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Several Subclasses of r-Fold Symmetric Bi-Univalent Functions possess Coefficient Bounds

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Abstract
In this paper we offer two new subclasses of an open unit disk of r-fold symmetric bi-univalent functions. The Taylor-Maclaurin coefficients $|e_{r+1}|$ *and* $|e_{2r+1}|$ have their coefficient bounds calculated. Furthermore, for functions in $\mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$, we have solved Fekete-Szegö functional issues. For the applicable classes, there are also a few particular special motivator results.

Keywords: Analytic Functions, Univalent Functions, Bi-Univalent Functions, Taylor-Maclaurin Series, Fekete-Szegö Coefficient.

تحديد المعاملات لأصناف جزئية من الدوال المتناظرة ثنائية التكافؤ *r−fold*

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الخلاصة
قدمنا في هذا العمل صنفين جزئيين جديدتين من الدوال المتكافئة ثنائية التكافؤ r−fold في قرص الوحدة المفتوح, وكذلك تم حساب معاملات تايلور− ماكلورين. وعلاوة على ذلك، قمنا بحل المشكلات الدوال Fekete−Szego بالنسبة للأصناف القابلة للتطبيق.

1. Introduction

Let $U=\{w\in\mathbb{C}:|w|<1\}$ be an open unit disc in $\mathbb{C}$. Let $H(U)$ be the class of analytic functions in $\mathfrak{I}$ and consider $U[a,i]$ be a subclass of $H(U)$ of the form

$$h(w)=\ e\ +\ e_i w^i\ +\ e_{i+1}\ w^{i+1}\ +\cdots\ ,$$

where $e\in\mathbb{C}$ and $i\in\mathbb{N}=\{1,2,\ldots\}$. The class $\mathcal{A}$ of normalizing functions is meeting the constraint $h(0)=h'(0)-1=0$ and its obtained by the next Taylor series expansion.

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$$h(w) = w + \sum_{i=2}^{\infty} e_i\, w^i\,, (w \in U). \quad (1.1)$$

The Hadamrd product of two functions in $\mathcal{A}$, can be defined by

$$k(w) = w + \sum_{i=2}^{\infty} d_i\, w^i\,, (w \in U), \quad (1.2)$$

which is given by

$$h(w) * k(w) = w + \sum_{i=2}^{\infty} e_i\, d_i\, w^i\,, (w \in U). \quad (1.3)$$

Furthermore, assume $\mathcal{H}$ be the class of all univalent functions of $\mathcal{A}$ in U.

The Kuebe one Qoarter Theorem [1] verifies that the image is correct of U each and every univalent function $h \in \mathcal{H}$ contains a disk with a radius $\frac{1}{4}$. As a result, each univalent function h has the inverse h^{-1} define $w = h^{-1}\big(h(w)\big), (w \in U)$

$$\omega = h^{-1}\big(h(\omega)\big), \left(|\omega| < \rho_0(h); \rho_0(h) \geq \frac{1}{4}\right), \quad (1.4)$$

whereas

$$h^{-1}(\omega) = \omega - s_2\omega^2 + (2s_2^2 - s_3)\omega^3 - (5s_2^3 - 5s_2s_3 + s_4)\omega^4 + \cdots. \quad (1.5)$$

If the functions h and h^{-1} are univalent in U, $h \in \mathcal{A}$ then both of them are known to be as the bi-univalent functions. Make a note of the class of bi-univalent functions in by $\sum$, that are normalized by (1.1).

Now, we assume $h(w)$ and $k(w)$ to be analytic functions in U. The function $h(w)$ is said to be subordinate to a function $k(w)$, or the $k(w)$ is said to be superordinate to $h(w)$, if there exists a Schwarz function $z(w)$ analytic in U, with $z(0) = 0$ and $|z(w)| < 1, (w \in U)$, such that

$$h(w) = k(z(w)),$$

written as

$$h \prec k \text{ or } h(w) \prec k(w), \quad (w \in U).$$

Furthermore, if the function h is univalent in U, then we get the following equivalence $h(w) \prec k(w)$ is obtained if and only if $h(0) = k(0)$ and $h(U) \subset k(U)$, this can be shown in [2-6].

Lewien [7] examined the class $\sum$ of bi-univalent functions and got a coefficient bound given by $|e_2| \leq 1.51$ for each $h \in \sum$. Following that, the work of Lewien [7], Clonie, and Branan [8] encouraged, we guessing$|e_2| \leq \sqrt{2}$ for all $h \in \sum$.

In recent times, in fact, Srivastva et. al, [9] have re-energized the study of bi-univalent and analytic functions, which was followed by Bulot [10]. Adegaeni and et. al, [11], Goney et al. [12], Srivastva and Wans [13] and other [14-16]. We observe that the class $\sum$ is a placeholder for a blank note. Consider, the functions w, $\frac{w}{1-w}, -\log(1-w)$ *and* $\frac{1}{2} log \frac{1+w}{1-w}$ belong to $\sum$. On the other hand, the Kuebe functions are not $\sum$ members. Until date, the coefficient estimation problem for each of the Taylor-Maclaurin coefficients $|e_i|, (\, i \in \mathbb{N} = \{1,2,3,4, \dots \dots\}, i \geq 3\,)$ has been a challenge, for functions $h \in \sum$ which is still a work in progress (see [9]).

The function l defined by $l(w) = \sqrt[r]{h(w^r)}, (r \in \{1,2,3, \dots\})$ maps and its is univalent in U into a place that r-fold symmetry for all $h \in \mathcal{H}$. If the condition of normalized is met, a function is said to be r-fold symmetric (see [17-19]).

$$h(w) = w + \sum_{i=1}^{\infty} s_{ri+1} w^{ri+1}, (r \in \mathbb{N} = \{1,2,3, \dots .\}, w \in U). \quad (1.6)$$

$\mathcal{H}_r$ indicates the class of r-fold symmetric univalent functions, which are normalized by the series expansion above (1.6). The function in class $\mathcal{H}$ is one-fold symmetric, especially if $r = 1$. The concept of r-fold symmetric biunivalent functions can be thought of in the same way as the concept of r-fold symmetric functions. Each function h in the class $\sum$ generates an r-fold symmetric bivalent function for all positive integers r, h^{-1} which is defined as follows:

$$k(\omega) = \omega - s_{r+1} w^{r+1} + [(r+1)s_{2r+1}^2 - s_{2r+1}]\omega^{2r+1} - \left[\frac{1}{2}(r+1)(3r+1)s_{r+1}^3 - (3r+1)s_{r+1}s_{2r+1}\right]\omega^{3r+1} + \cdots, (k = h^{-1}), \quad (1.7)$$

where $\sum_r$ denotes the class of r-fold symmetric bivalent functions. The formula (1.7) of the class $\sum$ is synchronized with the expression (1.5) for $r = 1$. $\mathcal{G}$ of the form's class function should be used to indicate:

$t(w) = 1 + t_1 w + t_2 w^2 + \cdots, (w \in U)$,

as a result

$\mathcal{R}e(t(w) > 0 \quad (w \in U)$.

Pommerenke [17] used a symmetric r-fold function t in the class $\mathcal{G}$ by having the following forms

$$t(w) = 1 + u_r w^r + u_{2r} w^{2r} + u_3 w^{3r} + \cdots. \quad (1.8)$$

Throughout this study, we assume that an analytic function ϑ with a positive real portion in U has the properties $\vartheta(0) = 0$ and $\vartheta'(0) > 0$, and $\vartheta(U)$ is symmetric in the true sense. A function of this type has a shape expansion in series:

$$\vartheta(w) = 1 + F_1 w + F_2 w^2 + F_3 w^3 + \cdots, (F_1 > 0). \quad (1.9)$$

Now, if we have two analytic functions $q(w)$ and $v(\omega)$ in U with

$$q(0) = v(0) \text{ and } \max\{|q(w)|, |v(\omega)|\} < 1.$$

If we consider the following scenario

$$q(w) = p_r w^r + p_{2r} w^{2r} + p_{3r} w^{3r} + \cdots \quad (1.10)$$

$$v(\omega) = u_r \omega^r + u_{2r} \omega^{2r} + u_{3r} \omega^{3r} + \cdots. \quad (1.11)$$

By taking,

$$|p_r| \le 1, \quad |p_{2r}| \le 1 - |p_r|^2, \quad |u_r| \le 1 \text{ and } |u_{2r}| \le 1 - |u_r|^2. \quad (1.12)$$

We have arrived to this conclusion using simple calculations.

$$\vartheta\big(h(w)\big) = 1 + F_1 p_r w^r + F_1 p_{2r} w^{2r} + F_2 p_r^2 w^r w^{2r} + \cdots, (|w| < 1), \quad (1.13)$$

and

$$\vartheta\big(v(\omega)\big) = 1 + F_1 u_r \omega^r + F_1 u_{2r} \omega^{2r} + F_2 u_r^2 \omega^{2r} + \cdots, (|\omega| < 1). \quad (1.14)$$

There are two classes of r-fold bivalent functions are proposed in this paper, and the Taylor-Maclauain coefficients' boundary values$|s_{r+1}|$ and $|s_{2r+1}|$ are obtained. We discussed another two new classes of Fekte-Szgö functional issues with a function.

Definition 1.1: Assume that $h(w)$ is a function in the class $\mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$ which is described in (1.6). If the following criteria are met

$$h \in \sum\nolimits_r \quad , \left(\frac{wh'(w)}{h(w)}\right)^{\gamma}\left(\frac{wh'(w)+(1+2\tau)w^2h''(w)}{(1-\tau)h(w)+\tau wh'(w)}\right) \prec \vartheta(w),$$

whereas $w \in U, 0 \le \tau \le 1, \gamma \ge 0$, and

$$\left(\frac{\omega k'(\omega)}{k(\omega)}\right)^{\gamma}\left(\frac{\omega k'(\omega)+(1+2\tau)\omega^2k''(\omega)}{(1-\tau)k(\omega)+\tau\omega k'(\omega)}\right) \prec \vartheta(\omega), \big(k(\omega)=h^{-1}(\omega)\big),$$

whereas $k(\omega)$ be a function by which (1.7).
The following theorem is demonstrated to a class $\mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$.

2. Main Results

Theorem 2.1: Assume that $h(w)$ be a function of the class $\mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$, which is defined by (1.6). Then

$$|s_{r+1}| \le \frac{F_1\sqrt{2\mathrm{F}_1}}{\sqrt{\left|\begin{matrix}\begin{bmatrix} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}F_1^2 - 2r^2(r\tau+\tau+r+\gamma+1)^2F_2 \\ +(r\tau+\tau+r+\gamma+1)^2F_1\end{matrix}\right|}}, \quad (2.15)$$

and

$$|s_{2r+1}| \le \left\{\begin{matrix} \dfrac{(r+1)F_1}{\begin{bmatrix} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}} & ;if\ |F_2| \le F_1 \\ \dfrac{(r+1)\left[\left\|\begin{matrix}\begin{bmatrix} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}F_1^2 \\ -2r^2(r\tau+\tau+r+\gamma+1)^2F_2\end{matrix}\right\|F_1\right] \\ +(r+1)(r\tau+\tau+r+\gamma+1)^2|F_2|F_1}{\begin{bmatrix} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}\times \left|\begin{matrix}\begin{bmatrix} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}F_1^2 \\ -2r^2(r\tau+\tau+r+\gamma+1)^2F_2 \\ +(r\tau+\tau+r+\gamma+1)^2F_1\end{matrix}\right|} & ;if\ |F_2| > F_1 \end{matrix}\right\}. \quad (2.16)$$

Proof. If $h \in \mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$. There are two analytic functions such that. $q: U \to U$ and $v: U \to U$ with $q(0)=v(0)=0$, satisfying the following criteria:

$$\left(\frac{wh'(w)}{h(w)}\right)^{\gamma}\left(\frac{wh'(w)+(1+2\tau)w^2h''(w)}{(1-\tau)h(w)+\tau wh'(w)}\right) = \vartheta\big(q(w)\big), \quad (2.17)$$

and

$$\left(\frac{\omega k'(\omega)}{k(\omega)}\right)^{\gamma}\left(\frac{\omega k'(\omega)+(1+2\tau)\omega^2 k''(\omega)}{(1-\tau)k(\omega)+\tau\omega k'(\omega)}\right)=\vartheta\big(v(w)\big). \tag{2.18}$$

We obtain

$$\left(\frac{wh'(w)}{h(w)}\right)^{\gamma}\left(\frac{wh'(w)+(1+2\tau)w^2h''(w)}{(1-\tau)h(w)+\tau wh'(w)}\right)=$$
$$1+r(\tau r+\tau+r+\gamma+1)s_{r+1}w^r+2r(2\tau r+\tau+2r+\gamma+1)s_{2r+1}w^{2r+1}$$
$$-\left(r^2(\tau r+\tau+r+1)(\tau r+\tau+2r-\gamma+1)-\frac{r}{2}(\gamma^2-3\gamma)\right)s_{r+1}^2w^{2r+1}+\cdots, \tag{2.19}$$

and

$$\left(\frac{\omega k'(\omega)}{k(\omega)}\right)^{\gamma}\left(\frac{\omega k'(\omega)+(1+2\tau)\omega^2k''(\omega)}{(1-\tau)k(\omega)+\tau\omega k'(\omega)}\right)=$$
$$1-r(\tau r+\tau+r+\gamma+1)s_{r+1}\omega^r-2r(2\tau r+\tau+2r+\gamma+1)s_{2r+1}\omega^{2r+1}$$
$$+\left(r^2(\tau r+\tau+r+1)(\tau r+\tau+2r-\gamma+1)+2r(r+1)+\frac{r}{2}(\gamma^2-3\gamma)\right)s_{r+1}^2\omega^{2r+1}$$
$$+\cdots, \tag{2.20}$$

We can deduce that from (1.13), (1.14), (2.19), and (2.20), that

$$r(\tau r+\tau+r+\gamma+1)s_{r+1}=F_1p_r, \tag{2.21}$$

$$2r(2\tau r+\tau+2r+\gamma+1)s_{2r+1}w^{2r+1}$$
$$-\left(r^2(r\tau+\tau+r+1)^2-r^2\gamma(r\tau+\tau+r+1)-\frac{1}{2}r(\gamma^2-3\gamma)\right)s_{r+1}^2$$
$$=F_1p_{2r}+F_2p_r^2, \tag{2.22}$$

$$r(\tau r+\tau+r-\gamma+1)s_{r+1}=F_1u_r, \tag{2.23}$$

and

$$-2r(2\tau r+\tau+2r+\gamma+1)s_{2r+1}w^{2r+1}$$
$$+\left(r^2(\tau r+\tau+r+1)(\tau r+\tau+2r-\gamma+1)+2r(r+1)+\frac{r}{2}(\gamma^2-3\gamma)\right)s_{r+1}^2$$
$$=F_1u_{2r}+F_2u_r^2. \tag{2.24}$$

We derive (2.21) and (2.23) from (2.21)

$$u_r=-p_r, \tag{2.25}$$

and

$$2r^2(\tau r+\tau+2r+\gamma+1)^2s_{r+1}^2=F_1^2(p_r^2+u_r^2).$$

We get (2.22) and (2.24) by putting them together and then by doing some calculations with (2.21) and (2.25), we have

$$\left(\begin{bmatrix}r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1)\\+r(r^3+2r^2+1)+r^2\gamma(r(\tau+)+\tau+2)\end{bmatrix}F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2\right)s_{r+1}^2$$
$$=F_1^3(p_{2r}+u_{2r}). \tag{2.26}$$

Furthermore, when the equations (2.25), (2.26), and (1.12) are combined, we have the following result

$$\left|\left(\begin{bmatrix}r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1)\\+r(r^3+2r^2+1)+r^2\gamma(r(\tau+)+\tau+2)\end{bmatrix}F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2\right)\right|$$
$$\leq 2F_1^3(1-|u_r|^2). \tag{2.27}$$

We now have (2.21) and (2.27) just like our starting points.

$$|s_{r+1}| \leq \frac{F_1\sqrt{2\mathrm{F}_1}}{\sqrt{\left|\begin{array}{c}\left[\begin{array}{l} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{array}\right]F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2 \\ +(r\tau+\tau+r+\gamma+1)^2F_1\end{array}\right|}},$$

By subtracting (2.24) from (2.22) and multiplying by (2.25) and (2.21), we obtain

$$\begin{gathered} r^2(\tau r+\tau+r+1)(\tau r+\tau+2r-\gamma+1)+2\gamma(r+1)+\frac{1}{2}(\gamma^2-3\gamma)F_1u_{2r} \\ +\left(r^2(r\tau+\tau+r+1)^2-r^2\gamma(r\tau+\tau+r+1)-\frac{1}{2}r(\gamma^2-3\gamma)\right)F_1p_{2r} \\ +2r^2(r\tau+\tau+r+\gamma+1)^2F_2p_r^2 \\ =2r^2(r\tau+\tau+r+\gamma+1)^2s_{2r+1}. \end{gathered} \tag{2.28}$$

As a result, applying equation (2.24) in (2.28), we get

$$\begin{gathered} -2r(2\tau r+\tau+2r+\gamma+1)|s_{2r+1}| \\ \leq r^2(\tau r+\tau+r+1)(\tau r+\tau+2r-\gamma+1)+2\gamma(r+1)+\frac{1}{2}(\gamma^2-3\gamma)F_1 \\ -r^2(r\tau+\tau+r+1)^2-r^2\gamma(r\tau+\tau+r+1)-\frac{1}{2}r(\gamma^2-3\gamma)F_1|p_r|^2 \\ =(r+1)(2\tau r+\tau+2r+\gamma+1)|F_2||p_r|^2. \end{gathered} \tag{2.29}$$

Since

$$|p_r|^2 \leq \frac{(r\tau+\tau+r+1)^2F_1}{\sqrt{\left|\begin{array}{c}\left[\begin{array}{l} r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{array}\right]F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2 \\ +(r\tau+\tau+r+\gamma+1)^2F_1\end{array}\right|}}. \tag{2.30}$$

We can simply get from Theorem 2.1's assertion the equation (2.16) by swapping from (2.30) into (2.29).

The following findings are obtained when one-fold symmetric functions of Theorem 2.1 are used.

Corollary (2.2): Assume $h(w)$ be a function of the class $\mathcal{M}_{\Sigma,1}(\gamma,\tau,\vartheta)$, which is defined by (1.6). Then

$$|s_2| \leq \frac{F_1\sqrt{2\mathrm{F}_1}}{\sqrt{\left|\begin{array}{c}[\tau^2(\tau^2+2)+6\tau+4+\gamma(2\tau+3)]F_1^2-2(2\tau+\gamma+2)^2F_2 \\ +(2\tau+\gamma+2)^2F_1\end{array}\right|}},$$

and

$|s_3| \leq$

$$\left\{\begin{array}{ll}\dfrac{2F_1}{\tau^2(\tau^2+2)+7\tau+3+\gamma(2\tau+3)} & ; if\ |F_2| \le F_1 \\ \dfrac{\begin{array}{c}2[|[\tau^2(\tau^2+2)+7\tau+3+\gamma(2\tau+3)]F_1^2-2(2\tau+\gamma+2)^2F_2|]F_1 \\ +2(2\tau+\gamma+2)^2|F_2|F_1\end{array}}{\begin{array}{c}[\tau^2(\tau^2+2)+7\tau+3+\gamma(2\tau+3)]\times \\ \left|\begin{bmatrix}\tau^2(\tau^2+2)+7\tau \\ +4+\gamma(2\tau+3)\end{bmatrix}F_1^2-2(2\tau+\gamma+2)^2F_2+(2\tau+\gamma+2)^2F_1\right|\end{array}} & ; if\ |F_2| > F_1\end{array}\right\}.$$

Theorem (2.3): Assume $h(w)$ be a function of the class $\mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$, which is defined by (1.6). Then

$$|s_{2r+1}-\beta s_{r+1}^2| \le \left\{\begin{array}{ll}\dfrac{F_1}{2r(r\tau+\tau+r+\gamma+1)} & \text{for } o \le |x(\beta)| < \dfrac{1}{2r(r\tau+\tau+r+\gamma+1)} \\ 2F_1|x(\beta)| & \text{for } |x(\beta)| \ge \dfrac{1}{2r(r\tau+\tau+r+\gamma+1)}\end{array}\right\}, \quad (2.31)$$

whereas

$$x(\beta) = \frac{(r+1-2\beta)F_1^2}{2\begin{bmatrix}r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2},$$

$$s_{r+1}^2 = \frac{F_1^3(p_{2r}+u_{2r})}{\begin{bmatrix}r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2}. \quad (2.32)$$

By subtracting (2.24) from (2.22) we arrive at

$$s_{2r+1} = \frac{(r+1)F_1^2(p_r^2+u_r^2)}{2r^2(r\tau+\tau+r+\gamma+1)^2} + \frac{F_1(p_{2r}-u_{2r})}{2r(r\tau+\tau+r+\gamma+1)}. \quad (2.33)$$

Consequently, from (2.32) and (2.33), we have

$$s_{2r+1}-\beta s_{r+1}^2 = F_1\left[\left(x(\beta)+\frac{1}{2r(r\tau+\tau+r+\gamma+1)}\right)p_{2r}+\left(x(\beta)-\frac{1}{2r(r\tau+\tau+r+\gamma+1)}\right)u_{2r}\right], \quad (2.34)$$

whereas

$$x(\beta) = \frac{(r+1-2\beta)F_1^2}{2\begin{bmatrix}r^2\tau^2(r^2\tau^2+r+1)+r\tau(r^3+4r^2+r+1) \\ +r(r^3+2r^2+1)+r^2\gamma(r(\tau+1)+\tau+2)\end{bmatrix}F_1^2-2r^2(r\tau+\tau+r+\gamma+1)^2F_2}.$$

Because each of $A_i \in \mathbb{R}$ and $A_1 > 0$, this implies we get the equation (2.31).
Theorem 2.3 has been reduced to the next level when one fold functions are symmetric.

Corollary 2.4: Assume $h(w)$ be a function of the class $\mathcal{M}_{\Sigma,1}(\gamma,\tau,\vartheta)$, which is defined by (1.6). Then

$$|s_3-\beta s_2| \le \left\{\begin{array}{ll}\dfrac{F_1}{2(2\tau+\gamma+2)} & \text{for } o \le |x(\beta)| < \dfrac{1}{2(2\tau+\gamma+2)} \\ 2F_1|x(\beta)| & \text{for } |x(\beta)| \ge \dfrac{1}{2(2\tau+\gamma+2)}\end{array}\right\}.$$

In Theorem 2.3 in the case where $\beta = 1$. As a result, we have the corollary.

Corollary 2.5: Assume $h(w)$ be a function of the class$\mathcal{M}_{\Sigma,r}(\gamma,\tau,\vartheta)$, which is defined by (1.6). Then

$$|s_{2r+1} - s_{r+1}^2| \leq \begin{cases} \dfrac{F_1}{2r(r\tau+\tau+r+\gamma+1)} & for \quad o \leq |x(\beta)| < \dfrac{1}{2r(r\tau+\tau+r+\gamma+1)} \\ 2F_1|x(\beta)| & for \quad |x(\beta)| \geq \dfrac{1}{2r(r\tau+\tau+r+\gamma+1)} \end{cases}.$$

Corollary 2.5 has been reduced to the following Corollary if there is a one fold symmetric.

Corollary 2.6: Assume $h(w)$ be a function of the class$\mathcal{M}_{\Sigma,1}(\gamma,\tau,\vartheta)$, which is defined by (1.6). Then

$$|s_3 - s_2^2| \leq \left(\frac{F_1}{2(2\tau+\gamma+2)}\right).$$

3. Conclusions

We conclude that when the two new classes r-fold symmetric bi-univalent are applied to geometric functions, it is possible to figure out$|s_{r+1}|$ and $|s_{2r+1}|$ for each class r-fold symmetric bi-univalent, this is advantageous in complex analysis.The Fekte-Szgo functional issues for functions are also obtained, and a large number of improver findings for these two new classes are published within a new U.

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