Semisimple Modules Relative to A Semiradical Property

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Abstract

In this paper, we introduce the concept of s.p-semisimple module. Let S be a semiradical property, we say that a module M is s.p-semisimple if for every submodule N of M, there exists a direct summand K of M such that K ≤ N and N / K has S. we prove that a module M is s.p-semisimple module if and only if for every submodule A of M, there exists a direct summand B of M such that A = B + C and C has S. Also, we prove that for a module M is s.p-semisimple if and only if for every submodule A of M, there exists an idempotent e ∈ End(M) such that e(M) ≤ A and (1 - e)(A) has S.

Keywords: Semiradical (radical) property, Semisimple modules, t-semisimple modules.

المقاسات البسيطة نسبة الى خاصية شبه جذرية

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في هذا البحث نقدم مفهوم المقاسات شبه بسيطة نسبة لخاصية شبه جذرية. لنفترض أن S خاصية شبه جذرية. فنحن نقول أن المقاس M هو شبه بسيطة نسبة لخاصية شبه جذرية إذا كان لكل مقاس جزئي N من N ≤ M يوجد متساوي القوى e ∈ End(M) بحيث e(M) ≤ A و (1 - e)(A) تملك S. كما أننا نرى ذلك. المقاس M يكون شبه بسيط نسبة لخاصةية شبه جذرية إذا وفقط إذا كان لكل مقاس جزئي A من A ≤ M يوجد متساوي القوى e ∈ End(M) بحيث e(M) ≤ A و (1 - e)(A) تملك S.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary left R-modules. Let A be a submodule of a module M. A is called an essential submodule of M (denoted by A ≤_e M) if A ∩ B ≠ 0, ∀ 0 ≠ B ≤ M. A submodule B of M is called a closed submodule of M if B has no proper essential extension. A module M is called an extending module if every submodule of M is essential in a direct summand. Equivalently, every closed submodule of M is a direct summand, see [1], [2], [3].

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Let M be a module. Recall that the socle of M (denoted by Soc(M)) is the sum of all simple submodules of M, a module M is called a semisimple if Soc(M) = M. Equivalently a module M is semisimple if and only if every submodule is a direct summand of M, see [1], [4]. Recall that the Jacobson radical of M (denoted by J(M)) is the intersection of all maximal submodules of M. If M has no maximal submodule, we write J(M) = M, see [5].

Let \( x \in M \). Recall that \( \text{ann} (x) = \{ r \in R: rx = 0 \} \). For a module M, the singular submodule is defined as follows \( Z(M) = \{ x \in M | \text{ann} x \leq \mathfrak{e} R \} \) or equivalently, \( Ix = 0 \) for some essential left ideal I of R. If \( Z(M) = M \), then M is called a singular module. If \( Z(M) = 0 \), then M is called a nonsingular module. The second singular (or Goldie torsion) submodule of a module M (denoted by \( Z_2(M) \)) is defined by \( Z(M / Z(M)) = Z_2(M) / Z(M) \), see [1],[6].

A submodule A of a module M is called t-essential submodule (denoted by \( A \leq \text{tes} M \)) if for any submodule B of M, \( A \cap B \leq Z_2(M) \) implies \( B \leq Z_2(M) \). A module M is called t-semisimple if for every submodule N of M there exists a direct summand K of M such that \( K \leq \text{tes} N \), see [5]. [7].

A property S is called a radical property if:
1- for every module M, there exists a submodule (denoted by \( S(M) \)) such that
   a- \( S(M) \) has S.
   b- \( A \leq S(M) \), for every submodule A of M such that A has S.
2- If \( f: M \rightarrow N \) is an epimorphism and M has S, then N has S.
3- \( S(M / S(M)) = 0 \) for every R-module M, see [8].

A property S is called a semiradical property if it satisfies conditions 1 and 2, see [8].

It’s known that each of the following two properties is a radical property, see [8].

1- \( S = Z_2 \). For a module M, \( S(M) = Z_2(M) \), the second singular of M.
2- \( S = \text{Snr} \). For a module M, \( \text{Snr}(M) \) is a submodule of M such that
   a1- \( J(\text{Snr}(M)) = \text{Snr}(M) \) {i.e. \( \text{Snr}(M) \) has no maximal submodule}.
   b2- \( A \leq \text{Snr} (M) \), for every submodule A of M such that \( J(A) = A \), see [8].

While each of the following two properties is a semiradical property (but it is not radical property), see [8].

1- \( S = Z \). For a module M, \( S(M) = Z(M) \), the singular submodule of M.
2- \( S = \text{Soc} \). For a module M, \( S(M) = \text{Soc}(M) = \sum_{A \text{ is simple}} A \).

Let S be a semiradical property. It is known that
1- M has S if and only if \( S(M) = M \).
2- \( S(S(M)) = S(M) \).
3- If \( M = \bigoplus_{i \in I} M_i \), then \( S(M) = \bigoplus_{i \in I} S(M_i) \), where I is any index set.
4- if \( S(M) = 0 \), then \( S(A) = 0, \forall A \leq M \).
5- For any short exact sequence \( 0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0 \), if \( S(M) = 0 \) and \( S(K) = 0 \), then \( S(N) = 0 \), see [8].

In this paper, S is a semiradical property, unless otherwise stated.
2- s.p - semisimple modules

In this section, we introduce the concept of s.p-semisimple modules and give the basic properties of this module. Also, we illustrate it with some examples.

Definition 2.1. Let S be a semiradical property. We say that a module M is s.p - semisimple module if for each submodule N of M, there exists a direct summand K of M such that K ⊆ N and N / K has S.

Remarks and Examples 2.2.

1- Every semisimple module is s.p - semisimple. The converse is not true in general.

Proof. Let N be a submodule of a semisimple module M, then N is a direct summand of M, by [4]. Let K = N, hence S(N / K) = S(N / N) = S(0) = 0 ≅ N / K. Thus M is s.p - semisimple. For example Z_6 as Z_6-module is s.p - semisimple module.

For the converse, Let S = Second singularity. Consider module Z_4 as Z-module. Since Z_4 is singular, then every submodule of Z_4 is singular, by [1]. Therefore, Z_2(N) = Z(N) = N, ∀ N ≤ Z_4. Let K = 0, hence Z_2(N / 0) ≅ Z_2(N) = Z(N) = N ≅ N / 0. So N / 0 has S, ∀ N ≤ Z_4. Thus Z_4 is s.p - semisimple. Clearly that Z_4 is not semisimple.

Recall that a semiradical property S is called hereditary if S is closed under submodules, see [8].

2- Let S be a hereditary property and M be a module. If M has S, then M is s.p - semisimple.

Proof. Let N be a submodule of M and S(M) = M. Since S is hereditary, then S(N) = N. Let K = 0, then S(N / 0) ≅ S(N / N) = N ≅ N / 0. Thus M is s.p - semisimple.

3- Let S = singularity. Consider module Q as Z-module. Clearly, that Q is nonsingular. Hence, Z(Q) = 0. Let N = 3Z. Since Q is indecomposable, then 0 is the only direct summand contained in 3Z. So S(3Z / 0 ) ≅ S(3Z) = Z(3Z) = 0. Thus Q is not s.p - semisimple module.

Proposition 2.3. Every submodule of s.p - semisimple module M is s.p – semisimple, For every property S.

Proof. Let N be a submodule of M and A ≤ N. Since M is s.p - semisimple, then there exists a direct summand K of M such that K ≤ N and N / K has S. By modular law, K is a direct summand of N. Thus N is s.p - semisimple.

Proposition 2.4. Let M be an indecomposable module and S be an assumed. Then M is s.p - semisimple if and only if every proper submodule of M has S.

Proof. ⇒) Let N be a proper submodule of M. Since M is s.p - semisimple, then there exists a direct summand K of M such that K ≤ N and N / K has S. But M is an indecomposable. Therefore, K = 0. Hence S(N) ≅ S(N / 0) = S(N / K) = N / K = N / 0 ≅ N. Thus N has S.

⇐) Clear.

Let S be a semiradical property. Recall that S is called a cohereditary property, if S(M) = 0 is closed under homomorphic images of M for every module M, see [8].

Proposition 2.5. Let S be a cohereditary property and let M be a module. If S(M) = 0. Then M is semisimple if and only if M is s.p - semisimple.
Proof. ⇒) Clear.

⇐) Let N be a submodule of M. Since M is s.p - semisimple, then there exists a direct summand K of M such that K ≤ N and N / K has S. But S(M) = 0, therefore S(N) = 0, by [8]. Since S is cohereditary property, then S(N / K) = 0. Hence N = K is a direct summand of M. Thus M is semisimple.

Remark 2.6. Let S be a hereditary property and M be a module. If S(M) = M, then M / N is s.p - semisimple module, for each submodule N of M.

Proof. Let N be a submodule of M and S(M) = M, then M / N has S, by [8]. Thus by. 2.2-2, M / N is s.p - semisimple module.

Proposition 2.7. Let M be s.p - semisimple module. Then every submodule N of M such that S(N) = 0 is a direct summand of M. The converse is true if S(M) = 0.

Proof. Assume that N is a submodule of M such that S(N) = 0. Then there exists a direct summand K of M such that K ≤ N and N / K has S. Let M = K ⊕ K₁, for some submodule K₁ of M. By modular law, N = K ⊕ (N ∩ K₁). Since N ∩ K₁ ≤ N and S(N)=0, then S(N∩K₁) = 0, by [8]. Since N / K = (K ⊕ (N∩K₁)) / K ≅ (N ∩ K₁) / 0 ≅ N ∩ K₁, by the second isomorphism theorem, then S(N / K) = 0. But S(N / K) = N / K, therefore N / K = 0. Thus N = K is a direct summand of M.

Conversely, let S(M) = 0 and N be a submodule of M. Then S(N) = 0, by [8]. By our assumption N, is a direct summand of M. Therefore M is semisimple. Thus by 2.2-1, M is s.p - semisimple module.

Proposition 2.8. Let M = A + S(M) be s.p - semisimple module. Then there exists a direct summand B of M such that B ≤ A, M = B + S(M) and A / B has S.

Proof. Assume that M is s.p - semisimple module. Then there exists a direct summand B of M such that B ≤ A and A / B has S. Let M = B ⊕ C, for some submodule C of M. Then A = B ⊕ (C ∩ A), by modular law. But A / B ≅ (C ∩ A), by the second isomorphism theorem, therefore (C ∩ A) has S. Since (C ∩ A) has S, then (C ∩ A) ≤ S(M). Thus M = A + S(M) = B + (C∩A) + S(M) and hence M = B + S(M).

Proposition 2.9. Let S be a hereditary property and M = M₁ ⊕ M₂ be a module such that M₁ has S and M₂ is semisimple. Then M is s.p - semisimple module.

Proof. Let N be a submodule of M. Since M₂ is semisimple, then N ∩ M₂ is a direct summand of M₂. But, M₂ is a direct summand of M, therefore N ∩ M₂ is a direct summand of M. By the second isomorphism theorem, M / M₂ = (M₁⊕M₂) / M₂ ≅ M₁. Since M₁ has S, then M / M₂ has S. But N / (N ∩ M₂) ≅ (N + M₂) / M₂ ≤ M / M₂ and S hereditary property. So N / (N ∩ M₂) has S. Thus M is s.p - semisimple module.

Corollary 2.10. Let S be a hereditary property and M be a module. If M = S(M) ⊕ M₁, where M₁ is a semisimple module, then M is s.p - semisimple module.
**Proof.** Clear.

**Proposition 2.11.** Let $M = M_1 \oplus M_2$ be a module such that $R = \text{Ann}(M_1) + \text{Ann}(M_2)$. If $M_1$ and $M_2$ are s.p - semisimple modules, then $M$ is s.p - semisimple module.

**Proof.** Let $N$ be a submodule of $M = M_1 \oplus M_2$. Since $R = \text{Ann}(M_1) + \text{Ann}(M_2)$, then by the same argument of the proof [9, prop.4.2, CH.1], $N = N_1 \oplus N_2$, where $N_1 \leq M_1$ and $N_2 \leq M_2$. Since $M_i$ is s.p - semisimple for $i=1, 2$, then there exist direct summands $K_i$ of $M_i$ such that $K_i$ is a submodule of $N_i$ and $N_i / K_i$ has $S$ $(i = 1, 2)$. Let $M_i = K_i \oplus L_i$, for some submodule $L_i$ of $M_i$. Therefore $M = M_1 \oplus M_2 = (K_1 \oplus L_1) \oplus (K_2 \oplus L_2) = (K_1 \oplus K_2) \oplus (L_1 \oplus L_2)$. Hence $(K_1 \oplus K_2)$ is a direct summand of $M$ and $(K_1 \oplus K_2) \leq N_1 \oplus N_2 = N$. Now since $N_i / K_i$ has $S$ $(i = 1, 2)$, then by [8], $(N_1 / K_1) \oplus (N_2 / K_2)$ has $S$. But $(N_1 / K_1) \oplus (N_2 / K_2) \cong ((N_1 \oplus N_2) / (K_1 \oplus K_2))$, by [10, p. 33], hence $(N_1 \oplus N_2) / (K_1 \oplus K_2) \cong N / (K_1 \oplus K_2)$ has $S$. Thus $M$ is s.p - semisimple module.

Let $M$ be an $R$- module. Recall that $M$ is called a duo-module if every submodule of $M$ is fully invariant, see [11].

**Proposition 2.12.** Let $M = \bigoplus_{i \in I} M_i$ be a duo module. Then $M$ is s.p - semisimple modules if and only if $M_i$ is s.p - semisimple module $\forall i \in I$.

**Proof.** Since $M$ is s.p - semisimple, then by prop.2.3, $M_i$ is s.p - semisimple, $\forall i \in I$. Conversely, let $M = \bigoplus_{i \in I} M_i$ be a module such that $M_i$ is s.p - semisimple, $\forall i \in I$. Let $N \leq M$, then $N = N \cap M = N \cap \left( \bigoplus_{i \in I} M_i \right) = \bigoplus_{i \in I} (N \cap M_i)$, by [12,lem.2.1]. Let $N_i = N \cap M_i$, $\forall i \in I$, then $N_i \leq M_i$, $\forall i \in I$. Since $M_i$ is s.p - semisimple, then there exists $K_i$ is a direct summand of $M_i$ such that $K_i$ is a submodule of $N_i$ and $N_i / K_i$ has $S$ $\forall i \in I$. Hence $((\bigoplus_{i \in I} N_i) / ((\bigoplus_{i \in I} K_i)) \cong \bigoplus_{i \in I} (N_i / K_i)$ has $S$, by [10]. Thus $M = \bigoplus_{i \in I} M_i$ is s.p - semisimple.

Let $M_1$ and $M_2$ be $R$- modules. $M_1$ is called $M_2$- projective if for every submodule $N$ of $M_2$ and any homomorphism $f: M_1 \rightarrow M_2 / N$, there is a homomorphism $g: M_1 \rightarrow M_2$ such that $\pi \circ g = f$, where $\pi: M_2 \rightarrow M_2 / N$ is the natural epimorphism, see [13].

\[ \begin{array}{ccc} M_1 & \xrightarrow{g} & M_2 / N \\ \downarrow{f} & & \downarrow{\pi} \\ M_2 & \xrightarrow{\pi} & 0 \end{array} \]

$M_1$ and $M_2$ are called relatively projective if $M_1$ is $M_2$- projective and $M_2$ is $M_1$- projective.

We know that for a module $M = A \oplus B$. $A$ is $B$-projective if and only if for every submodule $C$ of $M$ such that $M = B + C$, there exists a submodule $D$ of $C$ such that $M = B \oplus D$, see [14].

**Proposition 2.13.** Let $S$ be a hereditary property. Let $M_1$ and $M_2$ be s.p - semisimple modules such that $M_1$ and $M_2$ are relative projective. Then $M = M_1 \oplus M_2$ is s.p - semisimple.
Proof. Let N be a submodule of M. Since \((N + M_2) \cap M_1 \leq M_1\) and \(M_1\) is s.p. - semisimple, then there exists a direct summand \(A_1\) of \(M\) such that \(A_1 \leq (N + M_2) \cap M_1\) and \(((N + M_2) \cap M_1) / A_1\) has S. Let \(M_1 = A_1 \oplus B_1\), for some submodule \(B_1\) of \(M_1\). Hence \((N + M_2) \cap M_1 = A_1 \oplus ((N + M_2) \cap M_1) \cap B_1\), by modular law. Since by the second isomorphism theorem, \(((N + M_2) \cap M_1) / A_1 \cong (N + M_2) \cap M_1 \cap B_1\), then \((N + M_2) \cap B_1\) has S, by [8]. Therefore \(M = M_1 \oplus M_2 = A_1 \oplus B_1 \oplus M_2 = (N + M_2) \cap M_1 + B_1 + M_2 = N + M_2 + B_1 + M_2 = N + (M_2 \oplus B_1)\). Since \((N + B_1) \cap M_2 \leq M_2\) and \(M_2\) is s.p. - semisimple, then there exists a direct summand \(A_2\) of \(M_2\) such that \(A_2 \leq (N + B_1) \cap M_2\) and \(((N + B_1) \cap M_2) / A_2\) has S. Let \(M_2 = A_2 \oplus B_2\), for some submodule \(B_2\) of \(M_2\) then \((N + B_1) \cap M_2 = A_2 \oplus ((N + B_1) \cap M_2) \cap B_2\), by modular law. By the second isomorphism theorem, \(((N + B_1) \cap M_2) / A_2 \cong ((N + B_1) \cap M_2) \cap B_2\), then \((N + B_1) \cap M_2 \cap B_2 = (N + B_1) \cap B_2\) has S, by [8]. Thus \(M = N + (M_2 \oplus B_1) = N + A_2 + B_2 + B_1 = N + (B_1 \oplus B_2)\). Since \(M = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)\), and \(M_1\) and \(M_2\) are relative projective, then \(A_1\) is \(B_1\)-projective and \(A_2\) is \(B_2\)-projective for \(j = 1, 2\), by [9, prop. 2.1.6]. So by [15, prop. 2.1.7], \(A_1\) is \(B_1 \oplus B_2\)-projective and \(A_2\) is \(B_1 \oplus B_2\)-projective. Hence \(A_1 \oplus A_2\) is \(B_1 \oplus B_2\)-projective, by [15, prop. 2.1.6]. Hence, there exists \(X \leq N\) such that \(M = X \oplus B_1 \oplus B_2\), by [14, lem. 5].

Now, we want to show that \(N \cap (B_1 \oplus B_2)\) has S. Since \((N + M_2) \cap B_1\) has S and \((N + B_2) \cap B_1 = (N + (A_2 \oplus B_2)) \cap B_1\), then \((N + B_2) \cap B_1\) has S. Since \((N + B_1) \cap B_2\) has S, then \((N + B_2) \cap B_1 \oplus (N \oplus B_1) \cap B_2\) has S, by [8]. But by [15, lem. 3.2], \(N \cap (B_1 \oplus B_2) = (N \oplus B_2) \cap B_1 \oplus (N \oplus B_1) \cap B_2\). Therefore, \(N \cap (B_1 \oplus B_2)\) has S. Thus \(M\) is s.p. - semisimple module.

Let \(M\) be an R-module. \(M\) is said to have the summand intersection property (briefly SIP) if the intersection of any two direct summands of \(M\) is a direct summand of \(M\), see [16].

Proposition 2.14. Let \(M\) be s.p. - semisimple module. If for any two direct summand A and B of \(M\), \(S(A \cap B) = 0\), then \(M\) has SIP.

Proof. Let \(A\) and \(B\) be direct summands of \(M\). Since \(M\) is s.p. - semisimple, then there exists a direct summand \(N\) of \(M\) such that \(N \leq A \cap B\) and \((A \cap B) / N\) has S. Let \(M = N \oplus N_1\), for some submodule \(N_1\) of \(M\), then \(A \cap B = N \oplus (N_1 \cap (A \cap B))\). Hence by the second isomorphism theorem, \((A \cap B) / N = [N \oplus (N_1 \cap (A \cap B))] / N \cong N_1 \cap (A \cap B) \leq A \cap B\). Since \(S(A \cap B) = 0\), then \(S(N_1 \cap (A \cap B)) = 0\), by [8]. So \(S((A \cap B) / N) = 0\). But \((A \cap B) / N\) has S, therefore \(A \cap B\) is a direct summand of \(M\). Thus \(M\) has SIP.

Let \(R\) be an integral domain. Recall that an R-module \(M\) is called a torsion free module if ann \((x) = 0\), for all \(0 \neq x \in M\), see [1].

Theorem 2.15. Let \(R\) be an integral domain and \(M\) be a torsion free module and s.p. - semisimple module. Then for every \(m \in M\), either \(Rm\) is a direct summand of \(M\) or \(Rm\) has S.

Proof. Let \(0 \neq m \in M\). Then there exists a direct summand \(K\) of \(M\) such that \(K \leq Rm\) and \(Rm / K\) has S. Let \(M = K \oplus H\), for some submodule \(H\) of \(M\). Then \(Rm = K \oplus (Rm \cap H)\), by modular law. But \(Rm / K \cong Rm \cap H\), by the second isomorphism theorem. Therefore \(Rm \cap H\) has S.

Let \(f : R \rightarrow Rm\) be a map defined by \(f(r) = rm\), for each \(r \in R\). It is easy to see that \(f\) is an epimorphism and \(\text{Ker}(f) = \text{ann}(m)\). By the first isomorphism theorem, \(R / \text{ann}(m) \cong Rm\). Since \(M\) is torsion free module, then \(\text{ann}(m) = 0\). Thus \(R \cong Rm\). But \(R\) is indecomposable.
Therefore, Rm is indecomposable. Implies that either Rm = K or Rm = Rm ∩ H. Thus either Rm is a direct summand of M or Rm has S.

**Proposition 2.16.** Let R be an indecomposable ring and M be a projective module. If M is s.p - semisimple module, then for every m ∈ M, either Rm is a direct summand of M or Rm has S.

**Proof.** Assume that M is a projective and s.p - semisimple module and let m ∈ M. Then there exists a direct summand K of M such that K ≤ Rm and Rm / K has S. Let M = K ⊕ H for some submodule H of M, then Rm = K ⊕ (H ∩ Rm), by modular law. But Rm / K ≅ H ∩ Rm, by the second isomorphism theorem. Therefore, H ∩ Rm has S.

Now, let f: R → Rm be a map defined by f(r) = rm, for all r ∈ R. It is clear that f is an epimorphism map. Let P: Rm → K be the projection map. Clearly, Pof: R → K is an epimorphism. Since M is projective, then K is projective by [4]. Therefore, Ker (Pof) is a direct summand of R. Since R is indecomposable, then either Ker Pof = 0 or Ker Pof = R. Ker (Pof) = f⁻¹ (Rm ∩ H) = f⁻¹ (Rm ∩ H). So either Rm ∩ H = 0 or Rm ∩ H = R. Thus Rm = K or Rm ∩ H = Rm has S.

### 3- Characterization of s.p - semisimple Modules

In this section, we give various characterizations of s.p - semisimple modules.

We start with the following theorem.

**Theorem 3.1.** Let M be a module. Then the following statements are equivalent

1- M is s.p - semisimple module.

2- For every submodule A of M, there exists a decomposition M = B ⊕ C such that B ≤ A and A ∩ C has S.

3- For every submodule A of M, A = A₁ ⊕ A₂, where A₁ is a direct summand of M and A₂ has S.

**Proof.**

1⇒2) Let A be a submodule of M. Since M is s.p - semisimple, then there exists a direct summand B of M such that B ≤ A and A / B has S. Let M = B ⊕ C, where C is a submodule of M. Then A = B ⊕ (C ∩ A), by modular law. By the second isomorphism theorem, A / B ≅ (C ∩ A). Thus A / B ≅ C ∩ A.

2⇒3) Let A be a submodule of M. By (2), there exists a decomposition M = B ⊕ C such that B ≤ A and A ∩ C has S. By modular law, A = B ⊕ ( C ∩ A). Let A₂ = A ∩ C has S.

3⇒1) Let A be a submodule of M. By (3), A = A₁ ⊕ A₂, where A₁ is direct summand of M and A₂ has S. By the second isomorphism theorem, A / A₁ ≅ A₂. So A / A₁ has S. Thus M is s.p - semisimple.

**Proposition 3.2.** A module M is s.p - semisimple if and only if for every submodule A of M there exists a direct summand B of M such that A = B + C, where C is a submodule of M has S.

**Proof.** ⇒) It is clear by Theorem 3.1.

⇐) Let A be a submodule of M. By our assumption, there exists a direct summand B of M such that A = B + C and C has S. Let M = B ⊕ D, for some submodule D of M, then A = B ⊕ (A ∩ D), by modular law. Hence, (A / B) = (B + C) / B ≅ C / (B ∩ C), by the second isomorphism theorem. But C has S, then C / (B ∩ C) has S. This implies that A / B has S. Thus M is s.p - semisimple.
Proposition 3.3. A module $M$ is s.p - semisimple if and only if for each submodule $A$ of $M$, there exists an idempotent $e \in \text{End}(M)$ such that $e(M) \leq A$ and $(1-e)(A)$ has $S$.

Proof. $\Rightarrow$) Let $A$ be a submodule of $M$. Since $M$ is s.p - semisimple, then there exists a decomposition $M = B \oplus C$ such that $B \leq A$ and $\nabla C$ has $S$, by th.3.1, 1-2. Let $e : M \to B$ be the projection map. Clearly that $e^2 = e$ and $C = (1 - e)(M)$. Claim that $(1-e)(A) = (1-e)M \cap A$. To show that, let $m \in (1-e)(A)$, then there is $a \in A$ such that $m = (1-e)(a) = a - e(a)$. Therefore $m \in A$ and hence $m \in (1-e)(M) \cap A$. Thus $(1-e)(A) \leq (1-e)(M) \cap A$. Now, let $n \in (1-e)(M) \cap A$, then $n \in (1-e)(M)$ and $n \in A$. Hence, there is $k \in M$ such that $n = (1-e)(k) = k - e(k)$. So $n + e(k) = k \in A$. Then $n \in (1-e)(A)$. Thus $A \cap C = A \cap (1-e)(M) = (1-e)(A)$. Thus $(1-e)A$ has $S$.

$\Leftarrow$) Let $A$ be a submodule of $M$ and $e \in \text{End}(M)$ be an idempotent such that $e(M) \leq A$ and $(1-e)A$ has $S$. Claim that $M = e(M) \oplus (1-e)(M)$. To show that, let $x \in M$, then $x = x + e(x) - e(x) = e(x) + x - e(x) = e(x) + (1-e)(x)$. Thus $M = e(M) + (1-e)(M)$.

Now, let $y \in e(M) \cap (1-e)(M)$, then $y = e(m_1)$ and $y = (1-e)(m_2)$, for some $m_1, m_2 \in M$. So $y = e(m_1) = e(1-e)(m_2) = e(m_1) - e(m_2) = 0$, then $y = e(m_1) = 0$. Thus $M = e(M) \oplus (1-e)(M)$. Let $B = e(M) \leq A$ and $C = (1-e)(M)$. Therefore $M = B \oplus C$ and $A \cap C = A \cap (1-e)M = (1-e)A$ has $S$. Thus $M$ is s.p - semisimple, by Theorem 3.1.

Let $M$ be a module and $N$ be a submodule of $M$. Recall that a submodule $K$ of $M$ is called an S-generalized supplement of $N$ in $M$, if $M = N + K$ and $N \cap K \leq S(K)$, see [17].

Let $M$ be a module. Recall that $M$ is called an S-generalized supplemented module (or briefly S-GS module), if every submodule of $M$ has S-generalized supplement in $M$, where $S$ is semidirectional property on modules, see [17].

Proposition 3.4. Every s.p - semisimple module $M$ is S-GS supplemented module.

Proof. Let $M$ is s.p - semisimple module and $N$ be a submodule of $M$, then there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N / K$ has $S$. Hence, $M = K \oplus K_1$, for some submodule $K_1$ of $M$. But $K \leq N$, therefore $M = N + K_1$. So by modular law, $N = K \oplus (N \cap K_1)$, then by the second isomorphism theorem, $N / K \cong N \cap K_1$ has $S$. Thus $N \cap K_1 \leq S(K)$ by [8].

Proposition 3.5. Let $M$ be s.p-semisimple module. If $M = N + K$, where $N$ is a direct summand of $M$, then $N$ contains an S-generalized supplement submodule of $K$ in $M$.

Proof. Since $M$ is an s.p - semisimple, then by Theorem 3.1.1-3, $N \cap K = A \oplus B$, where $A$ is a direct summand of $M$ and $B$ has $S$. Let $M = A \oplus C$, for some submodule $C$ of $M$. Hence, $N = A \oplus (N \cap C)$, by modular law. Let $A_1 = N \cap C$, then $M = N + K = (A + A_1) + K$. But $A \leq K$. Therefore, $M = K + A_1$. Now we want to show $K \cap A_1 \leq S(A_1)$. Since $N \cap K = (A \oplus A_1) \cap K = A \oplus (K \cap A_1)$, by modular law. Let $N : A \oplus A_1 \to A_1$ be the projection map. So we have $K \cap A_1 = (A \oplus (K \cap A_1) = (N \cap K) = (A \oplus B) = B)$. But $B$ has $S$. Therefore, $K \cap A_1$ has $S$, by [8]. Hence, $K \cap A_1 \leq S(A_1)$. Thus $A_1$ is an S-generalized supplement submodule of $K$ in $M$ and $A_1$ is contained in $N$.

Proposition 3.6. Let $S$ be a hereditary property and $M$ be a module. Then the following statements are equivalent

1- $M$ is s.p - semisimple module.
2- Every submodule $N$ of $M$ has S-generalized supplement $K$ in $M$ such that $N \cap K$ is a direct summand of $N$. 
Proof. 1⇒2) Let N be a submodule of M. Then by the same argument of proof of Proposition 3.4, N has an S-generalized supplement. 2⇒1) Let N be a submodule of M. Then by our assumption N has an S-generalized supplement K in M such that N∩K is a direct summand of N. Hence M = N + K and N ∩ K ≤ S(K). Let N= (N ∩ K) ⊕ L, for some submodule L of N. Then M = (N∩K) + L + K = L + K. But, L ∩ K = N ∩ K ∩ L = 0. Therefore, M = L ⊕ K. By the second isomorphism theorem, N / L ≅ N ∩ K. Since N ∩ K ≤ S(K) and S is hereditary property, then N ∩ K has S by [8] and hence N / L has S. Thus M is s.p-semisimple.

Proposition 3.7. Let M be a module. If M is S-GS supplemented module, then M / S(M) is a semisimple module.

Proof. Let N / S(M) be a submodule of M / S(M). Since M is S-GS supplemented, then there exists a submodule K of M such that M = N + K and N ∩ K ≤ S(K). Then M / S(M) = (N+K)/S(M) = N / S(M) + (K+S(M))/S(M)). Since (N/S(M)) ∩ ((K+S(M))/S(M)) = [(N ∩ K) + S(M) / S(M)), by modular law and N ∩ K ≤ S(K) ≤ S(M), by [17]. Then (N ∩ K) + S(M) = S(M). Therefore M / S(M) = (N / S(M) ⊕ ((K + S(M) / S(M)). Thus M / S(M) is semisimple.

Corollary 3.8. Let M be a module. If M is S-GS supplemented module, then M / S(M) is s.p - semisimple module.

Proof. It is clear by Proposition. 3.7 and 2.2-1.

Proposition 3.9. Let M be s.p-semisimple module. Then every submodule N of M has an S-generalized supplement which is a direct summand of M.

Proof. Let N be a submodule of M, then there exists a decomposition M = A ⊕ B such that A ≤ N and N ∩ B has S, by Theorem 3.1, 1-2. Clearly M = N + B and N ∩ B ≤ S(B). Thus B is an S-generalized supplement of N which is a direct summand of M.

Let M be an R- module. Recall that M is called π-projective (or co-continuous) if for every two submodules U, V of M with U + V = M there exists f ∈ End(M) with Im (f) ≤ U and Im (1− f) ≤ V, see [18].

Proposition 3.10. Let S be a hereditary property and a module M be a π-projective module. Then M is s.p - semisimple if and only if M is S-GS module.

Proof. ⇒) It is clear by Proposition 3.4. ⇐) Let N be a submodule of M. Since M is S-GS module, then there exists a submodule K of M such that M = N + K and N ∩ K ≤ S(K). Since M is π- projective, then there exists an idempotent e ∈ End (M) such that Im (e) ≤ N and Im (1 − e) ≤ K. But by the same proof of Proposition 3.3 we have N(1- e) = N ∩ (1 − e)M ≤ N ∩ K ≤ S(K) and S is hereditary property, therefore N(1- e) has S. Thus by Proposition 3.3 M is s.p - semisimple.

Conclusion
In this work, the concept of s.p-semisimple module is introduced and studied. We also conclude the following:
1. Every semisimple module is s.p – semisimple. However, the converse is not true. Let S = Second singularity. Consider module Z₄ as Z- module. Since Z₄ is singular, then every submodules of Z₄ is singular, by [1]. Therefore, Z₂(N) = Z(N) = N, ∀ N ≤ Z₄, let K = 0, hence ZZ(N / 0) ≅ Z₂(N) = Z(N) = N ≅ N / 0. So N / 0 has S, ∀ N ≤ Z₄. Thus Z₄ is s.p - semisimple. Clearly, that Z₄ is not semisimple.

2. Let M = ⊕ᵢ∈I Mᵢ be a duo module. Then M is s.p - semisimple modules if and only if Mᵢ is s.p - semisimple module ∀ i ∈ I.

3. Let S be a hereditary property. If M₁ and M₂ are s.p - semisimple modules such that M₁ and M₂ are relative projective. Then M = M₁ ⊕ M₂ is s.p - semisimple.

4. Every s.p - semisimple module M is S-GS supplemented module.

5. Let S be a hereditary property and a module M be a π-projective module. Then M is s.p - semisimple if and only if M is S-GS module.

References