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On The extension Bi-Normality of Linear Operators

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Abstract
In this paper, we introduce the bi-normality set, denoted by $\mathcal{N}_{(A,B)}$, which is an extension of the normality set, denoted by $\mathcal{N}_A$ for any operators A, B in the Banach algebra $\mathfrak{B}(\mathcal{H})$. Furthermore, we show some interesting properties and remarkable results. Finally, we prove that it is not invariant via some transpose linear operators.

Keywords: Bi-normality set, Unitary operator, Quasi-similar operator, Similar operator, Aluthge transformation.

حول توسيع المجموعة القياسية الثنائية للمؤثرات الخطية

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الخلاصة
في هذا البحث, سنقدم المجموعة القياسية الثنائية من النمط $\mathcal{N}_{(A,B)}$ وهي مفهوم موسع للمجموعة القياسية من النمط $\mathcal{N}_A$ حيث كل من A, B مؤثران في فضاء باناخ الجبري $\mathfrak{B}(\mathcal{H})$. أكثر تحديداً, سوف نعطي بعض الخواص و النتائج الجوهرية للمجموعة. أضافة لذلك, سنبرهن أن المجموعة القياسية الثنائية لاتتغير مع بعض مؤثرات التحويل الخطي.

1. Introduction

Let $\mathfrak{B}(\mathcal{H})$ be the algebra of all bounded linear operators on Hilbert space $\mathcal{H}$. We studied the normality set of linear operators which is denoted by $\mathcal{N}_A = \{T \in \mathfrak{B}(\mathcal{H}): AT^* = T^*A\}$, which motivates this work. The normality set has been done by Shaakir, L. K. et al. in [1].

Now, we extend and improve the concept of the normality set to the bi-normality set which is defined by. $\mathcal{N}_{(A,B)} = \{T \in \mathfrak{B}(\mathcal{H}): AT^* = T^*B\} \neq \emptyset$. This definition has many properties, so we will focus on some of them for some transformation linear operators. In 1950 Fuglede B.

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proved that if H is normal such that $HA = HA$ for every bounded operator, then A commutes via the adjoint of operator H [2]. Putnam C. R. 1951 extends Fuglede's theorem [3]. Shihab, M. K. 2017 proved that extension of the Fuglede-Putnam theorem for two operators [4]. For more details see [5, 6, 7, 8, 9].

In the current study, the concept of bi-normality set of $A, B \in \mathfrak{B}(\mathcal{H})$ is given and studied. We also prove many properties, specifically product operators among them like $\widetilde{\mathcal{N}}_{(A,B)}$. Furthermore, it is not invariant for transpose linear operators and has a nontrivial-invariant subspace. Finally, the current work highlights some properties and basic concepts, as well as some results and theorems on the bi-normality set.

The paper's organization is as follows. Section 2 contains some preliminary properties, definitions and concepts which will be necessary to prove our main results. In section 3, we present some results on the possible connection by strong conditions between the normal operator, normality set and quasi-normality sets by using Fuglede-Putnam theorem their consequences in the form of many results. Section 4 exhibits several of relationships among the normality, bi-normality sets and many linear transformations.

2. Preliminaries

The aim of this section is to give and introduce some basic properties and concepts that are needed throughout this work.

Definition 2.1 [6, 7]: Let $T \in \mathfrak{B}(\mathcal{H})$. The adjoint of T is T^*, if $T^* = T, TT^* = T^*T, TT^* \geq T^*T$ and $TT^* = T^*T = I$, then T is called self-adjoint, normal, hyponormal and unitary operator , respectively, where I the identity operator.

Definition 2.2 [10]: For any $A, B \in \mathfrak{B}(\mathcal{H})$, the operator A is said to be similar to B, if there exists an invertible operator $X \in \mathfrak{B}(\mathcal{H})$, yields $A\mathrm{X} = \mathrm{X}B$, denoted by $A \approx B$.

Definition 2.3 [10]: Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces, and $A \in \mathfrak{B}(\mathcal{H}_1)$, $B \in \mathfrak{B}(\mathcal{H}_2)$, the operator A is said to be quasi-similar to B, if there exists injective via dense range $T_1: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ and $T_2: \mathcal{H}_2 \longrightarrow \mathcal{H}_1$, bounded operators such that. $T_1 A = BT_1$ and $AT_2 = T_2 B$, denoted by $A \simeq B$.

Definition 2.4 [10]: If there exists a unitary operator U that verifies $UA = BU$; that means $A = U^* BU$ or $B = UAU^*$. Then it is said to be a unitarily equivalent as denoted by $A \cong B$.

Definition 2.5 [11, 12]: Let $A = U|A|$ be the polar decomposition of A. Aluthge transformation of A is defined as follows: $\triangle(A) = \tilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$ and its adjoint is $\left(\tilde{A}\right)^* = |A|^{\frac{1}{2}} U^* |A|^{\frac{1}{2}}$.
Moreover, an operator $A \in \mathfrak{B}(\mathcal{H})$, such that $A^* = U^*|A^*|$ be the polar decomposition of A^*. Then $*$ $-$Aluthge transformation is defined as $\tilde{A}^{(*)} = \left(\widetilde{A^*}\right)^* = |A^*|^{\frac{1}{2}} U |A^*|^{\frac{1}{2}}$.
More generally, for any real number $\lambda \in [0, 1]$, the λ-Aluthge transformation is defined as $\triangle_\lambda(A) = |A|^{\lambda} U |A|^{1-\lambda}$.

Theorem 2.6 [2]: (Fuglede's Theorem) For any bounded linear operator A, let H be a normal operator on $\mathcal{H}$ if $AH = HA$, then $AH^* = H^*A$.

Theorem 2.7 [3]: (Fuglede-Putnam Theorem) The operators H, K are normal operators on $\mathcal{H}$ if $AH = KA$, then $AH^* = K^*A$ for each bounded linear operator A.

Definition 2.8 [1]: For each operator $A \in \mathfrak{B}(\mathcal{H})$, the normality set of A is defined by $\mathcal{N}_A = \{T \in \mathfrak{B}(\mathcal{H}): AT^* = T^*A\}$. It is clear that it is non-empty set, since $O, I \in \mathcal{N}_A$ and $\mathcal{N}_{\alpha I} = \mathfrak{B}(\mathcal{H})$ also $\mathcal{N}_O = \mathfrak{B}(\mathcal{H})$ for every $\alpha \in \mathbb{C}$, where O, I are the zero, identity operator on $\mathcal{H}$, respectively.

Theorem 2.9 [6]: If A and B are two operators on $\mathcal{H}$ such that $\sigma(A) \cap \sigma(B) = \emptyset$. Then $\Gamma = 0$ is the only solution to the operator equation $A\Gamma - \Gamma B = 0$, where $\sigma(A)$ is the spectrum of the operator A.
Notice that, since $\Gamma = 0$. We take the adjoint to get $\Gamma^* = 0$ as the only solution to the operator equation $A\Gamma^* - \Gamma^* B = 0$.

3. Main Results:

Definition 3.1: If $A, B \in \mathfrak{B}(\mathcal{H})$, we define the normality set of two operators (A, B) as follows$\mathcal{N}_{(A,B)} = \{T \in \mathfrak{B}(\mathcal{H}): AT^* = T^*B\}$. It is also said to be bi-normality set. It is clear that the bi-normality set is nonempty set, since $O \in \mathcal{N}_{(A,B)}$ and $\mathcal{N}_{(O,O)} = \mathfrak{B}(\mathcal{H})$ also, $\mathcal{N}_{(\alpha I, \beta I)} = \mathfrak{B}(\mathcal{H})$ for every $\alpha, \beta \in \mathbb{C}$.
The next proposition shows that $\mathcal{N}_{(A,B)}$ is a closed and linear subspace in $\mathfrak{B}(\mathcal{H})$.

Proposition 3.2: If $A, B \in \mathfrak{B}(\mathcal{H})$, then $\mathcal{N}_{(A,B)}$ is a closed and linear subspace on $\mathfrak{B}(\mathcal{H})$.
Proof: Assume that $X, Y \in \mathcal{N}_{(A,B)}$ and $a_1, a_2 \in \mathbb{C}$. We get $AX^* = X^*B$ via $AY^* = Y^*B$.
$A(a_1X + a_2Y)^* = \overline{a_1}AX^* + \overline{a_2}AY^* = \overline{a_1}X^*B + \overline{a_2}\, Y^*B = (a_1X + a_2Y)^*B$.
Thus, $a_1X + a_2Y \in \mathcal{N}_{(A,B)}$.
Take $\{T_n\} \in \mathcal{N}_{(A,B)}$ is convergent to T. $AT_n{}^* = T_n{}^*B$ for each $n \in Z^+$. Hence, $\{T_n{}^*\} \to T^*$, $\{AT_n{}^*\} \to AT^*$, and $\{T_n{}^*B\} \to T^*B$. Thus, $AT^* = T^*B$, this yields $T \in \mathcal{N}_{(A,B)}$. Therefore, $\mathcal{N}_{(A,B)}$ is a linear subspace in $\mathfrak{B}(\mathcal{H})$. □

Remark 3.3:
1- We denote that $\widetilde{\mathcal{N}}_{(A,B)} = \mathcal{N}_{(A,B)} \cap \mathcal{N}_{(B,A)} = \{T \in \mathfrak{B}(\mathcal{H}): AT^* = T^*B \text{ and } BT^* = T^*A\}$, where $A, B \in \mathfrak{B}(\mathcal{H})$. Moreover, we always notice that $\mathcal{N}_{(A,B)} \neq \mathcal{N}_{(B,A)}$, see example 2.
2- If $\mathcal{N}_{(A,B)} \neq \{0\}$ for any two operators A and $B \in \mathfrak{B}(\mathcal{H})$, then $\sigma(A) \cap \sigma(B) \neq \emptyset$, by Theorem 2.9.
3- $I \in \mathcal{N}_{(A,B)}$ if and only if $A = B$. This means $\mathcal{N}_{(A,B)} = \mathcal{N}_A$. As a result, we attempt to take A is not equal B throughout this paper.
In the following proposition, we prove the relationship of a composite of two sets with another set that means, if $T \in \mathcal{N}_{(B,C)}\mathcal{N}_{(A,B)}$, we have that $T = SK$, where $S \in \mathcal{N}_{(B,C)}$ and $K \in \mathcal{N}_{(A,B)}$.

Proposition 3.4: If $A, B, C \in \mathfrak{B}(\mathcal{H})$, then
1- $\mathcal{N}_{(A,0)} = \mathcal{N}_{(0,B)} = \{0\}$ for every injective operator in $\mathfrak{B}(\mathcal{H})$.
2- $\mathcal{N}_{(A+\beta I, B+\beta I)} = \mathcal{N}_{(A,B)}$ $for\ all\ \beta \in \mathbb{C}$.
3- $\mathcal{N}_{(B,C)}\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_{(A,C)}$. In particular, $\mathcal{N}_{(A,B)}\mathcal{N}_{(B,A)} \subseteq \mathcal{N}_A$.

Proof: The proof 1 and 2 are obtained by Definition 3.1. Now, it is enough to prove 3 by $\mathcal{N}_{(B,C)}\mathcal{N}_{(A,B)} = \{T_1T_2 : T_1 \in \mathcal{N}_{(B,C)} \text{ and } T_2 \in \mathcal{N}_{(A,B)}\}$, such $T_1T_2 \in \mathcal{N}_{(A,C)}$, there $BT_1{}^* = T_1{}^*C$ and $AT_2{}^* = T_2{}^*B$. Thus, $AT_2{}^*T_1{}^* = T_2{}^*BT_1{}^* = T_2{}^*T_1{}^*C$. Therefore, $A(T_1T_2)^* = (T_1T_2)^*C$. In particular, if $A = C$. □

Theorem 3.5: If $A, B \in \mathfrak{B}(\mathcal{H})$, then $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_{A+B}$.

Proof: By [7], we have. $\mathcal{N}_A \cap \mathcal{N}_B \subseteq \mathcal{N}_{A+B}$. So, it is enough to prove that $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$. Let $T, S \in \widetilde{\mathcal{N}}_{(A,B)}$, so $AT^* = T^*B$ via $BT^* = T^*A$, also, we have $AS^* = S^*B$ and $BS^* = S^*A$. Hence, $A(TS)^* = AS^*T^* = S^*BT^* = (TS)^*A$. Therefore, $TS \in N_A$.
That means $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A$. By the same way, we can see that $B(TS)^* = BS^*T^* = S^*AT^* = (TS)^*B$. Hence, $TS \in \mathcal{N}_B$. That means $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_B$.
Therefore, we have, $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$. □

Corollary 3.6: If $A, B \in \widetilde{\mathcal{N}}_{(A,B)}$, then, $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A + \mathcal{N}_B$.

Proof: Suppose that $A, B \in \widetilde{\mathcal{N}}_{(A,B)}$. So, we have $AA^* = A^*B$; $BA^* = A^*A$, and $AB^* = B^*B$; $BB^* = B^*A$, respectively. First, we have to prove that $A\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_B$, that is; $AY \in \mathcal{N}_B$, where $Y \in \widetilde{\mathcal{N}}_{(A,B)}$. Hence, $AY^* = Y^*B$ and $BY^* = Y^*A$. Since $AY^* = Y^*B$. So $A^*AY^* = A^*Y^*B$. Hence, $BA^*Y^* = A^*Y^*B$. So $B(YA)^* = (YA)^*B$. That means, $YA \in \mathcal{N}_B$. Also, since, $BY^* = Y^*A$. So $BY^*A^* = Y^*AA^*$. Hence, $BY^*A^* = Y^*A^*B$. So $B(AY)^* = (AY)^*B$. That means $AY \in \mathcal{N}_B$. Then, $A\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_B$, and $\widetilde{\mathcal{N}}_{(A,B)}A \subseteq \mathcal{N}_B$.
In the same way, we can prove that, $B\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A$, and $\widetilde{\mathcal{N}}_{(A,B)}B \subseteq \mathcal{N}_A$.
Now, it is clear that $A + B \in \widetilde{\mathcal{N}}_{(A,B)}$, by Proposition 3.2, so $A\widetilde{\mathcal{N}}_{(A,B)} + B\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_B + \mathcal{N}_A$. Hence, $(A + B)\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A + \mathcal{N}_B$. Then $\widetilde{\mathcal{N}}_{(A,B)}\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_A + \mathcal{N}_B$. □
The next example shows that Corollary 3.6 is not true if we replace $\widetilde{\mathcal{N}}_{(A,B)}$ with the set $\mathcal{N}_{(A,B)}$.

Example 1: We show that $T, T^* \notin \mathcal{N}_{(T,T^*)}$. Let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$ and let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of scalars. Define a bounded linear map $T: \mathcal{H} \to \mathcal{H}$ by $Te_i = a_i e_{i+1}$, foe each $i \in \mathbb{N}$ or ,
$Tx = \sum_{n=1}^{\infty} \langle x, e_n \rangle a_n e_{n+1}$, and $T^*x = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{a_n} e_{n+1}$.
Now, we show that T is hyponormal if and only if the sequence $(|a_n|)_{n=1}^{\infty}$ monotonically increasing.
On the other hand, T is normal if and only if $a_n = 0$, for all $n \in \mathbb{N}$, that is; $T = 0$. We know that every normal is hyponormal operator, however, the converse is not true. Therefore, $T, T^* \notin \mathcal{N}_{(T,T^*)}$.

Theorem 3.7: If $T_i \in \mathcal{N}_{(A,B)}$ for all $i = 1, \dots, n$, then
1- $T_1 \oplus T_2 \oplus \dots \oplus T_n \in \mathcal{N}_{(A,B)}$.
2- $T_1 \otimes T_2 \otimes \dots \otimes T_n \in \mathcal{N}_{(A,B)}$.

Proof: (1) Since $T_i \in \mathcal{N}_{(A,B)}, \forall i = \overline{1,n}$, and $A, B \in \mathfrak{B}(\mathcal{H})$. So that, $AT_i{}^* = T_i{}^*B$. Hence,
$A(T_1 \oplus T_2 \oplus \dots \oplus T_n)^* = (AT_1^* \oplus AT_2^* \oplus \dots \oplus AT_n^*)$
$= (T_1^*B \oplus T_2^*B \oplus \dots \oplus T_n^*B) = (T_1 \oplus T_2 \oplus \dots \oplus T_n)^*B.$
(2) Let $x_1, x_2, \dots, x_n \in \mathcal{H}$. Then $A(T_1 \otimes T_2 \otimes \dots \otimes T_n)^*(x_1 \otimes x_2 \otimes \dots \otimes x_n)$
$= (AT_1^* \otimes AT_2^* \otimes \dots \otimes AT_n^*)(x_1 \otimes x_2 \otimes \dots \otimes x_n)$
$= (AT_1^*(x_1) \otimes AT_2^*(x_2) \otimes \dots \otimes AT_n^*(x_n)) = (T_1^*B(x_1) \otimes T_2^*B(x_2) \otimes \dots \otimes T_n^*B(x_n))$
$= (T_1 \otimes T_2 \otimes \dots \otimes T_n)^*B(x_1 \otimes x_2 \otimes \dots \otimes x_n)$. □

Proposition 3.8: If $A, B \in \mathfrak{B}(\mathcal{H})$, then
1- $\mathcal{N}^*{}_{(A,B)} = \mathcal{N}_{(B^*,A^*)}$. However, $\widetilde{\mathcal{N}}^*{}_{(A,B)} = \widetilde{\mathcal{N}}_{(A^*,B^*)}$.
2- If A and B are normal operator, then $\mathcal{N}_{(A,B)} = \mathcal{N}^*{}_{(B,A)}$, where $\mathcal{N}^*{}_{(B,A)} = \{T^*: T \in \mathcal{N}_{(B,A)}\}$.

3- If A, B are invertible operators, then $\mathcal{N}_{(A,B)} = \mathcal{N}_{(A^{-1},B^{-1})}$, via $\widetilde{\mathcal{N}}_{(A,B)} = \widetilde{\mathcal{N}}_{(A^{-1},B^{-1})}$.

Proof: (1) Consider that $T \in \mathcal{N}_{(B^*,A^*)}$, were $B^*T^* = T^*A^*$. Hence, $TB = AT$ or $AT = TB$. Then, $T^* \in \mathcal{N}_{(A,B)}$; that is; $T \in \mathcal{N}^*_{(A,B)}$. By the same way, we can prove the converse. Moreover, if $T \in \widetilde{\mathcal{N}}_{(A^*,B^*)}$. Therefore, $A^*T^* = T^*B^*$ and $B^*T^* = T^*A^*$. Hence, $BT = TA$ and $AT = TB$. Then, $T^* \in \widetilde{\mathcal{N}}_{(A,B)}$; that is; $T \in \widetilde{\mathcal{N}}^*_{(A,B)}$. Also, in the same way we can prove the converse.
(2) Assume that $T \in \mathcal{N}_{(A,B)}$, were $AT^* = T^*B$, via adjoint. We see that $TA^* = B^*T$. By Theorem 2.7. Therefore, $TA = BT$, (i.e. $BT = TA$). Then, $T^* \in \mathcal{N}_{(B,A)}$; that is, $T \in \mathcal{N}^*_{(B,A)}$. The converse is similar.
(3) Let A and B have an invertible, and $X \in \mathcal{N}_{(A,B)}$. So $AX^* = X^*B$. Hence $X^*B^{-1} = A^{-1}X^*$ or $A^{-1}X^* = X^*B^{-1}$. Then $X \in \mathcal{N}_{(A^{-1},B^{-1})}$. The converse is similar.
By same way to prove, $\widetilde{\mathcal{N}}_{(A,B)} = \widetilde{\mathcal{N}}_{(A^{-1},B^{-1})}$. □

Lemma 3.9: If the operators$A, B \in \mathfrak{B}(\mathcal{H})$. Then
1- For every invertible operator $T \in \mathcal{N}_{(A,B)}$ if and only if $T^{-1} \in \mathcal{N}_{(B,A)}$. Moreover, $T \in \widetilde{\mathcal{N}}_{(A,B)}$ if and only if, $T^{-1} \in \widetilde{\mathcal{N}}_{(A,B)}$.
2- Let A and B be normal operators. Then $T \in \widetilde{\mathcal{N}}_{(A,B)}$ if and only if $T^* \in \widetilde{\mathcal{N}}_{(A,B)}$.

Theorem 3.10: If A, B, C and $D \in \mathfrak{B}(\mathcal{H})$. Then
1- $\mathcal{N}_{(A,B)} \cap \mathcal{N}_{(C,D)} \subseteq \mathcal{N}_{(AC,BD)}$.
2- $\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_{(A^n,B^n)}, \forall\, n \in \mathbb{Z}^+$.

Proof: (1) Suppose that $X \in \mathcal{N}_{(A,B)} \cap \mathcal{N}_{(C,D)}$. Therefore, $AX^* = X^*B$ and $CX^* = X^*D$. Hence, $AX^*D = X^*BD$. So $ACX^* = X^*BD$, that is, $X \in \mathcal{N}_{(AC,BD)}$.
(2) Assume that $Y \in \mathcal{N}_{(A,B)}$. Therefore, $AY^* = Y^*B$. Hence, $A^2Y^* = AY^*B = Y^*B^2$. by the mathematical induction, it is easy to prove that $A^nY^* = Y^*B^n$ for each positive integer number n. Then, $Y \in \mathcal{N}_{(A^n,B^n)}$. □

Corollary 3.11: If he operators A, B, C and $D \in \mathfrak{B}(\mathcal{H})$. Then, $\mathcal{N}_{(A,B)} \cap \mathcal{N}_{(C,D)} \subseteq \mathcal{N}_{(A+C,B+D)}$. In particular, $\widetilde{\mathcal{N}}_{(A,B)} \subseteq \mathcal{N}_{(A+B)}$.
Notice that, this property does not hold in general. We see that in the following example.

Example 2: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, $X = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2-i & 2+2i \\ 1+i & -1+2i \end{bmatrix}$.
First, since $X = X^*$ is self-adjoint, and we have, $AX^* = \begin{bmatrix} 1 & i \\ 1 & 2i \end{bmatrix} = X^*B$. But, we see that $(A+B) = \begin{bmatrix} 3-i & 2+2i \\ 1+i & -1+3i \end{bmatrix}$. So $(A+B)X^* = \begin{bmatrix} 5-3i & 5+7i \\ 4+2i & -3+7i \end{bmatrix}$ and $X^*(A+B) = \begin{bmatrix} 2 & -1+i \\ 3-i & 3i \end{bmatrix}$. Hence, $(A+B)X^* \neq X^*(A+B)$. It implies that $X \in \mathcal{N}_{(A,B)}$ and $X \notin \mathcal{N}_{(A+B)}$ This means $\mathcal{N}_{(A,B)} \nsubseteq \mathcal{N}_{(A+B)}$.
Also, $BX^* = \begin{bmatrix} 2 & 5+6i \\ 3+2i & -3+5i \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ -i & 2i \end{bmatrix} = X^*A$. We have $X \notin \widetilde{\mathcal{N}}_{(A,B)}$. That means, $\mathcal{N}_{(A,B)} \neq \widetilde{\mathcal{N}}_{(A,B)}$.

Corollary 3.12: The A is a nilpotent operator if and only if B is nilpotent for each injective operator in $\mathcal{N}_{(A,B)}$.

Proof: It is clear by Theorem 3.10, and let T is injective in $\mathcal{N}_{(A,B)}$. So $0 = A^n T^* = T^* B^n$. Hence, $T^* B^n = 0$. Since T is injective. Therefore, B is nilpotent for all positive integer number n. □

Lemma 3.13: If $T \in \mathcal{N}_{(A,B)}$ and $T \in \mathcal{N}_A$ or $T \in \mathcal{N}_B$, then $T^n \in \mathcal{N}_{(A,B)}$ for each positive integer number $n \geq 1$.

Remark 3.14: If $T \in \widetilde{\mathcal{N}}_{(A,B)}$, then $AT^* = T^* B$ and $BT^* = T^* A$. Hence, $A(T^*)^2 = (T^*)^2 A$ and $B(T^*)^2 = (T^*)^2 B$. Therefore, $\widetilde{\mathcal{N}}^2_{(A,B)} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$, where $\widetilde{\mathcal{N}}^2_{(A,B)} = \{T^2 : T \in \widetilde{\mathcal{N}}_{(A,B)}\}$.

Theorem 3.15: Let $A, B \in \mathfrak{B}(\mathcal{H})$. Then for each positive integer number $n \geq 1$
$\left(\mathcal{N}_A \cap \mathcal{N}_{(A,B)}\right)^n \subseteq \mathcal{N}_{(A^n,B^n)}$.

Proof: Since $A, B \in \mathfrak{B}(\mathcal{H})$. Therefore, by Lemma 3.13, we have $\left(\mathcal{N}_A \cap \mathcal{N}_{(A,B)}\right)^n \subseteq \mathcal{N}_{(A,B)}$, and by Theorem 3.10, we have $\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_{(A^n,B^n)}$ for all positive integer number n. Therefore, the result is got.
Moreover, it can be proven in the same way that $\left(\mathcal{N}_B \cap \mathcal{N}_{(A,B)}\right)^n \subseteq \mathcal{N}_{(A^n,B^n)}, \forall n \geq 1$. □

Corollary 3.16: Let $A, B \in \mathfrak{B}(\mathcal{H})$. Then for each positive integer number $n \geq 1$
$\left(\widetilde{\mathcal{N}}^2_{(A,B)} \cap \widetilde{\mathcal{N}}_{(A,B)}\right)^n \subseteq \widetilde{\mathcal{N}}_{(A^n,B^n)}$.

Proof: Obviously, by Remark 3.14, which that $\widetilde{\mathcal{N}}^2_{(A,B)} \subseteq \mathcal{N}_A \cap \mathcal{N}_B$ and by Theorem 3.15. It's not hard to verify the result. □

Lemma 3.17: If $A, B \in \mathfrak{B}(\mathcal{H})$, then $\mathcal{N}_B \mathcal{N}_{(A,B)} = \mathcal{N}_{(A,B)} \mathcal{N}_A$.

Proof: Assume that $T_1 \in \mathcal{N}_A$ and $T_2 \in \mathcal{N}_{(A,B)}$, respectively. Therefore, $AT_1^* = T_1^* A$ and $AT_2^* = T_2^* B$. Hence, $AT_1^* T_2^* = T_1^* A T_2^*$ implies that $A(T_2 T_1)^* = (T_2 T_1)^* B$, also $T_1^* A T_2^* = T_1^* T_2^* B$ implies that $\mathcal{N}_{(A,B)} \mathcal{N}_A \subseteq \mathcal{N}_{(A,B)}$. Since $IT = T \in \mathcal{N}_{(A,B)}$, $AI^* = I^* A$ and $AT^* = T^* B$. The converse holds. Thus, $\mathcal{N}_{(A,B)} = \mathcal{N}_{(A,B)} \mathcal{N}_A$. By same way, we can prove that $\mathcal{N}_{(A,B)} = \mathcal{N}_B \mathcal{N}_{(A,B)}$. □
In the following proposition, we will show a relationship between the normality set and the direct sum decomposition.

Proposition 3.18: Consider a direct sum decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if $A \in \mathfrak{B}(\mathcal{H}_1)$ and $B \in \mathfrak{B}(\mathcal{H}_2)$, then $\mathcal{N}_A \cup \mathcal{N}_B \subset \mathcal{N}_{A \oplus B}$.

Proof: Clearly, it suffices to prove that $\mathcal{N}_A \subseteq \mathcal{N}_{A \oplus B}$. Let $\in N_A$, it implies that $AT^* = T^* A$. Hence, $T = T \oplus 0 \in \mathfrak{B}(\mathcal{H})$. So $(A \oplus B)(T \oplus 0)^* = (T \oplus 0)^* (A \oplus B)$ it holds. Therefore, we have the result. □
However, we can see that whenever $0 \neq A \neq B$ the $\mathcal{N}_A \cup \mathcal{N}_B = \mathcal{N}_{A \oplus B}$ does not hold.

4. Equivalent relationships with Bi-Normality set

In this section, we deal with the following relational structures.
Theorem 4.1: If $A \approx B$, then $\mathcal{N}_{(A,B)}$=$\widetilde{\mathcal{N}}_{(A,B)}$.

Proof: Suppose that A is similar to$B$. Then there exists X is an invertible operator such that $A = XBX^{-1}$. First, we have to prove that $\mathcal{N}_{(A,B)} = \mathcal{N}_B X^* = X^*\mathcal{N}_A$. Let $T \in \mathcal{N}_{(A,B)}$. Therefore, $AT^* = T^*B$. So $(XBX^{-1})T^* = T^*B$. Hence, $B(TX^{*^{-1}})^* = (TX^{*^{-1}})^*B$. Then $TX^{*^{-1}} \in \mathcal{N}_B$ or $T \in \mathcal{N}_B X^*$, that is, $\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_B X^*$. By the same way, we can prove that $\mathcal{N}_B X^* \subseteq \mathcal{N}_{(A,B)}$. Then $\mathcal{N}_{(A,B)} = \mathcal{N}_B X^*$. Also, by [7, Theorem 12], we have $\mathcal{N}_B X^* = X^*\mathcal{N}_A$. By the similar way , we can prove that $\widetilde{\mathcal{N}}_{(A,B)} = \mathcal{N}_B X^* = X^*\mathcal{N}_A$. Therefore, we have the result. □

Theorem 4.2: If $A \cong B$, then $\mathcal{N}_{(A,B)}$=$\widetilde{\mathcal{N}}_{(A,B)}$.

Proof: Suppose that A and B are unitarily equivalent. Then there exists a unitary operator U such that $A = U^*BU$. Let $T \in \mathcal{N}_{(A,B)}$, then $AT^* = T^*B$. Hence, $(U^*BU)T^* = T^*B$. Thus, $B(TU^*)^* = (TU^*)^*B$. Then $TU^* \in \mathcal{N}_B$ or $T \in \mathcal{N}_B U$, that is $\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_B U$. By the same way, we can reverse the proof. Therefore, $\mathcal{N}_{(A,B)} = \mathcal{N}_B U$. Also, by [7, Theorem 13], we can see that $\mathcal{N}_{(A,B)} = \mathcal{N}_B U = U\mathcal{N}_A$. By the similar way, we can prove that $\widetilde{\mathcal{N}}_{(A,B)} = \mathcal{N}_B U = U\mathcal{N}_A$. Then we have the result. □
In the following theorem, we notice that, if the relation is quasi-similar, then the result does not necessarily equal.

Theorem 4.3: If $A \simeq B$, then
1- $T_1^*\mathcal{N}_{(B,A)}T_1^* \subseteq \mathcal{N}_{(A,B)}$.
2- $T_1^*\widetilde{\mathcal{N}}_{(A,B)}T_1^* \cap T_2^*\widetilde{\mathcal{N}}_{(A,B)}T_2^* \subseteq \widetilde{\mathcal{N}}_{(A,B)}$.

Proof: Assume A is quasi-similar to B, if there exists T_1 form $\mathcal{H}_1$ to $\mathcal{H}_2$ and T_2 form $\mathcal{H}_2$ to $\mathcal{H}_1$ are injective via dense range such that $T_1A = BT_1$ and $AT_2 = T_2B$.
(1) Assume that $X \in \mathcal{N}_{(A,B)}$, where $AX^* = X^*B$. Hence, $T_1AX^*T_1 = T_1X^*BT_1$. Thus, $BT_1X^*T_1 = T_1X^*T_1A$ or, $B({T_1}^*X{T_1}^*)^* = ({T_1}^*X{T_1}^*)^*A$, that is; ${T_1}^*X{T_1}^* \in \mathcal{N}_{(A,B)}$ implies that $T_1^*\mathcal{N}_{(B,A)}T_1^* \subseteq \mathcal{N}_{(A,B)}$.
(2) Suppose that $Y \in \widetilde{\mathcal{N}}_{(A,B)}$, where $AY^* = Y^*B$ and $BY^* = Y^*A$. So, $BT_1Y^*T_1 = T_1Y^*T_1A$ and $AT_2Y^*T_2 = T_2Y^*T_2B$. Hence, $B({T_1}^*Y{T_1}^*)^* = ({T_1}^*Y{T_1}^*)^*A$ and $A({T_2}^*Y{T_2}^*)^* = ({T_2}^*Y{T_2}^*)^*B$. Therefore, $T_1^*\widetilde{\mathcal{N}}_{(A,B)}T_1^* \subseteq \mathcal{N}_{(B,A)}$, and $T_2^*\widetilde{\mathcal{N}}_{(A,B)}T_2^* \subseteq \mathcal{N}_{(A,B)}$. By intersection, we have the result. □

Proposition 4.4: Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{A},\mathcal{B} \in \mathfrak{B}(\mathcal{H})$ be operators of the form $\mathcal{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$. If for all $i = 1, \dots, 4,\ j = 1,2.\ T_i \in \mathcal{N}_{(A_j,B_j)}$, then $T \in \mathcal{N}_{(\mathcal{A},\mathcal{B})}$, where $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$.

Proof: We have that $T_i \in \mathfrak{B}(\mathcal{H}_j)$ and $T_i \in N_{(A_j,B_j)}$ for all $i = 1, \dots, 4,\ j = 1,2$.
$\mathcal{A}T^* = \begin{bmatrix} A_1T_1^* & A_1T_3^* \\ A_2T_2^* & A_2T_4^* \end{bmatrix} = \begin{bmatrix} T_1^*B_1 & T_3^*B_1 \\ T_2^*B_1 & T_4^*B_1 \end{bmatrix} = T^*\mathcal{B}$. □

Theorem 4.5: Let $S = U|S|$ be the polar decomposition of $\mathcal{N}_{(A,B)}$ for an operator S, where U is a unitary and S is a quasi-normal if and only if $\tilde{S}$ is in $\mathcal{N}_{(A,B)}$.
Proof: Suppose that $S \in \mathcal{N}_{(A,B)}$ and $S = U|S|$ is the polar decomposition of S. Then, $AS^* = S^*B$.
$\Leftrightarrow\ A(U|S|)^* = (U|S|)^*B$,

$\Leftrightarrow\ A|S|^{\frac{1}{2}}U^*|S|^{\frac{1}{2}} = |S|^{\frac{1}{2}}U^*|S|^{\frac{1}{2}}B$,
$\Leftrightarrow\ A\tilde{S}^* = \tilde{S}^*B$. Therefore, $\tilde{S} \in \mathcal{N}_{(A,B)}$. □

Corollary 4.6: If A is the Aluthge transformation of B, then $\mathcal{N}_{(A,B)} = \mathcal{N}_A = \mathcal{N}_B$, where A and B are quasi normal.

Proof: Assume that $X \in N_{(A,B)}$ and $A = |B|^{\frac{1}{2}}U|B|^{\frac{1}{2}}$. So $AX^* = X^*B$. Thus, $|B|^{\frac{1}{2}}U|B|^{\frac{1}{2}}X^* = X^*B$. Since $B = U|B|$ is the polar decomposition of B and B is quasi normal. Hence, $U|B|X^* = X^*B$. Then, $BX^* = X^*B$. So $X \in \mathcal{N}_B$. The converse is the same way.
Now, if $A = U|A|$ is the polar decomposition of A, in the same way, we can get the result. □

Proposition 4.7: If A, B be a normal operators, then, $A\widetilde{\mathcal{N}}_{(A,B)} = \widetilde{\mathcal{N}}_{(A,B)}B$ and $B\widetilde{\mathcal{N}}_{(A,B)} = \widetilde{\mathcal{N}}_{(A,B)}A$.

Proof: Suppose that $T \in \widetilde{\mathcal{N}}_{(A,B)}$ and A, B are normal operators. So $AT^* = T^*B$ via $BT^* = T^*A$. We can see that $TA^* = B^*T$ via $TB^* = A^*T$.
Since A, B are normal operators and using Theorem 2.7. Therefore, $AT = TB$ and $BT = TA$. The result is proven. □

Proposition 4.8: If $A, B \in \mathfrak{B}(\mathcal{H})$, and $A \in \mathcal{N}_B$, then $A\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_{(A,B)}$.

Proof: Let $AT \in \mathcal{N}_{(A,B)}$, where $T \in \mathcal{N}_{(A,B)}$, so $AT^* = T^*B$. Thus, $A(AT)^* = AT^*A^* = T^*BA^* = T^*A^*B = (AT)^*B$. Then $A\mathcal{N}_{(A,B)} \subseteq \mathcal{N}_{(A,B)}$. It has a nontrivial invariant subspace, such that $A \neq \alpha I, \alpha \in \mathbb{C}$. So $\mathcal{N}_{(A,B)} \neq \mathfrak{B}(\mathcal{H})$. □
We will leave the following as an open problem.

Problem 1: If $A \in \mathfrak{B}(\mathcal{H}_1)$ and $B \in \mathfrak{B}(\mathcal{H}_2)$, is the formula $\mathcal{N}_{A\oplus B} = \mathcal{N}_A \oplus \mathcal{N}_B$ valid for every operator in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$?

5. Conclusion

The aim of this paper is to introduce the general concept of the normality set and to try to solve the problems of branching or splitting with some properties of the extended it, and showing that not invariant via some transpose linear operators, but has nontrivial invariant subspace. Founded some relationships that connect it and the reverse is not true always.

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