Stability Analysis of a Prey-Predator Model with Prey Refuge and Fear of Adult Predator

Salam Jasim Majeed[*1], Sarah Fawzi Ghafel[*2]

[1]*Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Thi-Qar, Iraq*

[2]*Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar ,Thi-Qar, Iraq*

Abstract

This paper is concerned with a Holling-II stage-structured predator-prey system in which predators are divided into an immature and mature predators. The aim is to explore the impact of the prey's fear caused by the dread of mature predators in a prey-predator model including intraspecific competitions and prey shelters. The theoretical study includes the local and global stability analysis for the three equilibrium points of the system and shows the prey's fear may lead to improving the stability at the positive equilibrium point. A numerical analysis is given to ensure the accuracy of the theoretical outcomes and to testify the conditions of stability of the system near the non-trivial equilibrium points.

Keywords: Equilibrium points, Stability analysis, Fear effect, Stage-structure

تحليل الاستقرار لنموذج فريسة المفترس مع ملجا الفريسة والخوف من المفترس البالغين

سلام جاسم مجيد[1]، ساره فوزي غافل[2]*

[1]قسم الرياضيات، كلية علوم الحاسوب والرياضيات ، جامعة ذي قار، ذي قار، العراق

[2]قسم الرياضيات ، كلية التربية للعلوم الصرفة ،جامعة ذي قار، ذي قار، العراق

الخلاصة:

يركز هذه البحث دراسة نظام فريسة ومفترس مع دالة استجابة من نوع هولنك الثاني ومراحل عمرية للمفترس.الهدف هو استكشاف تأثير خوف الفريسة الناجم عن الخوف من المفترس الناضج في نموذج فريسة مفترس يتضمن المنافسة الداخلية وملاجئ فريسة. اشتملت الدراسة النظرية على تحليل الاستقرارية المحلية والشاملة عند نقاط الاتزان الثلاث للنظام ، وأظهرت أن خوف الفريسة قد يؤدي إلى تحسين الاستقرارية عند نقطة الاتزان الموجبة.

1. Introduction

The real-world problems show, day by day, that ecological modelling becomes a highly demandable area of research that is considered by mathematicians and theoretical ecologists due to its universal existence and importance. Mathematical models may help to a better understanding of dynamic behaviors of real-life problems. Especially, the dynamic interrelationships between predators and their prey populations. In the last decade, many

_____

*Email: sarah_fawzi.math@utq.edu.iq

biologists have empirically observed that the prey-predator systems reflect impact not only merely direct and lethal (due to killing by predators) but also indirect and non-consumptive (caused by the fear of predators) [1, 2]. The ceaseless fear may propel prey to leave their zone and move gradually to a more secure regions [3, 4]. The first authentic prey-predator model was done by Lotka-Volterra [5, 6]. It is assumed that each individual predator admits the same ability to attack prey. But, in the natural world, the predatory conduct and reproductive conduct for all animals are mainly completed by adult individuals, and these abilities may be ignored for juveniles, who they being raised by their adult parents. So, the life cycle of most, if not all, animals exhibits two distinct stages, mature and immature and in these stages death rate and birth rate are different [7, 8]. As a consequence, in [9, 10, 11, 12, 13, 14] the authors analyzed the influence of a stage-structure for the predator on the stability of a prey-predator system. Nevertheless, there are not sufficient studies to determine the real effect of the stage-structure phenomena and prey fear level in the dynamic of the prey-predator relationship, when they act together in the interaction. Recently, Mondal et al (2020)[15] studied the influence of the behaviour of adult predator incited fear in a stage-structured predator-prey model with the linear form of functional response. Their study suggested that the cost of fear (due to adult predators) and the predator maturity rate may stabilize or destabilize the system dynamics. Many research papers by mathematicians and ecologists exhibited that prey-predator systems are affected by many other factors such as Alle effect, refuge, competition among predators or preys and functional responses of predators. In population ecology, the ecological refuges may provide some degree of protection for organisms by decreasing the opportunity of extinction due to predation, and then may help to preserve the ecological balance [7,16]. On the other hand, intra-specific competition is competition between the same species on the same resource in an ecosystem, mostly caused by limited the resources such as water, food etc. It may lead to a reduction in growth and fitness for both individuals [3, 7]. Moreover, Holling type two is a prey dependent functional response, in which a predator has to dedicate a certain time to search, capture, and ingest its prey, and is considered the most common Holling functional form [3, 4, 9, 10, 11, 13, 16].

2. Model Formulation

Zhang at el. [16] considered prey–predator model with anti-predator behavior resulting in the fear of predators and Holling type-II functional response including a prey refuge to formulate the following mathematical model

$$\frac{dx}{dt} = \frac{rx}{1+ay} - c_1 x^2 - \frac{\alpha(1-m)xy}{1+b(1-m)x}$$
$$\frac{dy}{dt} = \frac{\beta\alpha(1-m)xy}{1+b(1-m)x} - c_2 y^2 - n_1 y \tag{2.1}$$

where x is the density of prey; y is the density of predator; r represents the growth rate of prey; a is a level of fear; c_i represents the intra- specific competition rate of (prey and immature predator); α is the maximum attack rate; b represents the satiety rate of mature predator; m represents the prey refuge constant, $m \in [0,1)$; n_1 is the predator natural death rate.

Since adult predators may be capable of hunting the prey and this ability is not found in young predators, the prey species are afraid of adult predators and are not afraid in young predator. This motivates us to extend Zhang at el. [16] model by incorporating a stage-structured of predator into the model. Hence, the system (2.1) is transformed to

$$\frac{dx}{dt} = \frac{rx}{1+ay_2} - c_1x^2 - \frac{\alpha(1-m)xy_2}{1+b(1-m)x}$$
$$\frac{dy_1}{dt} = \frac{\alpha(1-m)xy_2}{1+b(1-m)x} - c_2y_1^2 - (d+n_1)y_1 \qquad (2.2)$$
$$\frac{dy_2}{dt} = dy_1 - n_2y_2 \ .$$

which satisfies initial conditions $x(0) \geq 0, y_1(0) \geq 0, y_2(0) \geq 0$.

Here, y_1 is the density of juvenile predators; y_2 is the density of adults; d represents the transformation rate from juvenile predators to adult predators and $n_i, i = 1,2$, are the natural death rates of predators.

Theorem (2.1): The domain of the system (2.2) R_+^3 is positively invariant.

Proof: Since the interaction functions that appear on the right-hand side of the system (2.2) and their partial derivatives are continuous on R_+^3. Thus these functions are Lipschitizian , and hence the solution $[x(t), y_1(t), y_2(t)]$ of system (2.2) with positive initial conditions $[x(0), y_1(0), y_2(0)]$ exists and is unique on $[0, \rho)$ where $0 < \rho \leq +\infty$.

Integrating the first and third equations in (2.2) give:

$$x(t) = x(0) \exp\left(\int_0^t \left[\frac{r}{1+ay_2} - c_1x - \frac{\alpha(1-m)y_2}{1+b(1-m)x}\right] ds\right), \quad t \in [0, \infty)$$

and

$$y_2(t) \geq y_2(0) \exp\left(\int_0^t -n_2 ds\right), t \in [0, \infty)$$

which shows that $x(t) \geq 0$ and $y_2(t) \geq 0$ if $x(0), y_2(0) \geq 0$ for $t \in [0, \infty)$.

Now, integrating the second equation in (2.2) gives:

$$y_1(t) \geq y_1(0)\exp\left(-\int_0^t [c_2y_1 + (d+n_1)]ds\right), \quad t \in [0, \infty)$$

then $y_1(t) \geq 0, \forall t \in [0, \infty)$, if $y_1(0) \geq 0$.

Hence, R_+^3 is a positive invariant for the prey-predator system (2.2).

Theorem (2.2): All the solutions $(x(t), y_1(t), y_2(t))$ of the system (2.2), which initiate in $R_+^3 - \{0\}$ are uniformly bounded.

Proof: Let $B(t)$ is the total population of the system (2.2), where $B(t) = x(t) + y_1(t) + y_2(t)$ here $x(t), y_1(t), y_2(t)$ are any solutions of system (2.2), clearly, $B(t)$ is bounded if and only if $x(t), y_1(t), y_2(t)$ are bounded. Deriving $B(t)$ with respect to the time. This gives

$\frac{dB(t)}{dt} = \frac{rx}{1+ay_2} - c_1x^2 - c_2y_1^2 - n_1y_1 - n_2y_2 \ .$

Now, for each $\mu > 0$, the below differential inequality is satisfied

$$\frac{dB(t)}{dt} + \mu B(t) \leq x(r + \mu - c_1x) - (n_1 - \mu)y_1 - (n_2 - \mu)y_2$$
$$= -\left\{\left(\sqrt{c_1}\,x - \frac{r+\mu}{2\sqrt{c_1}}\right)^2 - \left(\frac{r+\mu}{2\sqrt{c_1}}\right)^2\right\} - (n_1 - \mu)y_1 - (n_2 - \mu)y_2$$

Choosing $\mu < min\{r, n_1, n_2\}$, the last differential inequality becomes

$$\frac{dB(t)}{dt} + \mu B(t) \leq \left(\frac{r+\mu}{2\sqrt{c_1}}\right)^2 = \delta.$$

Applying the Garonwall lemma [15] gives $0 < B(t) \leq B(0)e^{-\mu t} + \frac{\delta}{\mu}(1 - e^{-\mu t})$ for $t \to \infty$

becomes $0 < B(t) < \frac{\delta}{\mu}$. Hence the total population of the system (2.2) in R_+^3 is uniformly bounded.

3. Existence of Equilibrium Points

In this section, we look at the local stability analysis of a nonlinear ordinary differential system. We simulate the dynamic behavior of a highly nonlinear complex system using the linearization technique We linearize the system of equations around each equilibrium point by applying a small perturbation. The system (2.2) has three biologically meaningful equilibrium points (EP).

(i) The trivial equilibrium point $\aleph_0 = (0,0,0)$ always exists.

(ii) The axial equilibrium point $\aleph_1 = (\bar{x}, 0,0)$, where $\bar{x} = \frac{r}{c_1}$, always exists.

(iii) The interior equilibrium point $\aleph_2 = (x^*, y_1^*, y_2^*)$ exists under some conditions.

Next, we discuss the existence of the interior equilibrium $\aleph_2$, where the values of x^*, y_1^* and y_2^* may be obtained by solving the following set of algebraic equations:

$$\frac{r}{1+ay_2} - c_1 x - \frac{\alpha(1-m)y_2}{1+b(1-m)x} = 0, \tag{3.1a}$$

$$\frac{\alpha(1-m)xy_2}{1+b(1-m)x} - c_2 y_1^2 - (d+n_1)y_1 = 0, \tag{3.1b}$$

$$dy_1 - n_2 y_2 = 0. \tag{3.1c}$$

From eq.(3.1c), one can get

$$y_2 = \frac{d}{n_2} y_1 \tag{3.2}$$

using eq.(3.2) in eq.(3.1a), then we obtain

$$h_1(x, y_1) = \frac{n_2 r}{n_2 + ady_1} - \frac{\alpha d(1-m)y_1}{n_2[1+b(1-m)x]} - c_1 x = 0. \tag{3.3}$$

Again using eq.(3.2) in eq.(3.1b) and we get

$$h_2(x, y_1) = \frac{\alpha d(1-m)x}{n_2[1+b(1-m)x]} - (d+n_1) - c_2 y_1 = 0. \tag{3.4}$$

In eq.(3.3), when $y_1 \to 0$, then $x \to x_a$ where x_a is the solution of
$h_1(x, 0) = r - c_1 x = 0$.
This gives

$$x_a = \frac{r}{c_1} > 0. \tag{3.5}$$

In eq.(3.4), if $y_1 \to 0, x \to x_b$ where x_b is a positive solution of
$h_2(x, 0) = [\alpha d(1-m) - n_2(d+n_1)b(1-m)]x - n_2(d+n_1) = 0$.
This gives

$$x_b = \frac{n_2(d+n_1)}{(1-m)[\alpha d - n_2 b(d+n_1)]}. \tag{3.6}$$

We see that x_b is positive under the following condition:
$$\alpha d > n_2 b(d+n_1).$$
Moreover, from eq.(3.3) we have

$$\frac{dx}{dy_1} = -\left(\frac{dh_1}{dy_1}\right)/\left(\frac{dh_1}{dx}\right) \tag{3.7}$$

where,
$$\frac{dh_1}{dy_1} = -\frac{adn_2 r}{(n_2+ady_1)^2} - \frac{\alpha d(1-m)}{n_2[1+b(1-m)x]} < 0,$$
and
$$\frac{dh_1}{dx} = \frac{\alpha dn_2 b(1-m)^2 y_1}{n_2^2[1+b(1-m)x]^2} - c_1.$$

So, $\frac{dx}{dy_1} < 0$, and the isocline (h_1) is decreasing passing through x_a if $\frac{dh_1}{dx} < 0$. Hence, under the following condition, the isocline (h_1) passes through x_a.
$\alpha db(1-m)^2 y_1 < c_1 n_2[1+b(1-m)x]^2$.
Moreover, from eq.(3.4) we notice that
$$\frac{dx}{dy_1} = -\left(\frac{dh_2}{dy_1}\right)/\left(\frac{dh_2}{dx}\right),$$
where

$$\frac{dh_2}{dy_1} = -c_2 < 0,$$

and

$$\frac{dh_2}{dx} = \frac{\alpha d(1-m)n_2}{n_2^2[1+b(1-m)x]^2} > 0.$$

So, $\frac{dx}{dy_1} > 0$, and then the isocline (h_2) is increasing passing through x_b. Thus, with the condition $x_b < x_a$, the two isoclines (h_1) and (h_2) intersect at a unique positive point (x^*, y_1^*) in the positive quadrant of $xy_1 -$ plane.

Now, substituting y_1^* in eq.(3.1c), then we get

$$y_2^* = \frac{d}{n_1} y_1^* \tag{3.8}$$

Accordingly, the (IEP) $\aleph_2 = (x^*, y_1^*, y_2^*)$ exists uniquely in the first octant of $xy_1y_2 -$ plane provided that:

$$x_a > x_b \tag{3.9}$$
$$\alpha d > n_2 b(d + n_1) \tag{3.10}$$
$$c_1 n_2[1 + b(1 - m)x^*]^2 > \alpha db(1 - m)^2 y_1^* \tag{3.11}$$

4. The Local Stability Analysis

The local stability analysis gives good information about the fate of the system (2.2). In this section, we will discuss the conditions of stability or instability for the aforementioned equilibrium points $\aleph_i, i = 1,2,3$. For that, the linearization of the system (2.2) about any point of equilibrium (x, y_1, y_2), yields the following variation matrix:

$$J = \begin{bmatrix} m_{11} & 0 & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix} \tag{4.1}$$

where ,

$$m_{11} = \frac{r}{1 + ay_2} - 2c_1 x - \frac{\alpha(1-m)y_2}{[1 + b(1-m)x]^2},$$

$$m_{13} = \frac{-rxa}{(1 + ay_2)^2} - \frac{\alpha(1-m)x}{[1 + b(1-m)x]},$$

$$m_{21} = \frac{\alpha(1-m)y_2}{[1+b(1-m)x]^2},$$

$$m_{22} = -2c_2 y_1 - (d + n_1),$$

$$m_{23} = \frac{\alpha(1-m)x}{[1+b(1-m)x]},$$

$$m_{32} = d,$$

$$m_{33} = -n_2.$$

The eigenvalues that locate the local stability at each equilibrium point are calculated as follows:

Theorem(4.1): The system (2.2) around (TEP) $\aleph_0 = (0,0,0)$ is always an unstable saddle point.

Proof: The variation matrix (4.1) of the system (2.2) at (TEP) is written as:

$$J_{\aleph_0} = \begin{bmatrix} r & 0 & 0 \\ 0 & -(d + n_1) & 0 \\ 0 & d & -n_2 \end{bmatrix} \tag{4.2}$$

Obviously, matrix $J_{\aleph_0}$ has the following characteristic equation:
$(\lambda - r)(\lambda + d + n_1)(\lambda + n_2) = 0.$

Hence, the eigenvalues of $J_{\aleph_0}$ are $\lambda_1 = r > 0; \lambda_2 = -(d + n_1) < 0,$ and $\lambda_3 = -n_2 < 0.$ Thus, (TEP) is an unstable saddle point.

Theorem (4.2): The system (2.2) around (AEP) $\aleph_1 = (\frac{r}{c_1}, 0, 0)$ is locally asymptotically stable if the following condition holds:

$$\alpha d(1 - m)r < n_2(d + n_1)[c_1 + b(1 - m)r] \qquad (4.3)$$

Proof: The variation matrix of the system (2.2) around AEP is as follows:

$$J_{\aleph_1} = \begin{bmatrix} -r & 0 & -\frac{r^2 a}{c_1} - \frac{\alpha(1-m)r}{c_1 + b(1-m)r} \\ 0 & -(d + n_1) & \frac{\alpha(1-m)r}{c_1 + b(1-m)r} \\ 0 & d & -n_2 \end{bmatrix} \qquad (4.4)$$

Thus, the characteristic equation of matrix $J_{\aleph_1}$ is given by:

$$(\lambda + r)(\lambda^2 + k_1\lambda + k_2) = 0,$$

where,

$$k_1 = d + n_1 + n_2 > 0,$$
$$k_2 = n_2(d + n_1) - \frac{d\alpha(1 - m)r}{c_1 + b(1 - m)r}.$$

Clearly, $J_{\aleph_0}$ has a negative eigenvalue which is given by $\lambda_1 = -r$. While the other two eigenvalues λ_2 and λ_3 are the roots of:

$$\lambda^2 + k_1\lambda + k_2 = 0 \qquad (4.5)$$

According to the Routh-Hurwitz criterion, eq.(4.5) have two roots with a negative real part if and only if $k_i > 0$, for $i = 1, 2$. Thus, if condition (4.3) holds, AEP is a locally asymptotically stable point for system (2.2), and the proof is finished.

Theorem(4.3): Assume that:

$$2c_1x^* + \frac{\alpha(1-m)y_2^*}{[1+b(1-m)x^*]^2} - \frac{r}{(1+ay_2^*)} > 0, \qquad (4.6a)$$

$$[2c_2y_1^* + (d + n_1)]n_2 > \frac{d\alpha(1-m)x^*}{1+b(1-m)x^*}, \qquad (4.6b)$$

$$\Delta > 0 \qquad (4.6c)$$

where Δ is defined in the proof, then (IEP) $\aleph_2 = (x^*, y_1^*, y_2^*)$ is locally asymptotically stable for the system (2.2).

Proof: The variation matrix of the system (2.2) around (IEP) is given by

$$J_{\aleph_2} = (n_{ij})_{3\times3} \qquad (4.7)$$

where, $n_{ij} = m_{ij}$ at (x^*, y_1^*, y_2^*) for all $i, j = 1, 2, 3.$
Evaluating the characteristic equation of $J_{\aleph_2}$ gives

$$\lambda^3 + \emptyset_1\lambda^2 + \emptyset_2\lambda + \emptyset_3 = 0 \qquad (4.8)$$

where

$$\emptyset_1 = -(n_{11} + n_{22} + n_{33})$$
$$\emptyset_2 = n_{11}n_{22} + n_{11}n_{33} + n_{22}n_{33} - n_{23}n_{32}$$
$$\emptyset_3 = n_{11}n_{23}n_{32} - n_{11}n_{22}n_{33} - n_{13}n_{21}n_{32}$$

The Routh-Hurwitz conditions for eq.(4.8) that grantee all roots with negative real part are: $\emptyset_1 > 0, \emptyset_3 > 0,$ and $\Delta = \emptyset_1\emptyset_2 - \emptyset_3 > 0$. Therefore, $\aleph_2$ is locally asymptotically stable if the conditions (4.6a-4.6c) hold.

5. The Analysis of Global Stability

In this section, we study the global stability analysis for the equilibrium points that are L.A.S. of the system (2.2) by helping the Lyapunov method.

Theorem(5.1):): The (AEP) $\aleph_1 = (\frac{r}{c_1}, 0, 0)$ of system (2.2) is global asymptotically stable whenever it is local asymptotically stable and the following condition holds

$$n_2 c_1 > (ra + \alpha(1-m))r \tag{5.1}$$

Proof: Consider the following function

$$v_1(x, y_1, y_2) = (x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}) + y_1 + y_2$$

It is clearly to see $v_1(x, y_1, y_2) \in c^1(R_+^3, R), v_1(\aleph_1) = 0$, and $v_1(x, y_1, y_2) > 0$; $\forall (x, y_1, y_2) \neq \aleph_1$. Now, by differentiating v_1 with respect to time t with some algebraic simplifications, we get

$$\frac{dv_1}{dt} = \frac{(x - \bar{x})}{x} x \left[\frac{r}{1 + ay_2} - c_1 x - \frac{\alpha(1-m)y_2}{1 + b(1-m)x} \right]$$
$$+ \frac{\alpha(1-m)xy_2}{1 + b(1-m)x} - c_2 y_1^2 - dy_1 - n_1 y_1 + dy_1 - n_2 y_2$$

$$\frac{dv_1}{dt} = (x - \bar{x})\left[\left(\frac{r}{1 + ay_2} - r\right) - c_1(x - \bar{x}) - \frac{\alpha(1-m)y_2}{1 + b(1-m)x} \right]$$
$$+ \frac{\alpha(1-m)xy_2}{1 + b(1-m)x} - c_2 y_1^2 - n_1 y_1 - n_2 y_2$$

$$\leq -[\frac{raxy_2}{1 + ay_2} + c_1(x - \bar{x})^2 + c_2 y_1^2 + n_1 y_1 + \frac{1}{c_1}(n_2 c_1 - (ra + \alpha(1-m))r)y_2]$$

So, $\frac{dv_1}{dt} \leq 0$, and hence v_1 is Lyapunov function. Therefore, $\aleph_1$ is globally asymptotically stable in R_+^3.

Theorem (5.2): The (IEP) $\aleph_2 = (x^*, y_1^*, y_2^*)$ of the system (2.2) is a global asymptotically stable if the following conditions hold:

$$\alpha b(1-m)^2 y_2^* < c_1(1 + b(1-m)x^*) \tag{5.2}$$
$$\Lambda_3 x^* + y_2^* < \frac{\Lambda_1 + \Lambda_2 - d}{\Lambda_2} \tag{5.3}$$

where, $\Lambda_1 = \frac{ar}{(1 + ay_2^*)(1 + ay_2)}$, $\Lambda_2 = \frac{\alpha(1-m)}{(1 + b(1-m)x)(1 + b(1-m)x^*)}$ and $\Lambda_3 = b(1-m)(x-1) + 1$.

Proof: First, the Lyapunov function can be defined as:

$$v_2(x, y_1, y_2) = \int_{x^*}^{x} \frac{x - x^*}{x} dx + \int_{0}^{y_1 - y_1^*} \omega \, d\omega + \int_{0}^{y_2 - y_2^*} \varpi \, d\varpi.$$

So that $v_2(\aleph_2) = 0$ and $v_2(x, y_1, y_2) > 0$; $\forall (x, y_1, y_2) \neq \aleph_2$. Substituting for $\dot{x}, \dot{y}_1$, and $\dot{y}_2$ in the time derivative of v_2 along the solution of the system (2.2) gives:

$$\dot{v}_2 = -\left(c_1 - \frac{\alpha b(1-m)^2 y_2^*}{(1 + b(1-m)x^*)} \right)(x - x^*)^2 - n_2(y_2 - y_2^*)^2$$
$$-(c_2(y_1 + y_1^*) + d + n_1)(y_1 - y_1^*)^2 + \Lambda_2 y_2(x - x^*)(y_1 - y_1^*)$$
$$+[d + \Lambda_2 x^*(1 + b(1-m)x)](y_1 - y_1^*)(y_2 - y_2^*)$$
$$-[\Lambda_1 + \Lambda_2(1 + b(1-m)x^*)](x - x^*)(y_2 - y_2^*)$$

Now, from the boundedness theorem, a constant $\Pi = \frac{\delta}{\mu}$ can be defined which satisfies $(t), y_1(t), y_2(t) < \Pi$, where δ and μ are defined in Theorem (2.2). Therefore, the above equation is reduced to

$$\dot{v}_2 = -\left(c_1 - \frac{\alpha b(1-m)^2 y_2^*}{(1+b(1-m)x^*)}\right)(x-x^*)^2 - n_2(y_2 - y_2^*)^2$$
$$-(c_2(y_1 + y_1^*) + d + n_1)(y_1 - y_1^*)^2 - (\Lambda_1 + \Lambda_2 - (y_2 + \Lambda_3 x^*)\Lambda_2 - d)\Pi^2$$

The condition (5.3) implies

$$\dot{v}_2 \leq -\left(c_1 - \frac{\alpha b(1-m)^2 y_2^*}{(1+b(1-m)x^*)}\right)(x-x^*)^2 - n_2(y_2 - y_2^*)^2$$
$$- (c_2(y_1 + y_1^*) + d + n_1)(y_1 - y_1^*)^2$$

Thus it follows from condition (5.3), $\frac{dv_2}{dt} \leq 0$. This means $v_2(x, y_1, y_2)$ is negative definite a globally stable under the conditions (5.2) and (5.3) hold. Moreover, $\frac{dv_2}{dt} = 0$ if and only if $x = x^*, y_1 = y_1^*$ and $y_2 = y_2^*$. Therefore, (IEP) is global asymptotically stable due to the LaSalle theorem and this completes the proof.

5. Numerical Simulation

In order to confirm the analytical results and clarify the impact of varying the parameters on the global dynamics of the suggested system, some numerical simulations have been displayed in detail. Many phase diagrams have been given to show the dynamics properties of the system (2.2), with the help of Matlab 8.1.

Table 1: Hypotheses parameters for system (2.2)

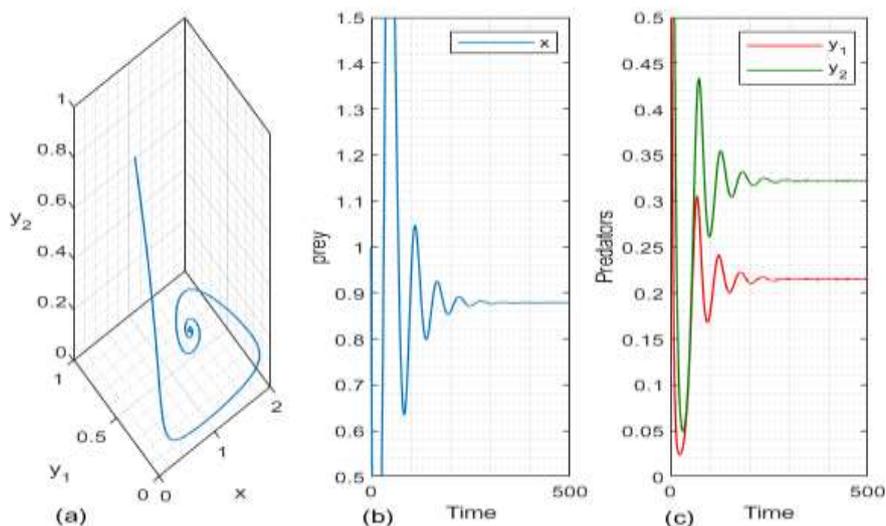| Parameter	Value	Parameter	Value
r	0.24	b	0.5
a	0.8	d	0.3
c_1	0.1	m	0.11
c_2	0.1	n_1	0.1
α	0.5	n_2	0.2

Figure 1: Dynamical behavior of the system (2.2) for the values that are given in Table 1. Graph (a) represents the phase portrait, while graphs (b) and (c) represent the time series of the solutions around $\aleph_2(0.8782, 0.2146, 0.3220)$:

Choosing the parameters of system (2.2) as in Table 1 and $(x(0), y_1(0), y_2(0)) = (1,1,1)$. Then the conditions of theorem (4.3) are satisfied as $\emptyset_1 = 0.7018 > 0, \emptyset_2 = 0.0421 > 0$, $\emptyset_3 = 0.0089 > 0$ and $\Delta = 0.0089 > 0$, and consequently IEP $\aleph_2(x^*, y_1^*, y_2^*) = (0.8782, 0.2146, 0.3220)$ is locally asymptotically stable. Figure 1(a) shows that the solution of the system (2.2) is a stable spiral converging to $\aleph_2$. Figure 1(b) and (c) show that x, y_1 and y_2 populations converge to their steady state solutions x^*, y_1^* and y_2^*, respectively.

For checking the global stability of $\aleph_2$, in Figure 2, the phase portrait and the time series of the solutions of system (2.2) with different initial points have been plotted. Here, it is found that all the solutions corresponding to these initial points converge to steady state solution $\aleph_2(0.8782, 0.2146, 0.3220)$. So, we say that system (2.2) shows its global stability near the positive equilibrium state.
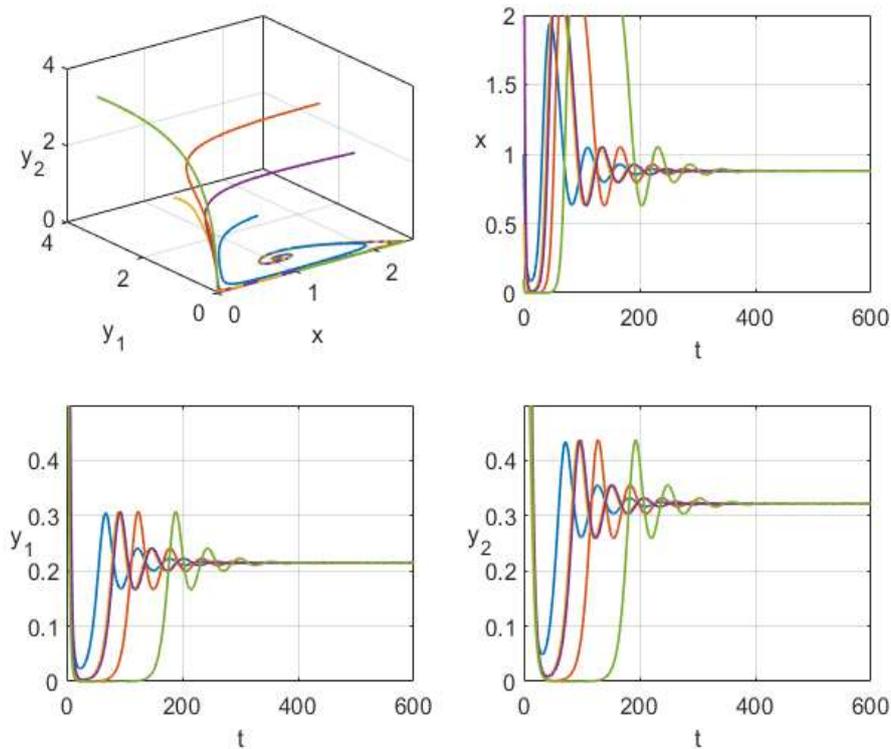

Figure 2: Global stability of IEP $\aleph_2 = (0.8782, 0.2146, 0.3220)$ for different choices of initial values $(x(0), y_1(0), y_2(0))$.

Also, it is observed from Figure 2 that $\aleph_0(0,0,0)$ is an unstable equilibrium state for system (2.2) since the eigenvalues of the variation matrix at this point are $\lambda_1 = 0.24, \lambda_2 = -0.4$ and $\lambda_3 = -0.2$.

According to Figure 3, the prey and predator species density changes due to increases in prey refuge. Figure 3(a),(b) and (c) at $m = 0.2$, $m = 0.5$ and $m = 0.7$ respectively, clearly indicate that increasing the value of m causes increasing in prey density and decreasing in the predator density gradually. Further, for m > 0.698, system (2.2) has no IEP and settles at AEP $\aleph_1 = (2.4, 0, 0)$, as displayed in Figure 3 (c). Moreover, at m=0.7, the local stability condition of theorem (4.2) is satisfied and the eigenvalues of variation matrix (2.7) at $\aleph_1$ are $\lambda_1 = -0.24$, $\lambda_2 = -0.599$ and $\lambda_3 = -0.001$. Therefore, AEP $\aleph_1 = (2.4, 0, 0)$ is globally asymptotically stable which may also be confirmed in Figure 3(d).
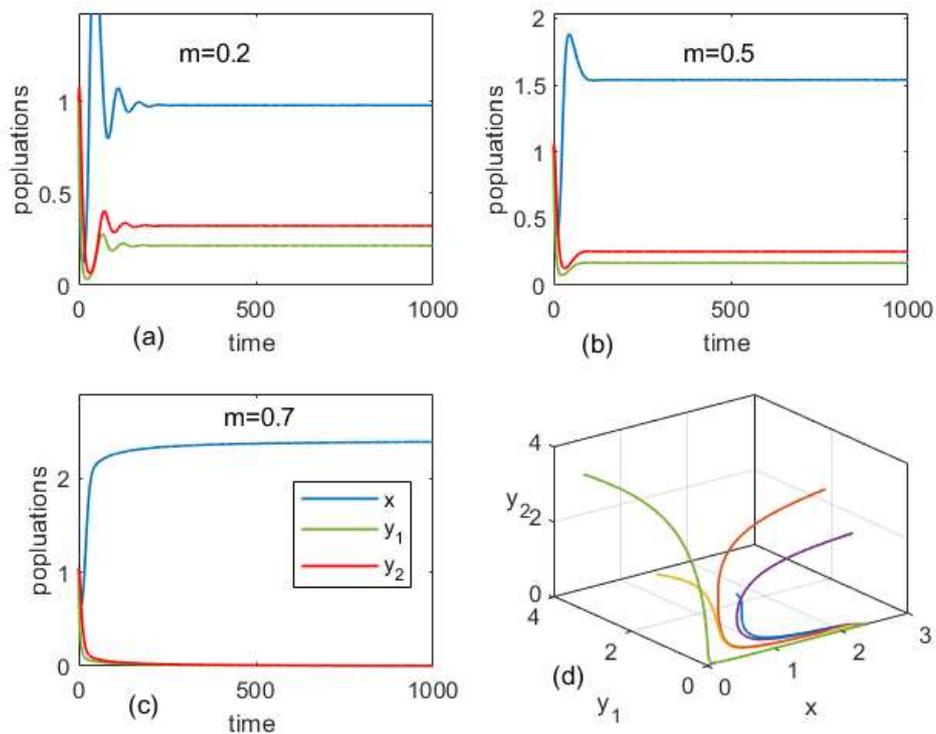
Figure 3: The dynamical behavior of the system (2.2) for different values of m. Graph (a) represents the time series for $m = 0.2$, the graph (b) represents the time series for $m = 0.5$, graph (c) represents the time series for $m = 0.7$, while graph (d) represents the 3D phase portrait around $\aleph_1 = (2.4, 0, 0)$.

Studying the impact of fear due to the mature predator on the dynamics of the system (2.2) requires increasing the value of the intrinsic growth rate. At $r = 0.6$, the local asymptotic stability condition (4.6c) of Theorem 4.3 is not satisfied as $\Delta = -0.0576 < 0$. So, system (2.2) shows unstable IEP $\aleph_2 = (0.8335, 0.0583, 0.0874)$ as seen in Figure 4 (a) and (b). If we increase the level of fear to $a = 2$, the dynamical behavior switches to stable IEP, this can be seen in Figures 4 (c) and (d).
Similarly, in Figure 5, the numerical solution of the system (2.2) is plotted with parameters used in Table 5.1 and at a specific value $c_1 = 0.05$. It is clear that the system (2.2) with decreasing intraspecific competition rate exhibits periodic dynamics (see Figure 5(a-b) at $a = 0.8$). This periodic oscillation can be controlled through the fear parameter as we gradually increase the value of a (see Figure 5(c-d) at the specific value $a = 4$).
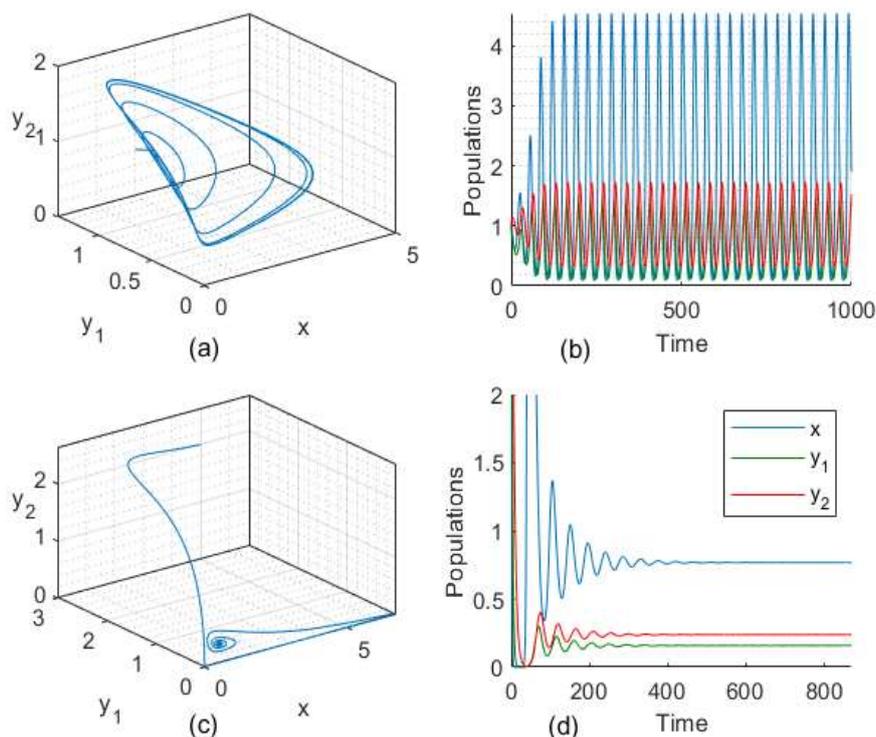
Figure 4: The dynamical behavior of the system (2.2) for r = 0.6 is presented. The graphs of (a) and (c) represent the 3D phase portrait at a =0.8 and a =2, respectively, whereas, the graph of (b) and (d) represent the time series for a =0.8 and a =2 ,respectively.
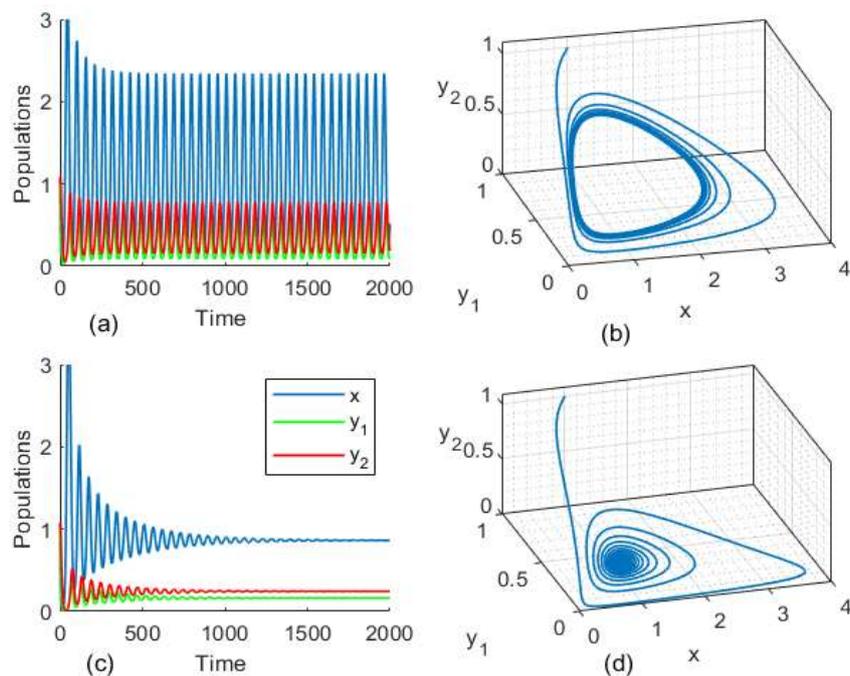

Figure 5: Dynamical behavior of the system (2.2) for c_1=0.05. The graphs (a) and (c) represent the time series for a =0.8 and a =4 respectively, whereas, the graphs of (b) and (d) represent the 3D phase portrait at a =0.8 and a =4 respectively,

Moreover, the fear cost has an effect on decreasing the immature and mature predator density but may not induce their extinction of them this is illustrated in Figure 6. In Figure 5.6(a), we used r=0.6 and the fear level a is taken from 0 to 10, whereas, in Figure 6(b), we

used c_1=0.05 and the fear level a is taken from 0 to 10. Figure 6 shows that, after the oscillations behavior at the low fear level, the dynamics of the system (2.2) continue stable steady-state behavior in the high fear level.
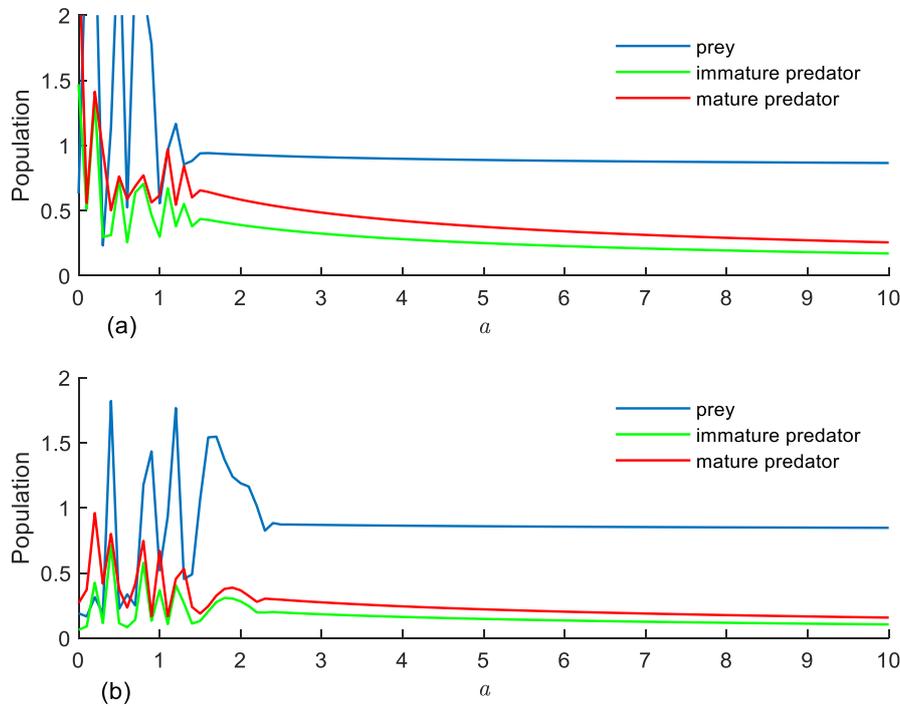

Figure 6: The biomass for prey, predator and mature predator with varying the fear effect a. The graph of (a) at $r = 0.6$, and the graph of (b) at $c_1 = 0.05$.

Now, to explore the influence of the transition rate d from immature predator to adult predator on the dynamic of the system (2.2). First, for any value of a and the value of d in the range $0 \le d < 0.0379$, i.e. the condition (4.3) holds, the solution of system (2.2) approaches asymptotically to AEP, as shown in Figure 7 for the typical value $d = 0.03$. At $a = 0.8$ and $d > 0.037$, the solution of the system (2.2) transmission from AEP to IEP as seen in Figure 7 (Top plots at $d = 0.15, 0.27, 0.5$). Clearly, varying the value of d with $a = 0.8$ the dynamical behavior switches from stable AEP to stable IEP and then to unstable IEP. Further, at $a = 4$ and d > 0.037, the solution of the system (2.2) still converges to a stable IEP due to the fear effect as demonstrated in Figure 7 (bottom plots at $d = 0.15, 0.27, 0.5$).
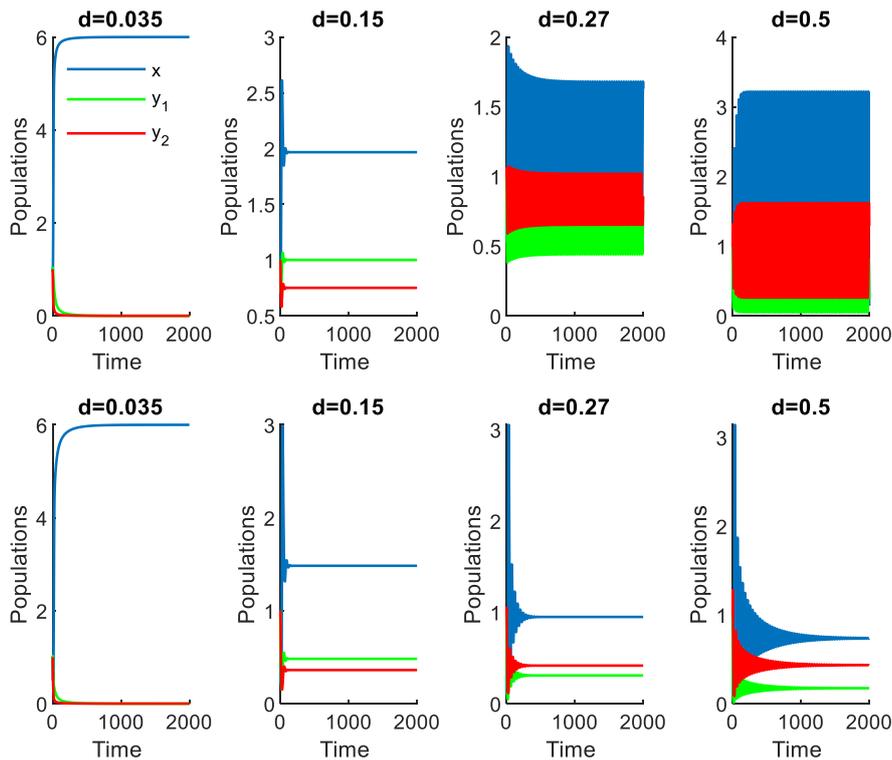
Figure 7: The time series of system (2.2) for different values of d. The top graphs at $a = 0.8$, whereas the bottom graphs at $a = 4$.

6. Conclusions:

In this paper, we have studied an ecological prey-predator model with prey refuge and fear induced by a mature predators. Intraspecific competition for prey and for an immature predators is also considered in the system. The objective is to investigate the role of fear induced by an adult predators on the dynamic behavior of this system. The properties of the solution of the model (such as positivity and boundedness) are discussed. From the theoretical analysis of the model, it is found that the system (2.2) consists of three biologically valid equilibrium points, and the nontrivial points are locally asymptotically stable, under certain conditions. Again, theoretically, it is proved that the system (2.2) is conditionally globally asymptotically stable. The numerical simulations are given to exemplify the efficacy of the theoretical results of the system (2.2), and we found that:

1-If the system (2.2) has a unique IEP, it is found that the IEP $\aleph_2 = (0.8335, 0.0583, 0.0874)$ is globally asymptotically stable. In this case, we have noticed that at small values of intrinsic growth rate r and prey refuge m, the system (2.2) has a steady-state behavior (Figures 1-2).

2- The fear influence can reduce the density of predators: as the level of fear a caused by the large predator increases, the immature predator density gradually decreases, and then the mature predator density. But in this case, the cost of fear cannot produce the extinction of predators, see Figure 6.

3-The prey refuge m has an impact on the stability of IEP in the system (2.2), It can decrease both densities of immature and mature predators due to lack of food. Here, it is observed that more increases in the amount of prey refuge $m \geq 0.7$, leads to the extinction of the predator population from the system, and the AEP is globally asymptotically stable, see Figure 3.

4- The prey intrinsic growth rate r has an impact on prey density. It is found that at $r = 0.6$ with a low level of fear among preys, the system (2.2) loses its stability, and goes to a

periodic solution. But with a high level of fear of prey, system (2.2) has no periodic behavior, and $\aleph_2$ changes from unstable to a stable equilibrium point, see Figure 4.

5-The intraspecific competition between preys has an appositive effect on the intrinsic growth rate, that is, decreasing its value may lead to a periodic solution of the system (2.2). In this case, again, increasing the level of fear changes the dynamic of the system to a stable steady-state behavior, see Figure 5.

6- The transition rate d has more influence on the dynamic of system (2.2). First, for its value in the range $0 \leq d < 0.0379$, the mature and immature predator gradually decreases to extinct. But for $d \geq 0.0379$ the dynamic of the system change to a steady-state behavior (stable or periodic solution). Again, increasing the value of fear can vanish the periodic behavior of the system (2.2) (Figure 7).

References

[1] S. Creel and D. Christianson. "Relationships between direct predation and risk effects." *Trends in ecology & evolution* , vo.l23, no.4 , pp. 194-201,2008.

[2] M. Clinchy, M.J. Sheriff, and L. Y. Zanette. "Predator-induced stress and the ecology of fear." *Functional Ecology* , vol 27, no.1 ,pp.56–65,2013

[3] X. Wang, L. Zanette, and X. Zou. "Modelling the fear effect in predator–prey interactions." *Journal of mathematical biology* , vol 73, no.5 , pp. 1179-1204,2016.

[4] K. Sarkar, and S. Khajanchi. "Impact of fear effect on the growth of prey in a predator-prey interaction model." *Ecological Complexity* , no.42 , pp 100826, 2020.

[5] MC. Anisiu , "Lotka, Volterra and their model." *Didactica Mathematica,* vol. 32, pp. 9-17, 2014.

[6] S. Al-Momen, and R.K. Naji. "The Dynamics of Sokol-Howell Prey-Predator Model Involving Strong Allee Effect." *Iraqi Journal of Science*, vol 62, no.9, pp.3114-3127, Sep 2021.

[7] R.K. Naji, and S.J. Majeed, "The dynamical analysis of a prey-predator model with a refuge-stage structure prey population." *International Journal of Differential Equations* , vol. 2016, pp10, 2016

[8] S. Khajanchi. "Modeling the dynamics of stage-structure predator-prey system with Monod–Haldane type response function." *Applied Mathematics and Computation*, vol.302, pp .122-143, 2017.

[9] Y.N. Xiao, and L .S. Chen. "Global stability of a predator-prey system with stage structure for the predator." *Acta Mathematica Sinica* , vol, 20, no.1 , pp. 63-70, Feb 2004.

[10] P. Georgescu, and Y.H. Hsieh. "Global dynamics of a predator-prey model with stage structure for the predator." *SIAM Journal on Applied Mathematics,* vol.67, no. 5 , pp 1379-1395, Jul 2007.

[11] X. Tian, and R. Xu. "Global dynamics of a predator-prey system with Holling type II functional response." *Nonlinear Analysis: Modelling and Control*, vol.16, no. 2, pp.242-253, May 2011.

[12] S. Khajanchi. "Dynamic behavior of a Beddington–DeAngelis type stage structured predator–prey model." *Applied Mathematics and Computation*, no.244, pp 344-36, 2014.

[13] Q.Liu, D. Jiang, T. Hayat, and A. Alsaedi, "Dynamics of a stochastic predator–prey model with stage structure for predator and Holling type II functional response." *Journal of Nonlinear Science* , vol 3, no. 28, pp. 1151-1187, Jan,2018.

[14] R.k. Naji, and S.J. Majeed. "A study of prey-predator model with stage-structre in predator." *World Journal of Modelling and Simulation*, vol. 16, no.2 , pp. 95-106, 2020.

[15] N. Mondal, D. Barman, and S. Alam. "Impact of adult predator incited fear in a stage-structured prey–predator model." *Environment, Development and Sustainability*, vol.23, no .6 , pp.9280-9307,Sep2021.

[16] H. Zhang ,Y. Cai, S. Fu, and W. Wang. "Impact of the fear effect in a prey-predator model incorporating a prey refuge," *Applied Mathematics and Computation*, no. 356 , pp. 328-337,2019.