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Semisimple Modules Relative to A Semiradical Property

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Abstract
In this paper, we introduce the concept of s.p-semisimple module. Let S be a semiradical property, we say that a module M is s.p - semisimple if for every submodule N of M, there exists a direct summand K of M such that $K \leq N$ and N / K has S. we prove that a module M is s.p - semisimple module if and only if for every submodule A of M, there exists a direct summand B of M such that A = B + C and C has S. Also, we prove that for a module M is s.p - semisimple if and only if for every submodule A of M, there exists an idempotent $e \in End(M)$ such that $e(M) \leq A$ and (1- e)(A) has S.

Keywords: Semiradical (radical) property, Semisimple modules, t- semisimple modules.

المقاسات البسيطة نسبة الى خاصية شبه جذرية

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الخلاصة
في هذا البحث نقدم مفهوم المقاسات شبه بسيطة نسبة لخاصية شبه جذرية. لنفترض أن S خاصية شبه جذرية فنحن نقول أن المقاس M هو شبه بسيط نسبة لخاصية شبه جذرية إذا كان لكل مقاس جزئي N من M، يوجد جمع مباشر K في M بحيث يكون $K \leq N$ و N / K تمتلك S. برهنا ان المقاس M هو شبه بسيط نسبة لخاصية شبه جذرية إذا وفقط إذا كان لكل مقاس جزئي A من M ، يوجد جمع مباشر B من M بحيث يكون A = B + C و C تمتلك S. كما أننا برهنا ذلك. المقاس M يكون شبه بسيط نسبة لخاصية شبه جذرية إذا وفقط إذا كان لكل مقاس جزئي A من M ، يوجد متساوي القوى $e\in End(M)$ بحيث ان $e(M) \leq A$ و (A −1) (e) تمتلك S.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary left R-modules. Let A be a submodule of a module M. A is called an essential submodule of M (denoted by $A \leq_e M$) if $A \cap B \neq 0$, $\forall\ 0 \neq B \leq M$. A submodule B of M is called a closed submodule of M if B has no proper essential extension. A module M is called an extending module if every submodule of M is essential in a direct summand. Equivalently, every closed submodule of M is a direct summand, see [1], [2], [3].

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Let M be a module. Recall that the socle of M (denoted by Soc(M)) is the sum of all simple submodules of M, a module M is called a semisimple if Soc(M) = M. Equivalently a module M is semisimple if and only if every submodule is a direct summand of M, see [1], [4]. Recall that the Jacobson radical of M (denoted by J(M)) is the intersection of all maximal submodules of M. If M has no maximal submodule, we write J(M) = M, see [5].

Let $x \in M$. Recall that ann (x) = $\{r \in R: rx = 0\}$. For a module M, the singular submodule is defined as follows $Z(M) = \{x \in M \mid \text{ann } x \leq_e R\}$ or equivalently, Ix = 0 for some essential left ideal I of R. If Z(M) = M, then M is called a singular module. If Z(M) = 0, then M is called a nonsingular module. The second singular (or Goldie torsion) submodule of a module M (denoted by $Z_2(M)$) is defined by $Z(M / Z(M)) = Z_2(M) / Z(M)$, see [1],[6].

A submodule A of a module M is called t- essential submodule (denoted by $A \leq_{tes} M$) if for any submodule B of M, $A \cap B \leq Z_2(M)$ implies $B \leq Z_2(M)$. A module M is called t-semisimple if for every submodule N of M there exists a direct summand K of M such that $K \leq_{tes} N$, see [5]. [7].

A property S is called a radical property if:
1- for every module M, there exists a submodule (denoted by S(M)) such that
a- S(M) has S.
b- $A \leq S(M)$, for every submodule A of M such that A has S.
2- If $f: M \rightarrow N$ is an epimorphism and M has S, then N has S.
3- S(M / S(M)) = 0 for every R- module M, see [8].
A property S is called a semiradical property if it satisfies conditions 1 and 2, see [8].

It's known that each of the following two properties is a radical property, see [8].

1- $S = Z_2$. For a module M, $S(M) = Z_2(M)$, the second singular of M.
2- S = Snr. For a module M, Snr(M) is a submodule of M such that
a_1- J(Snr(M)) = Snr(M) {i.e. Snr(M) has no maximal submodule}.
b_2- $A \leq$ Snr (M), for every submodule A of M such that J(A) = A, see [8].

While each of the following two properties is a semiradical property (but it is not radical property), see [8].
1- S = Z. For a module M, S(M) = Z (M), the singular submodule of M.
2- S = Soc. For a module M, $S(M) = Soc(M) = \sum_{\substack{A \leq M \\ A \text{ is simple}}} A$.
Let S be a semiradical property. It is known that
1- M has S if and only if S(M) = M.
2- S(S(M)) = S(M).
3- If $M = \bigoplus_{i \in I} M_i$, then $S(M) = \bigoplus_{i \in I} S(M_i)$, where I is any index set.
4- if S(M) = 0, then S(A) = 0, $\forall A \leq M$.
5- For any short exact sequence $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$, if S(M) = 0 and S(K) = 0, then S(N) = 0, see [8].

In this paper, S is a semiradical property, unless otherwise stated.

2- s.p - semisimple modules

In this section, we introduce the concept of s.p-semisimple modules and give the basic properties of this module. Also, we illustrate it with some examples.

Definition2.1. Let S be a semiradical property. We say that a module M is s.p - smisimple module if for each submodule N of M, there exists a direct summand K of M such that $K \leq N$ and N / K has S.

Remarks and Examples2.2.
1- Every semisimple module is s.p - semisimple. The converse is not true in general.
Proof. Let N be a submodule of a semisimple module M, then N is a direct summand of M, by [4]. Let K = N, hence $S(N / K) = S(N / N) = S(0) = 0 \cong N / K$. Thus M is s.p - semisimple. For example Z_6 as Z_6- module is s.p - semisimple module.

For the converse, Let S = Second singularity. Consider module Z_4 as Z- module. Since Z_4 is singular, then every submodules of Z_4 is singular, by [1]. Therefore, $Z_2(N) = Z(N) = N$, $\forall N \leq Z_4$. Let K = 0, hence $Z_2(N / 0) \cong Z_2(N) = Z(N) = N \cong N / 0$. So N / 0 has S, $\forall N \leq Z_4$. Thus Z_4 is s.p - semisimple. Cleary that Z_4 is not semisimple.

Recall that a semiradical property S is called hereditary if S is closed under submodules, see [8].
2- Let S be a hereditary property and M be a module. If M has S, then M is s.p - semisimple.

Proof. Let N be a submodule of M and S(M) = M. Since S is hereditary, then S(N) = N. Let K = 0, then $S(N / 0) \cong S(N) = N \cong N / 0$. Thus M is s.p - semisimple.

3- Let S = singularity. Consider module Q as Z-module. Clearly, that Q is nonsingular. Hence, Z(Q) = 0. Let N = 3Z. Since Q is indecomposable, then 0 is the only direct summand contained in 3Z. So $S(3Z / 0) \cong S(3Z) = Z(3Z) = 0$. Thus Q is not s.p - semisimple module.

Proposition2.3. Every submodule of s.p - semisimple module M is s.p – semisimple, For every property S.
Proof. Let N be a submodule of M and $A \leq N$. Since M is s.p - semisimple, then there exists a direct summand K of M such that $K \leq A$ and A / K has S. By modular law, K is a direct summand of N. Thus N is s.p - semisimple.

Proposition2.4. Let M be an indecomposable module and S be an assumed. Then M is s.p - semisimple if and only if every proper submodule of M has S.
Proof. $\Rightarrow$) Let N be a proper submodule of M. Since M is s.p - semisimple, then there exists a direct summand K of M such that $K \leq N$ and N / K has S. But M is an indecomposable. Therefore, K = 0. Hence $S(N) \cong S(N / 0) = S(N / K) = N / K = N / 0 \cong N$. Thus N has S.
$\Leftarrow$) Clear.
Let S be a semiradical property. Recall that S is called a cohereditary property, if S(M) = 0 is closed under homomorphic images of M for every module M, see [8].

Proposition2.5. Let S be a cohereditary property and let M be a module. If S(M) = 0. Then M is semisimple if and only if M is s.p - semisimple.

Proof. ⇒) Clear.
⇐) Let N be a submodule of M. Since M is s.p - semisimple, then there exists a direct summand K of M such that $K \leq N$ and N / K has S. But S(M) = 0, therefore S(N) = 0, by [8]. Since S is cohereditary property, then S(N / K) = 0. Hence N = K is a direct summand of M. Thus M is semisimple.

Remark2.6. Let S be a hereditary property and M be a module. If S(M) = M, then M / N is s.p - semisimple module, for each submodule N of M.

Proof. Let N be a submodule of M and S(M) = M, then M / N has S, by [8]. Thus by. 2.2-2, M / N is s.p - semisimple module.

Proposition2.7. Let M be s.p - semisimple module. Then every submodule N of M such that S(N) = 0 is a direct summand of M. The converse is true if S(M) = 0.

Proof. Assume that N is a submodule of M such that S(N) = 0. Then there exists a direct summand K of M such that $K \leq N$ and N / K has S. Let $M = K \oplus K_1$, for some submodule K_1 of M. By modular law, $N = K \oplus (N \cap K_1)$. Since $N \cap K_1 \leq N$ and S(N)=0, then $S(N \cap K_1) = 0$, by [8]. Since $N / K = (K \oplus (N \cap K_1)) / K \cong (N \cap K_1) / 0 \cong N \cap K_1$, by the second isomorphism theorem, then S(N / K) = 0. But S(N / K) = N / K, therefore N / K = 0. Thus N = K is a direct summand of M.

Conversely, let S(M) = 0 and N be a submodule of M. Then S(N) = 0, by [8]. By our assumption N, is a direct summand of M. Therefore M is semisimple. Thus by 2.2-1, M is s.p - semisimple module.

Proposition 2.8. Let M = A + S(M) be s.p - semisimple module. Then there exists a direct summand B of M such that $B \leq A$, M = B + S(M) and A / B has S.

Proof. Assume that M is s.p - semisimple module. Then there exists a direct summand B of M such that $B \leq A$ and A / B has S. Let $M = B \oplus C$, for some submodule C of M. Then $A = B \oplus (C \cap A)$, by modular law. But $A / B \cong (C \cap A)$, by the second isomorphism theorem, therefore $(C \cap A)$ has S. Since $(C \cap A)$ has S, then $(C \cap A) \leq S(M)$. Thus $M = A + S(M) = B + (C \cap A) + S(M)$ and hence M = B + S(M).

Proposition2.9. Let S be a hereditary property and $M = M_1 \oplus M_2$ be a module such that M_1 has S and M_2 is semisimple. Then M is s.p - semisimple module.

Proof. Let N be a submodule of M. Since M_2 is semisimple, then $N \cap M_2$ is a direct summand of M_2. But, M_2 is a direct summand of M, therefore $N \cap M_2$ is a direct summand of M. By the second isomorphism theorem, $M / M_2 = (M_1 \oplus M_2) / M_2 \cong M_1$. Since M_1 has S, then M / M_2 has S. But $N / (N \cap M_2) \cong (N + M_2) / M_2 \leq M / M_2$ and S hereditary property. So $N / (N \cap M_2)$ has S. Thus M is s.p - semisimple module.

Corollary 2.10. Let S be a hereditary property and M be a module. If $M = S(M) \oplus M_1$, where M_1 is a semisimple module, then M is s.p - semisimple module.

Proof. Clear.

Proposition 2.11. Let $M = M_1 \oplus M_2$ be a module such that $R = Ann(M_1) + Ann(M_2)$. If M_1 and M_2 are s.p - semisimple modules, then M is s.p - semisimple module.

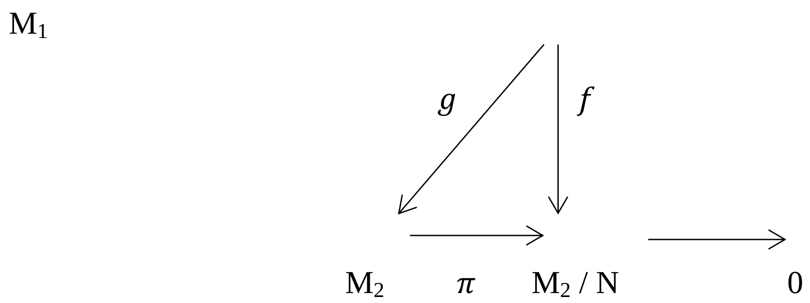
Proof. Let N be a submodule of $M = M_1 \oplus M_2$. Since $R = Ann(M_1) + Ann(M_2)$, then by the same argument of the proof [9, prop.4.2, CH.1], $N = N_1 \oplus N_2$, where $N_1 \leq M_1$ and $N_2 \leq M_2$. Since M_i is s.p - semisimple for i= 1, 2, then there exist direct summands K_i of M_i such that K_i is a submodule of N_i and N_i / K_i has S (i = 1, 2). Let $M_i = K_i \oplus L_i$, for some submodule L_i of M_i. Therefore $M = M_1 \oplus M_2 = (K_1 \oplus L_1) \oplus (K_2 \oplus L_2) = (K_1 \oplus K_2) \oplus (L_1 \oplus L_2)$. Hence $(K_1 \oplus K_2)$ is a direct summand of M and $(K_1 \oplus K_2) \leq N_1 \oplus N_2 = N$. Now since N_i / K_i has S (i = 1, 2), then by [8], $(N_1 / K_1) \oplus (N_2 / K_2)$ has S. But $(N_1 / K_1) \oplus (N_2 / K_2) \cong ((N_1 \oplus N_2) / (K_1 \oplus K_2))$, by [10, p. 33], hence $(N_1 \oplus N_2) / (K_1 \oplus K_2) = N / (K_1 \oplus K_2)$ has S. Thus M is s.p - semisimple module.

Let M be an R- module. Recall that M is called a duo-module if every submodule of M is fully invariant, see [11].

Proposition 2.12. Let $M = \oplus_{i \in I} M_i$ be a duo module. Then M is s.p - semisimple modules if and only if M_i is s.p - semisimple module $\forall$ i $\in$ I.

Proof. Since M is s.p - semisimple, then by prop.2.3, M_i is s.p - semisimple, $\forall$ i$\in$ I.
Conversely, let $M = \oplus_{i\in I} M_i$ be a module such that M_i is s.p - semisimple, $\forall$ i $\in$ I. Let $N \leq M$, then $N = N \cap M = N \cap (\oplus_{i\in I} M_i) = \oplus_{i\in I} (N \cap M_i)$, by [12,lem.2.1]. Let $N_i = N \cap M_i$, $\forall$ i $\in$ I, then $N_i \leq M_i$ $\forall$ i$\in$ I. Since M_i is s.p - semisimple, then there exists K_i is a direct summand of M_i such that K_i is a submodule of N_i and N_i / K_i has S $\forall$ i $\in$ I. Hence $((\oplus_{i\in I} N_i) / (\oplus_{i\in I} K_i)) \cong \oplus_{i\in I} (N_i / K_i)$ has S, by [10]. Thus $M = \oplus_{i\in I} M_i$ is s.p - semisimple.

Let M_1 and M_2 be R- modules. M_1 is called M_2- projective if for every submodule N of M_2 and any homomorphism $f: M_1 \to M_2 / N$, there is a homomorphism $g : M_1 \to M_2$ such that $\pi o g = f$. where $\pi: M_2 \to M_2 / N$ is the natural epimorphism, see [13].



M_1 and M_2 are called relatively projective if M_1 is M_2- projective and M_2 is M_1- projective.

We know that for a module $M = A \oplus B$. A is B-projective if and only if for every submodule C of M such that $M = B + C$, there exists a submodule D of C such that $M = B \oplus D$, see [14].

Proposition2.13. Let S be a hereditary property. Let M_1 and M_2 be s.p - semisimple modules such that M_1 and M_2 are relative projective. Then $M = M_1 \oplus M_2$ is s.p - semisimple.

Proof. Let N be a submodule of M. Since $(N + M_2) \cap M_1 \leq M_1$ and M_1 is s.p - semisimple, then there exists a direct summand A_1 of M such that $A_1 \leq (N + M_2) \cap M_1$ and $((N + M_2) \cap M_1) / A_1$ has S. Let $M_1 = A_1 \oplus B_1$, for some submodule B_1 of M_1. Hence $(N + M_2) \cap M_1 = A_1 \oplus ((N + M_2) \cap M_1) \cap B_1$, by modular law. Since by the second isomorphism theorem, $((N + M_2) \cap M_1) / A_1 \cong (N + M_2) \cap M_1 \cap B_1)$, then $(N + M_2) \cap B_1$ has S, by [8]. Therefore $M = M_1 \oplus M_2 = A_1 \oplus B_1 \oplus M_2 = (N + M_2) \cap M_1 + B_1 + M_2 = N + M_2 + B_1 + M_2 = N + (M_2 \oplus B_1)$. Since $(N + B_1) \cap M_2 \leq M_2$ and M_2 is s.p - semisimple, then there exists a direct summand A_2 of M_2 such that $A_2 \leq (N + B_1) \cap M_2$ and $((N + B_1) \cap M_2) / A_2$ has S. Let $M_2 = A_2 \oplus B_2$, for some submodule B_2 of M_2 then $(N + B_1) \cap M_2 = A_2 \oplus ((N + B_1) \cap M_2) \cap B_2$, by modular law. By the second isomorphism theorem, $((N + B_1) \cap M_2) / A_2 \cong ((N + B_1) \cap M_2) \cap B_2$, then $(N + B_1) \cap M_2 \cap B_2 = (N + B_1) \cap B_2$ has S, by [8]. Thus $M = N + (M_2 \oplus B_1) = N + A_2 + B_2 + B_1 = N + (B_1 \oplus B_2)$. Since $M = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$ and M_1 and M_2 are relative projective, then A_1 is B_j - projective and A_2 is B_j - projective for j = 1, 2, by [9, prop. 2.1.6]. So by [15, prop.2.1.7], A_1 is $B_1 \oplus B_2$-projective and A_2 is $B_1 \oplus B_2$ - projective. Hence $A_1 \oplus A_2$ is $B_1 \oplus B_2$-projective, by [15, prop.2.1.6]. Hence, there exists $X \leq N$ such that $M = X \oplus B_1 \oplus B_2$, by [14, lem. 5].

Now, we want to show that $N \cap (B_1 \oplus B_2)$ has S. Since $(N+M_2) \cap B_1 = ((N + (A_2 \oplus B_2)) \cap B_1$ has S and $(N + B_2) \cap B_1 \leq ((N + (A_2 \oplus B_2)) \cap B_1$, then $(N \oplus B_2) \cap B_1$ has S. Since $(N + B_1) \cap B_2$ has S, then $(N \oplus B_2) \cap B_1 \oplus (N \oplus B_1) \cap B_2$ has S, by [8]. But by [15,lem.3.2], $N \cap (B_1 \oplus B_2) \leq (N \oplus B_2) \cap B_1 \oplus (N \oplus B_1) \cap B_2$. Therefore, $N \cap (B_1 \oplus B_2)$ has S. Thus M is s.p - semisimple module.

Let M be an R-module. M is said to have the summand intersection property (briefly SIP) if the intersection of any two direct summands of M is a direct summand of M, see [16].

Proposition 2.14. Let M be s.p - semisimple module. If for any two direct summand A and B of M, $S(A \cap B) = 0$, then M has SIP.

Proof. Let A and B be direct summands of M. Since M is s.p - semisimple, then there exists a direct summand N of M such that $N \leq A \cap B$ and $(A \cap B) / N$ has S. Let $M = N \oplus N_1$, for some submodule N_1 of M, then $A \cap B = N \oplus (N_1 \cap (A \cap B))$. Hence by the second isomorphism theorem, $(A \cap B) / N = [N \oplus (N_1 \cap (A \cap B))] / N \cong N_1 \cap (A \cap B) \leq A \cap B$. Since $S(A \cap B) = 0$, then $S(N_1 \cap (A \cap B)) = 0$, by [8]. So $S((A \cap B) / N) = 0$. But $(A \cap B) / N$ has S, therefore $A \cap B = N$. Hence $A \cap B$ is a direct summand of M. Thus M has SIP.

Let R be an integral domain. Recall that an R- module M is called a torsion free module if ann (x) = 0, for all $0 \neq x \in M$, see [1].

Theorem 2.15. Let R be an integral domain and M be a torsion free module and s.p - semisimple module. Then for every $m \in M$, either Rm is a direct summand of M or Rm has S.

Proof. Let $0 \neq m \in M$. Then there exists a direct summand K of M such that $K \leq Rm$ and Rm / K has S. Let $M = K \oplus H$, for some submodule H of M. Then $Rm = K \oplus (Rm \cap H)$, by modular law. But $Rm / K \cong Rm \cap H$, by the second isomorphism theorem. Therefore $Rm \cap H$ has S.

Let : $R \rightarrow Rm$ be a map defined by $f(r) = rm$, for each $r \in R$.It is easy to see that f is an epimorphism and Ker (f) = ann (m). By the first isomorphism theorem, R / ann(m) $\cong$ Rm. Since M is torsion free module, then ann(m)= 0. Thus $R \cong Rm$. But R is indecomposable.

Therefore, Rm is indecomposable. Implies that either Rm = K or Rm = Rm ∩ H. Thus either Rm is a direct summand of M or Rm has S.

Proposition2.16. Let R be an indecomposable ring and M be a projective module. If M is s.p - semisimple module, then for every m ∈ M, either Rm is a direct summand of M or Rm has S.

Proof. Assume that M is a projective and s.p - semisimple module and let m ∈ M. Then there exists a direct summand K of M such that K ≤ Rm and Rm / K has S. Let M = K ⊕ H for some submodule H of M, then Rm = K ⊕ (H ∩ Rm), by modular law. But Rm / K ≅ H ∩ Rm, by the second isomorphism theorem. Therefore, H ∩ Rm has S.

Now, let f: R → Rm be a map defined by f(r) = rm, for all r ∈ R. It is clear that f is an epimorphism map. Let P: Rm → K be the projection map. Clearly, Pof: R → K is an epimorphism. Since M is projective, then K is projective by [4]. Therefore, Ker (Pof) is a direct summand of R. Since R is indecomposable, then either Ker Pof = 0 or Ker Pof = R. Ker (Pof) = f^{-1} (Rm ∩ H) = f^{-1} (Rm ∩ H). So either Rm ∩ H = 0 or Rm ∩ H = R. Thus Rm = K or Rm ∩ H = Rm has S.

3- Characterization of s.p - semisimple Modules

In this section, we give various characterizations of s.p - semisimple modules.

We start with the following theorem.

Theorem 3.1. Let M be a module. Then the following statements are equivalent

1- M is s.p - semisimple module.

2- For every submodule A of M, there exists a decomposition M = B ⊕C such that B ≤ A and A ∩ C has S.

3- For every submodule A of M, A= A_1 ⊕A_2, where A_1 is a direct summand of M and A_2 has S.

Proof. 1⇒2) Let A be a submodule of M. Since M is s.p - semisimple, then there exists a direct summand B of M such that B ≤ A and A / B has S. Let M = B ⊕ C, where C is a submodule of M. Then A = B ⊕ (C ∩ A), by modular law. By the second isomorphism theorem, A / B ≅ (C ∩ A). Thus A / B ≅ C ∩ A.

2 ⇒3) Let A be a submodule of M. By (2), there exists a decomposition M = B ⊕ C such that B ≤ A and A ∩ C has S. By modular law, A =B ⊕ (C ∩ A). Let A_2 = A ∩ C has S.

3⇒1) Let A be a submodule of M. By (3), A = A_1 ⊕ A_2, where A_1 is direct summand of M and A_2 has S. By the second isomorphism theorem, A / A_1 ≅ A_2. So A / A_1 has S. Thus M is s.p - semisimple.

Proposition 3.2. A module M is s.p - semisimple if and only if for every submodule A of M there exists a direct summand B of M such that A = B + C, where C is a submodule of M has S.

Proof. ⇒) It is clear by Theorem3.1.

⇐) Let A be a submodule of M. By our assumption, there exists a direct summand B of M such that A = B + C and C has S. Let M = B ⊕ D, for some submodule D of M, then A = B ⊕ (A ∩ D), by modular law. Hence, (A / B) = (B + C) / B ≅ C / (B ∩ C), by the second isomorphism theorem. But C has S, then C / (B ∩ C) has S. This implies that A / B has S. Thus M is s.p - semisimple.

Proposition3.3. A module M is s.p - semisimple if and only if for each submodule A of M, there exists an idempotent $e \in End(M)$ such that $e(M) \leq A$ and $(1-e)(A)$ has S.

Proof. ⇒) Let A be a submodule of M. Since M is s.p - semisimple, then there exists a decomposition $M = B \oplus C$ such that $B \leq A$ and $A \cap C$ has S, by th.3.1, 1-2. Let $e : M \to B$ be the projection map. Clearly that $e^2 = e$ and $C = (1 - e)(M)$. Claim that $(1-e)(A) = (1-e) M)\cap A$. To show that, let $m \in (1-e)(A)$, then there is $a \in A$ such that $m = (1 - e)(a) = a - e(a)$. Therefore $m \in A$ and hence $m \in (1-e)(M) \cap A$. Thus $(1-e)(A) \leq (1-e)(M) \cap A$. Now, let $n \in (1-e)(M) \cap A$, then $n \in (1-e)(M)$ and $n \in A$. Hence, there is $k \in M$ such that $n = (1 - e)(k) = k - e(k)$. So $n + e(k) = k \in A$. then $n \in (1-e)(A)$. Thus $A \cap C = A \cap (1-e)(M) = (1-e)(A)$. Thus $(1-e)A$ has S.
⇐) Let A be a submodule of M and $e \in End(M)$ be an idempotent such that $e(M) \leq A$ and $(1-e)A$ has S. Claim that $M = e(M)\oplus(1-e)(M)$. To show that, let $x \in M$, then $x = x+ e(x)- e(x) = e(x) + x - e(x) = e(x) + (1 - e)(x)$. Thus $M = e(M) + (1-e)(M)$.
Now, let $y \in e(M) \cap (1-e)(M)$, then $y = e(m_1)$ and $y = (1 - e)(m_2)$, for some $m_1, m_2 \in M$. So $y = e(m) = e(e(m_1)) = e((1- e)(m_2)) = e(m_2) - e(m_2) = 0$, then $y = e(m_1) = 0$. Thus $M = e(M)\oplus(1-e)(M)$. Let $B = e(M) \leq A$ and $C = (1-e)(M)$. Therefore $M = B \oplus C$ and $A \cap C = A \cap(1-e)M = (1-e)A$ has S. Thus M is s.p - semisimple, by Theorem 3.1.

Let M be a module and N be a submodule of M. Recall that a submodule K of M is called an S-generalized supplement of N in M, if $M = N + K$ and $N \cap K \leq S(K)$, see [17].

Let M be a module. Recall that M is called an S-generalized supplemented module (or briefly S-GS module), if every submodule of M has S-generalized supplement in M, where S is semiradical property on modules, see [17].

Proposition3.4. Every s.p - semisimple module M is S-GS supplemented module.

Proof. Let M is s.p - semisimple module and N be a submodule of M, then there exists a direct summand K of M such that $K \leq N$ and N / K has S. Hence, $M = K \oplus K_1$, for some submodule K_1 of M. But $K \leq N$, therefore $M = N + K_1$. So by modular law, $N = K\oplus(N \cap K_1)$, then by the second isomorphism theorem, $N / K \cong N \cap K_1$ has S. Thus $N \cap K_1 \leq S(K)$ by [8].

Proposition3.5. Let M be s.p-semisimple module. If $M = N + K$, where N is a direct summand of M, then N contains an S-generalized supplement submodule of K in M.

Proof. Since M is an s.p - semisimple, then by Theorem 3.1.1-3, $N \cap K = A \oplus B$, where A is a direct summand of M and B has S. Let $M = A \oplus C$, for some submodule C of M. Hence, $N = A \oplus(N\cap C)$, by modular law. Let $A_1 = N\cap C$, then $M = N + K = (A+A_1) + K$. But $A \leq K$. Therefore, $M = K + A_1$. Now we want to show $K \cap A_1 \leq S(A_1)$. Since $N \cap K = (A \oplus A_1) \cap K = A \oplus (K\cap A_1)$, by modular law. Let : $N = A \oplus A_1 \to A_1$ be the projection map. So we have $K \cap A_1 = (A \oplus (K\cap A_1) = (N \cap K) = (A \oplus B) = (B)$. But B has S. Therefore, $K \cap A_1$ has S, by [8]. Hence, $K\cap A_1 \leq S(A_1)$. Thus A_1 is an S-generalized supplement submodule of K in M and A_1 is contained in N.

Proposition3.6. Let S be a hereditary property and M be a module. Then the following statements are equivalent
1- M is s.p - semisimple module.
2- Every submodule N of M has S-generalized supplement K in M such that $N \cap K$ is a direct summand of N.

Proof. 1⇒2) Let N be a submodule of M. Then by the same argument of proof of Proposition 3.4. N has an S-generalized supplement.
2⇒1) Let N be a submodule of M. Then by our assumption N has an S-generalized supplement K in M such that N∩K is a direct summand of N. Hence M = N + K and N ∩ K ≤ S(K). Let N= (N ∩ K) ⊕ L, for some submodule L of N. Then M = (N∩K) + L + K = L + K. But, L ∩ K = N ∩ K ∩ L = 0. Therefore, M = L ⊕ K. By the second isomorphism theorem, N / L ≅ N ∩ K. Since N ∩ K ≤ S(K) and S is hereditary property, then N ∩ K has S by [8] and hence N / L has S. Thus M is s.p-semisimple.

Proposition3.7. Let M be a module. If M is S-GS supplemented module, then M / S(M) is a semisimple module.

Proof. Let N / S(M) be a submodule of M / S(M). Since M is S-GS supplemented, then there exists a submodule K of M such that M = N + K and N ∩ K ≤ S(K). Then M /S(M) = (N+K)/S(M) = N /S(M)) + (K+S(M))/S(M)). Since (N/S(M)) ∩ ((K+S(M))/ S(M)) = [(N ∩ K) + S(M)] / S(M)), by modular law and N ∩ K ≤ S(K) ≤ S(M), by [17]. Then (N ∩ K) + S(M) = S(M). Therefore M / S(M) = (N / S(M)) ⊕ ((K + S(M) / S(M)). Thus M / S(M) is semisimple.

Corollary3.8. Let M be a module. If M is S-GS supplemented module, then M / S(M) is s.p - semisimple module.

Proof. It is clear by Proposition. 3.7 and 2.2-1.

Proposition3.9. Let M be s.p-semisimple module. Then every submodule N of M has an S-generalized supplement which is a direct summand of M.

Proof. Let N be a submodule of M, then there exists a decomposition M = A ⊕ B such that A ≤ N and N ∩ B has S, by Theorem 3.1, 1-2. Clearly M = N + B and N ∩ B ≤ S(B). Thus B is an S-generalized supplement of N which is a direct summand of M.

Let M be an R- module. Recall that M is called π-projective (or co-continuous) if for every two submodules U, V of M with U + V = M there exists f ∈ End(M) with Im (f) ≤ U and Im (1− f) ≤ V, see [18].

Proposition3.10. Let S be a hereditary property and a module M be a π-projective module. Then M is s.p - semisimple if and only if M is S-GS module.

Proof. ⇒) It is clear by Proposition 3.4.
⇐) Let N be a submodule of M. Since M is S-GS module, then there exists a submodule K of M such that M = N + K and N ∩ K ≤ S(K). Since M is π- projective, then there exists an idempotent e ∈ End (M) such that Im (e) ≤ N and Im (1 − e) ≤ K. But by the same proof of Proposition 3.3 we have N(1- e) = N ∩ (1 – e)M ≤ N ∩ K ≤ S(K) and S is hereditary property, therefore N(1- e) has S. Thus by Proposition 3.3 M is s.p - semisimple.

Conclusion

In this work, the concept of s.p-semisimple module is introduced and studied. We also conclude the following:

1. Every semisimple module is s.p – semisimple. However, the converse is not true. Let S = Second singularity. Consider module Z_4 as Z- module. Since Z_4 is singular, then every submodules of Z_4 is singular, by [1]. Therefore, $Z_2(N) = Z(N) = N$, $\forall N \leq Z_4$. let K = 0, hence $Z2(N / 0) \cong Z_2(N) = Z(N) = N \cong N / 0$. So N / 0 has S, $\forall N \leq Z_4$. Thus Z_4 is s.p - semisimple. Clearly, that Z_4 is not semisimple.
2. Let $M = \bigoplus_{i \in I} M_i$ be a duo module. Then M is s.p - semisimple modules if and only if M_i is s.p - semisimple module $\forall i \in I$.
3. Let S be a hereditary property. If M_1 and M_2 are s.p - semisimple modules such that M_1 and M_2 are relative projective. Then $M = M_1 \bigoplus M_2$ is s.p - semisimple.
4. Every s.p - semisimple module M is S-GS supplemented module.
5. Let S be a hereditary property and a module M be a π-projective module. Then M is s.p - semisimple if and only if M is S-GS module.

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