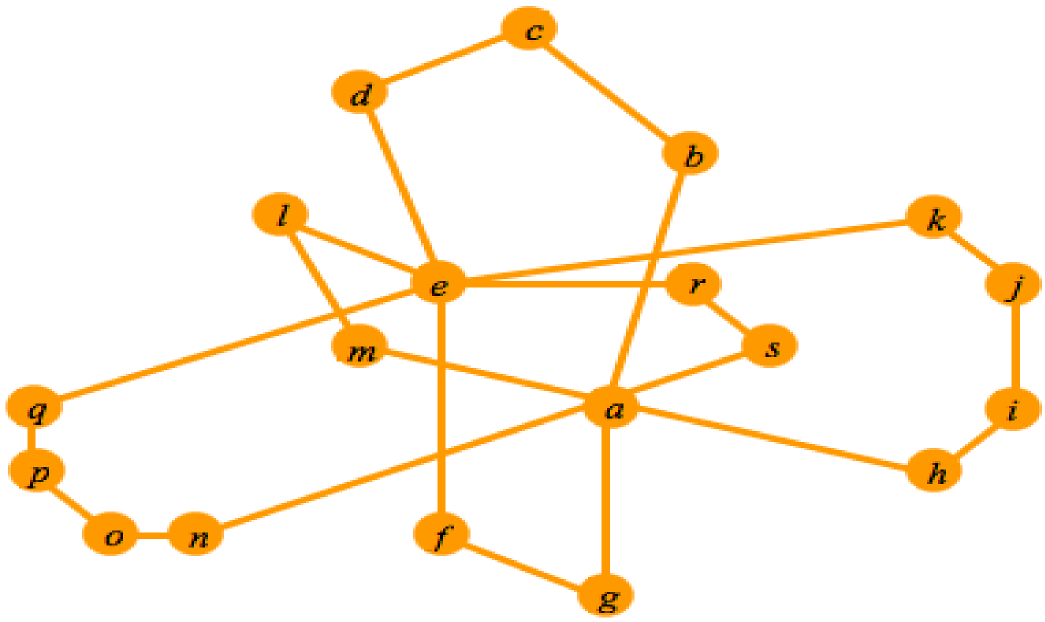

Figure 3- Isomorphic Graph H contains 19 vertices and 23 edges.

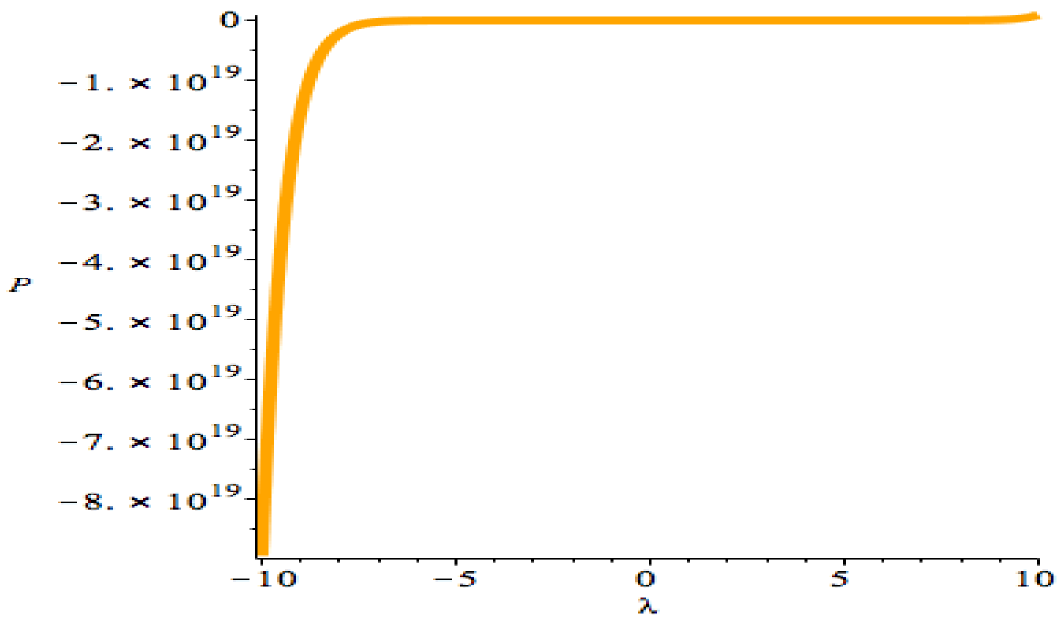

Figure 4- 2-D Plot of Graph H

4- Discussion

The report data of chromatic number on graph G indicated that the computing of chromatic number is 3. From this number, it was fairly evident that the first set of vertices [1, 3, 5, 7, 9, 12, 14, 16, 18] was coloured by a certain colour, the second set of vertices [2, 4, 6, 8, 10, 11, 13, 15, 17] was coloured by another colour, while the third set of vertices [19] was coloured different than both previous sets. Moreover, the main factor of this result was the number of vertices located in the bridge number 4 which were 5 vertices, especially the vertex number 10 which caused that the vertex number 19 was coloured by different colour than others. Regarding to the previous results which have been found chromatic number. Corollary of Zykov's theorem was the main key for findings which stated that the chromatic number of a graph G can be expressed as the minimum number of vertices of complete graphs [7].

The present data of chromatic polynomial on graph G referred to the computing of chromatic polynomial was 112410. From this number, it was obvious that there were 112410 different ways to colour a graph G such that any two adjacent vertices have different colour by using 3 different colour.

Moreover, the main factor of this results was chromatic number after substituted in (Equation 2). When it was review a previous results regarding to the chromatic polynomial it noticed that this results relied on fundamental reduction theorem (FRT). This theorem stated that the chromatic polynomial of a graph G can be expressed in terms of the chromatic polynomials of two graphs the first one with an extra edge, and another with fewer vertices. Then the theorem can be applied again to these graphs, and so on. The end of process must be done when none of these graphs has a pair of non-adjacent vertices. Therefore, the chromatic polynomial of the given graph has been expressed as the sum of the chromatic polynomials of complete graphs [8].

Finally, the outcomes of chromaticity on graph H gave a graph H an isomorphic property with graph G such that the first main factor of this result is a bijection function α from the set $V(G)$ to a set $V(H)$ as the following:
$[1 = a, 2 = g, 3 = f, 4 = m, 5 = l, 6 = s, 7 = r, 8 = b, 9 = c, 10 = d, 11 = h, 12 = i, 13 = j, 14 = k, 15 = n, 16 = o, 17 = p, 18 = q, 19 = e]$.

Furthermore, the computing of chromatic polynomial was 112410. From this number, it was clear that there were 112410 different ways to colour a graph H such that any two adjacent vertices have different colour by using 3 different colour. Moreover, the second main factor of this result was chromatic number after substituted in (Equation 4). This number of chromatic polynomial made $P(H,\lambda) = P(G,\lambda)$, then the condition is completed to say that G is chromatically unique. The previous results for chromatically unique relied on (Theorem 8) in [9]. Whose proof also depend on some theorems and lemmas to determine the chromatically unique of graph.

5- Conclusion

This study focuses on how to use the maple software on some kinds of 6-bridge graphs to find out the chromatic number, chromatic polynomial and chromaticity. during the research results it was observed that chromatic number of this sample was 3, chromatic polynomial was 112410. As for the chromaticity it has been shown that G was chromatically unique. One of the most important effect to make the chromatic number 3 was the odd number of vertices located in one of these bridges. Concerning for chromatic polynomial the important factor that made the number of way which is 112410 to colour a graph G such that any two vertices has different colour was chromatic number such that whenever increasing the colours offset by an increase in the number of ways. Then the chromatic number depends on the minimum colour to colour G. On the other side this made the number of way is minimum. As talking about the factors that help the G to be chromatically unique were the bijection function α though which it was created isomorphic graph H to graph G and the chromatic polynomial of H which made $P(H,\lambda) = P(G,\lambda)$. For researchers in this field will follow this method cause it rely on the software which is considered as one of the best way for solving. In addition to it characterized by a correct solution at a high rate with a little mistake.

6- References

1. Bernardin, L., Chin, P., Demarco, P., Geddes, K., Hare, D., Heal, K., Labahn, G., May, J., Mccarron, J. & Monagan, M. **2011**. *Maple Programming Guide.* Maplesoft Waterloo.
2. Abell, M. L. & Braselton, J. P. **2005**. *Maple by Example.* Academic Press.
3. Dong, F. M., Koh, K. M. & Teo, K.-L. **2005**. *Chromatic Polynomials and Chromaticity of Graphs.* World Scientific.
4. Gross, J. L. & Yellen, J. **2004**. *Handbook of Graph Theory.* CRC press.
5. Chartrand, G. & Zhang, P. **2008**. *Chromatic Graph Theory.* CRC press.
6. Koh, K. & Teo, K. L. **1990**. The Search for Chromatically Unique Graphs. *Graphs and Combinatorics*, **6**(3): 259-285.
7. Corneil, D. G. & Graham, B. **1973**. An Algorithm for Determining the Chromatic Number of a Graph. *SIAM Journal on Computing*, **2**(4): 311-318.
8. Read, R. C. **1968**. An Introduction to Chromatic Polynomials. *Journal of Combinatorial Theory* **4**(1): 52-71.
9. Khalaf, A. & Peng, Y. **2009**. The Chromatic Uniqueness of a Family of 6-Bridge Graphs. *AKCE Journal of Graphs and Combinatorics*, **6**(3): 393-400.