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Periodic Solutions For Nonlinear Systems of Multiple Integro-differential Equations that Contain Symmetric Matrices with Impulsive Actions

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Abstract

This paper examines a new nonlinear system of multiple integro-differential equations containing symmetric matrices with impulsive actions. The numerical-analytic method of ordinary differential equations and Banach fixed point theorem are used to study the existence, uniqueness and stability of periodic solutions of impulsive integro-differential equations with piecewise continuous functions. This study is based on the Hölder condition in which the ordering α, β and γ are real numbers between 0 and 1.

Keywords: Nonlinear T-system; existence, uniqueness and stability solution; numerical-analytic method; impulsive actions; Hölder condition.

الحلول الدورية للانظمة غير الخطية للمعادلات التكاملية–التفاضلية الحاوية على مصفوفات متناظرة مع التاثيرات النبظية

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الخلاصة

تبحث هذه المخطوطة دراسة جديدة للانظمة غير الخطية للمعادلات التكاملية –التفاضلية الحاوية على مصفوفات مع التاثيرات النبية. تم استخدام الاسلوب التحليلي– العددي للمعادلات التفاضلية العادية ومبرهنة باناخ للنقطة الثابتة لدراسة وجود ووحدانية واستقرارية الحلول الدورية للمعادلات التكاملية –التفاضلية غير الخطية المندفعة ذات الدوال المستمرة مقطعيا. في هذه الدراسة تم الاعتماد على شرط هولدرللاعداد الحقيقية الفا وبيتا وكاما المحصورة بين الصفر والواحد.

1. Introduction

The importance of studying impulsive differential equations stems from the fact that they are useful mathematical machinery for modelling a variety of processes and phenomena investigated in mechanics, electronics, economics, optimal control, biotechnologies and other fields [1]. Impulsive differential equations (or differential equations with impulse actions) have a long history in control and mechanical systems, which could be explored using a mathematical tool based on impulsive differential equations for real-world applications in

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pure and applied mathematics. The application of the classical methods of the theory of ordinary differential equations is also used in impulsive differential equations [2-3].

The works of Samoilenko describe the broad ideas of the theory of systems with impulsive actions from a novel point of view and examine their essential particular properties. His study can be considered as the first work in the area of numerical analytic technique [4-5].

Later, many mathematicians' efforts were devoted to studying the problems of the existence, uniqueness and stability of solutions of differential equations with pulse action, as well as the creation of the periodic theory of impulsive systems. The theory of impulsive differential equations has advanced rapidly in recent decades due to its theoretical and practical importance [6-10].

We concentrate on nonlinear systems of differential equations with impulsive and describe the initial and impulsive conditions using the physical meaning of initial conditions as defined in Butris [11].

1.1 Preliminaries

The essential definitions, theorems, lemmas and conditions have been illustrated in this part using the numerical analytical technique that has been investigated for periodic T-system solutions. In this article, we investigate the numerical-analytical method analytically through some theorems and the theorems that have been proven by Samoilenko. The method is concerned with the analytical technique and demonstrated the theorems in pure form.

This study focuses on the following nonlinear T-system with impulse actions:

$$\left.\begin{aligned}
\frac{dx(t,x_0,y_0)}{dt} &= \big(A_1+B_1(t)\big)x(t,x_0,y_0)+\big(A_2+B_2(t)\big)y(t,x_0,y_0)\\
&\quad +f\big(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big),t\neq\tau_i,\\
\Delta x|_{t=\tau_i} &= I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big),\\
\frac{dy(t,x_0,y_0)}{dt} &= \big(C_1+D_1(t)\big)x(t,x_0,y_0)+\big(C_2+D_2(t)\big)y(t,x_0,y_0)\\
&\quad +g\big(t,x(t,x_0,y_0),y(t,x_0,y_0),v(t)\big),t\neq\tau_i\\
\Delta y|_{t=\tau_i} &= J_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),v(\tau_i)\big)
\end{aligned}\right\} \tag{1}$$

where $0<\tau\leq t\leq T$, $x\in G_0$, $y\in G_1$, $u\in G_u$ and $v\in G_v$. G_0 and G_1 are compact subset of R^n, while G_v and G_u are bounded subsets of R^m.

Define the piecewise continuous vector functions $f(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t))$ and $g(t,x(t,x_0,y_0),y(t,x_0,y_0),v(t))$ on the following domains:

$$\left.\begin{aligned}
(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)) \in R^n\times G_0\times G_1\times G_u\\
(t,x(t,x_0,y_0),y(t,x_0,y_0),v(t)) \in R^n\times G_0\times G_1\times G_v
\end{aligned}\right\} \tag{2}$$

while $G_0\colon \|x(t,x_0,y_0)-x_0\|\leq r_x$, $G_1\colon \|y(t,x_0,y_0)-y_0\|\leq r_y$, $G_u\colon \|u(t)\|\leq d_u$ and $G_v\colon \|v(t)\|\leq d_v$ are continuous in $x(t,x_0,y_0)$, $y(t,x_0,y_0)$. The functions $u(t)$ and $v(t)$ and periodic in t with period T.

The periodic solutions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ that are obtained from (1) are defined as follows:

$$x(t,x_0,y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)}\Big(z(s) - \Delta_f\big(s, x(s,x_0,y_0), y(s,x_0,y_0), u(s)\big)\Big)\,ds + \sum_{0<\tau_i<t} e^{A_1(t-\tau_i)} I_i\big(x(\tau_i,x_0,y_0), y(\tau_i,x_0,y_0), u(\tau_i)\big) \quad (3)$$

$$\Delta_f\big(t, x(t,x_0,y_0), y(t,x_0,y_0), u(t)\big) = A_1 x_0 + \frac{A_1}{e^{A_1 T} - I}\int_0^T e^{A_1(T-s)} z(s)\,ds + \frac{A_1}{e^{A_1 T} - I}\sum_{i=1}^{p} e^{A_1(t-\tau_i)} I_i\big(x(\tau_i,x_0,y_0), y(\tau_i,x_0,y_0), u(\tau_i)\big) \quad (4)$$

where $det(A_1)$ and $det(e^{A_1 T} - I)$ are defined and nonzero. The initial condition $x(0,x_0,y_0) = x_0$ and $z(t) = B_1(t)x(t,x_0,y_0) + \big(A_2 + B_2(t)\big)y(t,x_0,y_0) + f\big(t, x(t,x_0,y_0), y(t,x_0,y_0), u(t)\big)$.

$$y(t,x_0,y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)}\Big(w(s) - \Delta_g\big(s, x(s,x_0,y_0), y(s,x_0,y_0), v(s)\big)\Big)\,ds + \sum_{0<\tau_i<t} e^{C_2(t-\tau_i)} J_i\big(x(\tau_i,x_0,y_0), y(\tau_i,x_0,y_0), v(\tau_i)\big) \quad (5)$$

$$\Delta_g\big(t, x(t,x_0,y_0), y(t,x_0,y_0), v(t)\big) = C_2 y_0 + \frac{C_2}{e^{C_2 T} - I}\int_0^T e^{C_2(T-s)} w(s)\,ds + \frac{C_2}{e^{C_2 T} - I}\sum_{i=1}^{q} e^{C_2(t-\tau_i)} J_i\big(x(\tau_i,x_0,y_0), y(\tau_i,x_0,y_0), v(\tau_i)\big) \quad (6)$$

where $det(C_2)$ and $det(e^{C_2 T} - I)$ are defined and nonzero. The initial condition is $y(0,x_0,y_0) = y_0$ and $w(t) = \big(C_1 + D_1(t)\big)x(t,x_0,y_0) + D_2(t)y(t,x_0,y_0) + g\big(t, x(t,x_0,y_0), y(t,x_0,y_0), v(t)\big)$.
Furthermore, we have

$$\left.\begin{aligned} u(t) &= \int_{-\infty}^{t}\int_a^b K_1(t,s)\psi_1\big(t,s,x(s,x_0,y_0),y(s,x_0,y_0),\rho(s)\big)dtds \\ \rho(s) &= \int_{h_1(s)}^{h_2(s)}\big(x(\tau) - y(\tau)\big)\,d\tau \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} v(t) &= \int_a^b\int_{-\infty}^{t} K_2(t,s)\psi_2\big(t,s,x(s,x_0,y_0),y(s,x_0,y_0),\omega(s)\big)dsdt \\ \omega(s) &= \int_{h_3(s)}^{h_4(s)}\big(x(\tau) - y(\tau)\big)\,d\tau \end{aligned}\right\} \quad (8)$$

Assume that the piecewise continuous vector functions $f\big(t, x(t,x_0,y_0), y(t,x_0,y_0), u(t)\big)$, $g\big(t, x(t,x_0,y_0), y(t,x_0,y_0), v(t)\big)$, $\psi_1\big(t, s, x(t,x_0,y_0), y(t,x_0,y_0), \rho(t)\big)$ and $\psi_2\big(t, s, x(t,x_0,y_0), y(t,x_0,y_0), \omega(t)\big)$ which include the required conditions and inequalities:

$$\|z(t)\| \le \vartheta_1, \quad \|w(t)\| \le \vartheta_2\} \quad (9)$$

$$\big\|I_i\big(x(t,x_0,y_0), y(t,x_0,y_0), u(t)\big)\big\| \le M, \quad \big\|J_i\big(x(t,x_0,y_0), y(t,x_0,y_0), v(t)\big)\big\| \le N\} \quad (10)$$

$$\begin{aligned}&\left\|f\big(t,x_2(t,x_0,y_0),y_2(t,x_0,y_0),u_2(t)\big)-f\big(t,x_1(t,x_0,y_0),y_1(t,x_0,y_0),u_1(t)\big)\right\|\\&\qquad\le\Gamma_1\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|^{\alpha}+\Gamma_2\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\ \|^{\beta}\\&\qquad+\Gamma_3\|u_2(t)-u_1(t)\|^{\gamma}\end{aligned}\tag{11}$$

$$\begin{aligned}&\left\|g\big(t,x_2(t,x_0,y_0),y_2(t,x_0,y_0),v_2(t)\big)-g\big(t,x_1(t,x_0,y_0),y_1(t,x_0,y_0),v_1(t)\big)\right\|\\&\qquad\le\Sigma_1\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|^{\alpha}+\Sigma_2\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\|^{\beta}\\&\qquad+\Sigma_3\|v_2(t)-v_1(t)\|^{\gamma}\end{aligned}\tag{12}$$

$$\begin{aligned}&\left\|I_i\big(x_2(t,x_0,y_0),y_2(t,x_0,y_0),u_2(t)\big)-I_i\big(x_1(t,x_0,y_0),y_1(t,x_0,y_0),u_1(t)\big)\right\|\\&\qquad\le\varpi_1\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|^{\alpha}+\varpi_2\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\ \|^{\beta}\\&\qquad+\varpi_3\|u_2(t)-u_1(t)\|^{\gamma}\end{aligned}\tag{13}$$

$$\begin{aligned}&\left\|J_i\big(x_2(t,x_0,y_0),y_2(t,x_0,y_0),v_2(t)\big)-J_i\big(x_1(t,x_0,y_0),y_1(t,x_0,y_0),v_1(t)\big)\right\|\\&\qquad\le\phi_1\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|^{\alpha}+\phi_2\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\ \|^{\beta}\\&\qquad+\phi_3\|v_2(t)-v_1(t)\|^{\gamma}\end{aligned}\tag{14}$$

$$\begin{aligned}&\big\|\psi_1\big(t,s,x_2(t,x_0,y_0),y_2(t,x_0,y_0),\rho_2(t,x_0,y_0)\big)\\&\qquad-\psi_1\big(t,s,x_1(t,x_0,y_0),y_1(t,x_0,y_0),\rho_1(t,x_0,y_0)\big)\big\|\\&\qquad\le h_1\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|^{\alpha}+h_2\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\ \|^{\beta}\\&\qquad+h_3\|\rho_2(t)-\rho_1\ (t)\|^{\gamma}\end{aligned}\tag{15}$$

$$\begin{aligned}&\big\|\psi_2\big(t,s,x_2(t,x_0,y_0),y_2(t,x_0,y_0),\omega_2(t,x_0,y_0)\big)\\&\qquad-\psi_2\big(t,s,x_1(t,x_0,y_0),y_1(t,x_0,y_0),\omega_1(t,x_0,y_0)\big)\big\|\\&\qquad\le l_1\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|^{\alpha}+l_2\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\ \|^{\beta}\\&\qquad+l_3\|\omega_2(t)-\omega_1(t)\ \|^{\gamma}\end{aligned}\tag{16}$$

for all $x,x_1,x_2\in G_0$, $y,y_1,y_2\in G_1$, $u,u_1,u_2\in G_u$ and $v,v_1,v_2\in G_v$ where ϑ_1,ϑ_2, $\Gamma_1,\Gamma_2,\Gamma_3$, $\Sigma_1,\Sigma_2,\Sigma_3$, h_1,h_2,h_3, l_1,l_2,l_3, $\phi_1,\phi_2,\phi_3,\varpi_1$ and ϖ_2 are positive constants. $t\in[0,T]$ and $0<\alpha,\beta,\gamma<1$.

The isolated singular kernels $K_1(t,s)$ and $K_2(t,s)$ are positive matrices that is:

$$\|K_1(t,s)\|\le\delta_1e^{-\gamma_1(t-s)},\quad\|K_2(t,s)\|\le\delta_2e^{-\gamma_2(t-s)}\}\tag{17}$$

where δ_1, δ_2, γ_1 and γ_2 are positive constants. The matrices $A_1=(A_{1ij})$, $A_2=(A_{2ij})$, $B_1=(B_{1ij})$, $B_2=(B_{2ij})$, $C_1=(C_{1ij})$, $C_2=(C_{2ij})$, $D_1=(D_{1ij})$ and $D_2=(D_{2ij})$ are symmetric where $i,j=1,2,\dots,n$ and $\|.\|=\max_{t\in[0,T]}|\,.\,|$.

The non-empty sets are defined as:-

$$\left.\begin{aligned}G_f&=G_0-r_x=G_0-(\mu_1(t)R_1\vartheta_1+\xi_1(t)R_1Mp)\\G_g&=G_1-r_y=G_1-(\mu_2(t)R_2\vartheta_2+\xi_2(t)R_1Nq)\end{aligned}\right\}\tag{18}$$

where $\mu_1(t)$, $\mu_2(t)$, $\xi_1(t)$ and $\xi_2(t)$ are functions of t and defined in (22) and (23). M, N, p, q, R_1, R_2, ϑ_1 and ϑ_1 are constants. $\sum_{i=1}^{p}\left\|I_i\big(x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big)\right\|=Mp$ and $\sum_{i=1}^{q}\left\|J_i\big(x_m(t,x_0,y_0),y_m(t,x_0,y_0),v_m(t)\big)\right\|=Nq$.

$$\left\|e^{A_1(t-s)}\right\|=\mathrm{R}_1,\quad\left\|e^{C_2(t-s)}\right\|=R_2\}\tag{19}$$

$$H_1(t)=\|h_2(t)-h_1(t)\|,\quad H_2(t)=\|h_4(t)-h_3(t)\ \|\}\tag{20}$$

where $\rho_m(t)=\int_{h_1(s)}^{h_2(s)}\big(x_m(\tau)-y_m(\tau)\big)\,d\tau$ and $\omega_m(t)=\int_{h_3(s)}^{h_4(s)}\big(x_m(\tau)-y_m(\tau)\big)\,d\tau$.

The functions $h_1(t)$, $h_2(t)$, $h_3(t)$ and $h_4(t)$ are limits of integral and defined on G_u and G_v.

$$\left.\begin{aligned}H^*{}_1(t)&=\|B_1(t)\|\|x(t,x_0,y_0)\ \|+\|A_2+B_2(t)\|\|y(t,x_0,y_0)\|+\vartheta_1\\H^*{}_2(t)&=\|C_1+D_1(t)\|\|x(t,x_0,y_0)\ \|+\|D_2(t)\|\|y(t,x_0,y_0)\|+\vartheta_2\end{aligned}\right\}\tag{21}$$

$$\mu_1(t)=\frac{t\left(e^{\|A_1\|T}-2e^{\|A_1\|t}+\|I\|\right)+T\left(e^{\|A_1\|t}-\|I\|\right)}{e^{\|A_1\|T}-\|I\|},\quad \xi_1(t)=\frac{e^{\|A_1\|T}+e^{\|A_1\|t}-2\|I\|}{e^{\|A_1\|T}-\|I\|}\Bigg\} \tag{22}$$

$$\mu_2(t)=\frac{t\left(e^{\|C_2\|T}-2e^{\|C_2\|t}+\|I\|\right)+T\left(e^{\|C_2\|t}-\|I\|\right)}{e^{\|C_2\|T}-\|I\|},\quad \xi_2(t)=\frac{e^{\|C_2\|T}+e^{\|C_2\|t}-2\|I\|}{e^{\|C_2\|T}-\|I\|}\Bigg\} \tag{23}$$

$$\Upsilon_1(t)=\frac{\|A_1\|}{e^{\|A_1\|T}-\|I\|},\quad \Upsilon_2(t)=\frac{\|C_2\|}{e^{\|C_2\|T}-\|I\|}\Bigg\} \tag{24}$$

Let $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ be the sequences from (3)-(6) that are defined as follows:

$$\begin{aligned}x_{m+1}(t,x_0,y_0)&\\ &=x_0e^{A_1t}\\ &+\int_0^t e^{A_1(t-s)}\left(z_m(s)-\Delta_{f,m}\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),u_m(s)\big)\right)ds\\ &+\sum_{0<\tau_i<t} e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big)\end{aligned} \tag{25}$$

$$\begin{aligned}\Delta_{f,m}(t,x_0,y_0)&=A_1x_0+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}z_m(s)\,ds\\ &+\frac{A_1}{e^{A_1T}-I}\sum_{i=1}^{p} e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big),\end{aligned} \tag{26}$$

while $x(0,x_0,y_0)=x_0$ and $m=0\,,1,2,\dots$.

$$\begin{aligned}y_{m+1}(t,x_0,y_0)&\\ &=y_0e^{C_2t}\\ &+\int_0^t e^{C_2(t-s)}\left(w_m(s)-\Delta_{g,m}\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0),v_m(s)\big)\right)ds\\ &+\sum_{0<\tau_i<t} e^{C_2(t-\tau_i)}J_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),v(\tau_i)\big)\end{aligned} \tag{27}$$

$$\begin{aligned}\Delta_{g,m}(t,x_0,y_0)&=C_2y_0+\frac{C_2}{e^{C_2T}-I}\int_0^T e^{C_2(T-s)}w_m(s)\,ds\\ &+\frac{C_2}{e^{C_2T}-I}\sum_{i=1}^{q} e^{C_2(t-\tau_i)}J_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),v(\tau_i)\big).\end{aligned} \tag{28}$$

while $y(0,x_0,y_0)=y_0$ and $m=0,1,2,\dots$.

Consider the case when the maximum Eigenvalue of the matrix $\Phi(T)$ is less than one where

$$\begin{aligned}\Phi(T)&=\begin{bmatrix}\Phi_1(T) & \Phi_2(T)\\ \Phi_3(T) & \Phi_4(T)\end{bmatrix}\\ &=\begin{bmatrix}\mu_1(t)R_1\varphi_1(t)+\xi_1(t)pR_1\chi_1(t) & \mu_1(t)R_1\varphi_2(t)+\xi_1(t)pR_1\chi_2(t)\\ \mu_2(t)R_2\varphi_3(t)+\xi_2(t)qR_1\chi_3(t) & \mu_2(t)R_2\varphi_4(t)+\xi_2(t)qR_1\chi_4(t)\end{bmatrix}.\end{aligned} \tag{29}$$

Thus

$$\max\big(\Phi(T)\big)=\frac{\Phi_1(T)+\Phi_4(T)}{2}+\frac{\sqrt{\big(\Phi_1(T)+\Phi_4(T)\big)^2-4\big(\Phi_1(T)\Phi_4(T)-\Phi_2(T)\Phi_3(T)\big)}}{2}<1\,, \tag{30}$$

where

$$\left.\begin{aligned}
\varphi_1(t) &= \|B_1(t)\| + \Gamma_1 + \Gamma_3\left(\left(h_1 + h_3\left(H_1(t)\right)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^{\gamma}\\
\varphi_2(t) &= \|A_2 + B_2(t)\| + \Gamma_2 + \Gamma_3\left(\left(h_2 + h_3\left(H_1(t)\right)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^{\gamma}\\
\varphi_3(t) &= \|C_1 + D_1(t)\| + \Sigma_1 + \Sigma_3\left(\left(l_1 + l_3\left(H_2(t)\right)^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\\
\varphi_4(t) &= \|D_2(t)\| + \Sigma_2 + \Sigma_3\left(\left(l_2 + l_3\left(H_2(t)\right)^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}
\end{aligned}\right\} \quad (31)$$

$$\left.\begin{aligned}
\chi_1(t) &= \varpi_1 + \varpi_3\left(\left(h_1 + h_3\left(H_1(t)\right)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^{\gamma}\\
\chi_2(t) &= \varpi_2 + \varpi_3\left(\left(h_2 + h_3\left(H_1(t)\right)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}\left(e^{-\gamma_1 a} - e^{-\gamma_1 b}\right)e^{\gamma_1 t}\right)\right)^{\gamma}\\
\chi_3(t) &= \phi_1 + \phi_3\left(\left(l_1 + l_3\left(H_2(t)\right)^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}\\
\chi_4(t) &= \phi_2 + \phi_3\left(\left(l_2 + l_3\left(H_2(t)\right)^{\gamma}\right)\left(\frac{\delta_2(b-a)}{\gamma_2}\right)\right)^{\gamma}
\end{aligned}\right\} \quad (32)$$

Definition 1.1.1. A continuous vector function $f: S \to \mathbb{R}$ that satisfies the Hölder condition of order α where $0 < \alpha < 1$ on $[a, b] \in \mathbb{R}$. Thus for all $K > 0$ and for all $x, y \in [a, b]$, $|f(x) - f(y)| \leq K|x - y|^{\alpha}$ [12].

Definition 1.1.2. Let T_t be a certain set, which is given in the extended phase space M and an operator A_t defined on the set T_t which is mapped into the set $T'_t = A_t T_t$ in the extended phase space. Then from the system of differential equations

$\frac{dx}{dt} = f(t, x), \quad x \in R^n, t \in \mathbb{R}$

the impulsive differential system is

$$\frac{dx}{dt} = f(t, x), \qquad (t, x) \notin T_t$$
$$\Delta x|_{(t,x)\in T_t} = T_t x - x$$

where the solutions of the impulsive differential system are functions that satisfy the above system of differential equations [13].

Lemma 1.1.1. Let the sequence $\alpha_i \in R$ and $r \in (0, \infty)$, then we have the following:

1. If $\alpha_i \geq 0$ and $r \geq 1$, then $\sum_{i=1}^{m} \alpha_i^r \leq \left(\sum_{i=1}^{m} \alpha_i\right)^r \leq m^{r-1} \sum_{i=1}^{m} \alpha_i^r$, for $1 \leq i \leq m$. The reverse holds if $0 < r < 1$. Hence for $1 \leq i \leq m$ we obtain that $\left(\sum_{i=1}^{m} \alpha_i\right)^r \leq \sum_{i=1}^{m} \alpha_i^r$.
2. If $\alpha_i, \beta_i \in R$ and $0 < r \leq 1$. Then, we obtain that $\|\alpha_i - \beta_i\|^r \leq \|\alpha_i - \beta_i\|$ for $1 \leq i \leq m$.

Lemma 1.1.2. Assume the continuous vector function $f\left(t, x(t, x_0, y_0), y(t, x_0, y_0)\right)$ on [0, T] defined. Then $\left\|\int_0^t \left(f\left(s, x(s, x_0, y_0), y(s, x_0, y_0)\right) - \Delta(T)\right) ds\right\| \leq \alpha(t) M$ where $\alpha(t) = 2t\left(1 - \frac{t}{T}\right)$, $M = \max_{t\in[0,T]}\left\|f\left(t, x(t, x_0, y_0), y(t, x_0, y_0)\right)\right\|$, $\forall \; t \in [0, T]$ and $\Delta(T) = \frac{1}{T}\int_0^T f\left(\tau, x(\tau, x_0, y_0), y(\tau, x_0, y_0)\right) d\tau$ [14].

Proof. Since

$$\int_0^t \Big(f\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)-\Delta(T)\Big)\,ds$$
$$=\int_0^t \Big(f\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)-\tfrac{1}{T}\int_0^T f\big(\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)\big)\,d\tau\Big)\,ds$$
$$=\int_0^t f\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)\,ds-\tfrac{t}{T}\int_0^T f\big(\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)\big)\,d\tau$$
$$=\Big(1-\tfrac{t}{T}\Big)\int_0^t f\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)\,ds-\tfrac{t}{T}\int_t^T f\big(\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)\big)\,d\tau.$$

Then, we have

$$\Big\|\int_0^t \Big(f\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)-\Delta(T)\Big)\,ds\Big\|$$
$$\le\Big(1-\tfrac{t}{T}\Big)\int_0^t\big\|f\big(s,x(s,x_0,y_0),y(s,x_0,y_0)\big)\big\|\,ds+\tfrac{t}{T}\int_t^T\big\|f\big(\tau,x(\tau,x_0,y_0),y(\tau,x_0,y_0)\big)\big\|\,d\tau$$
$$\le\Big(1-\frac{t}{T}\Big)t\max_{t\in[0,T]}\big\|f\big(t,x(t,x_0,y_0),y(t,x_0,y_0)\big)\big\|+\frac{t}{T}(T-t)\max_{t\in[0,T]}\big\|f\big(t,x(t,x_0,y_0),y(t,x_0,y_0)\big)\big\|$$
$$\le\Big(1-\frac{t}{T}\Big)t\max_{t\in[0,T]}\big\|f\big(t,x(t,x_0,y_0),y(t,x_0,y_0)\big)\big\|+t\Big(1-\frac{t}{T}\Big)\max_{t\in[0,T]}\big\|f\big(t,x(t,x_0,y_0),y(t,x_0,y_0)\big)\big\|$$
$$\le\Big(1-\tfrac{t}{T}\Big)tM+t\Big(1-\tfrac{t}{T}\Big)M=2t\Big(1-\tfrac{t}{T}\Big)M=\alpha(t)M.$$

Lemma 1.1.3. The following inequalities hold according to lemma (2.1) and equations (7), (8) with condition (20).

1. $\|\rho_m(t)-\rho_{m-1}(t)\|^\gamma\le\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^\gamma\big(H_1(t)\big)^\gamma+\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^\gamma\big(H_1(t)\big)^\gamma$
2. $\|\omega_m(\mathrm{t})-\omega_{m-1}(\mathrm{t})\|^\gamma\le\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^\gamma\big(H_2(t)\big)^\gamma+\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^\gamma\big(H_2(t)\big)^\gamma$.

Proof. From equation (7) we subtract the sequences as

$$\rho_m(t)-\rho_{m-1}(t)=\int_{h_1(s)}^{h_2(s)}\big(x_m(\tau)-y_m(\tau)\big)\,d\tau-\int_{h_1(s)}^{h_2(s)}\big(x_{m-1}(\tau)-y_{m-1}(\tau)\big)\,d\tau$$
$$\le\int_{h_1(s)}^{h_2(s)}\big(x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\big)\,d\tau+\int_{h_1(s)}^{h_2(s)}\big(y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\big)\,d\tau.$$

Thus, we obtain the norm and from equation (20) as follows:

$$\|\rho_m(t)-\rho_{m-1}(t)\|\le\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|H_1(t)+\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|H_1(t)$$

From lemma 1.1.1 and by Holder condition we have

$$\|\rho_m(t)-\rho_{m-1}(t)\|^\gamma\le\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^\gamma\big(H_1(t)\big)^\gamma+\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^\gamma\big(H_1(t)\big)^\gamma.$$

Also from the equation (8) and the equation (20) we obtain the second item.

Theorem 1.1.1. Let $\psi_1\big(t,s,x(t,x_0,y_0),y(t,x_0,y_0),w(t)\big)$, $\psi_2\big(t,s,x(t,x_0,y_0),y(t,x_0,y_0),v(t)\big)$, $u(t)$ *and* $v(t)$ be vector functions that are defined and continuous in (2) and fulfil the inequalities and conditions (15), (16), (17) and (20). Then the following inequalities hold.

1. $\|u_m(t)-u_{m-1}(t)\|^{\gamma} \le$
$\left(\left(h_1+h_3\big(H_1(t)\big)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^{\gamma}+$
$\left(\left(h_2+h_3\big(H_1(t)\big)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^{\gamma},$
2. $\|v_m(t)-v_{m-1}(t)\|^{\gamma} \le$
$\left(\left(l_1+l_3\big(H_2(t)\big)^{\gamma}\right)\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^{\gamma}+$
$\left(\left(l_2+l_3\big(H_2(t)\big)^{\gamma}\right)\frac{\delta_2(b-a)}{\gamma_2}\right)^{\gamma}\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^{\gamma}$
for all $t\in[0,T]$ and $m=1,2,3,\dots$.

Proof.

From equality (7), the sequence $u_m(t)$ belongs to G_u where $m=0,1,2,\dots,$, then $u_m(t)=\int_{-\infty}^{t}\int_a^b K_1(t,s)\psi_1\big(t,s,x_m(s),y_m(s),\rho_m(t)\big)\,dt\,ds$.
Take the norm of subtract as follows: -

$$\|u_m(t)-u_{m-1}(t)\| \le \int_{-\infty}^{t}\int_a^b \|K_1(t,s)\|\big\|\psi_1\big(t,s,x_m(t),y_m(t),\rho_m(t)\big) - \psi_1\big(t,s,x_{m-1}(t),y_{m-1}(t),\rho_{m-1}(t)\big)\big\|\,dt\,ds$$

From inequalities (15) and (17) we simplify as follows:-

$$\|u_m(t)-u_{m-1}(t)\| \le \left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\big(h_1\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^{\alpha} + h_2\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^{\beta}+h_3\|\rho_m(t)-\rho_{m-1}(t)\|^{\gamma}\big).$$

Thus, from previous lemma 1.1.3 and from lemma 1.1.1, we obtain that

$$\begin{aligned}\|\mathrm{u_m(t)-u_{m-1}}\| &\le \left(h_1+h_3\big(H_1(t)\big)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| \\ &+ \left(h_2+h_3\big(H_1(t)\big)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\|.\end{aligned}$$

With the power of γ we achieve the following form
$\|u_m(t)-u_{m-1}(t)\|^{\gamma} \le$
$\left(\left(h_1+h_3\big(H_1(t)\big)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|^{\gamma}+$
$\left(\left(h_2+h_3\big(H_1(t)\big)^{\gamma}\right)\left(\frac{\delta_1}{(\gamma_1)^2}(e^{-\gamma_1 a}-e^{-\gamma_1 b})e^{\gamma_1 t}\right)\right)^{\gamma}\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|^{\gamma}.$

From equation (8) with applying the same previous steps, the second part of the theorem is achieved.

Lemma 1.1.4. Let $f\big(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big)$ and $g\big(t,x(t,x_0,y_0),y(t,x_0,y_0),v(t)\big)$ be piecewise continuous vector functions with respect to t on [0, T] with period T. Then the periodic solutions with impulsive actions of the system (1) are given in equations (3) and (5).

Proof. From system (1) and by assumption $x(t,x_0,y_0)=u(t,x_0,y_0)e^{A_1t}$ we get the differential equation which has the form: -

$$\frac{du(t,x_0,y_0)}{dt}=B_1(t)u(t,x_0,y_0)$$
$$+\big(A_2+B_2(t)\big)v(t,x_0,y_0)e^{C_2t}e^{-A_1t}+e^{-A_1t}f(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)).$$

Take the integral of both sides and put in $x(t,x_0,y_0)=u(t,x_0,y_0)e^{A_1t}$ to obtain the form

$$x(t,x_0,y_0)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}z(s)\,ds.$$

which is the solution of $\frac{dx(t,x_0,y_0)}{dt}$ without impulsive actions in system (1). So that for $t\in[\tau_i,\tau_{i+1}]$,

$$x(t,x_0,y_0)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}z(s)\,ds+e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big).$$

Afterwards for all $t\in[0,T]$, we obtain the solution with impulsive actions as:-

$$x(t,x_0,y_0)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}z(s)\,ds$$
$$+\sum_{0<\tau_i<t}e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big).$$

To find the periodic solution with impulsive actions we subtract the Δ-constant to the function $f\big(t,x(t),y(t),u(t)\big)$ inside $z_1(t)$ as follows to obtain equation (3) as:

$$x(t)=x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\Big(z(s)-\Delta_f\big(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big)\Big)\,ds$$
$$+\sum_{0<\tau_i<t}e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big).$$

Since from initial condition $x(0)=x(T)=x_0$, where $t\in[0,T]$, we obtain that

$$x(T)=x_0e^{A_1T}+\int_0^T e^{A_1(T-s)}z(s)\,ds-\Delta_f\big(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big)\int_0^T e^{A_1(T-s)}\,ds$$
$$+\sum_{0<\tau_i<t}e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big).$$

After simplifying we achieve the equation (4)

$$\Delta_f\big(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big)=A_1x_0+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}z(s)\,ds$$
$$+\frac{A_1}{e^{A_1T}-I}\sum_{i=1}^{p}e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big).$$

Therefore the equations (3)-(4) are proved from $\frac{dx(t,x_0,y_0)}{dt}$ with impulsive actions. By following the same steps where $y(t,x_0,y_0)=v(t,x_0,y_0)e^{C_2t}$ we receive the equation (5)-(6).

Lemma 1.1.5. Suppose that the vector functions$f\big(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t)\big)$ and $g\big(t,x(t,x_0,y_0),y(t,x_0,y_0),v(t)\big)$ are defined and continuous on [0, T] in t of period T. Then, the vector function

$$\begin{pmatrix}\|E_1(t,x_0,y_0)\|\\ \|E_2(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\mu_1(t)R_1\vartheta_1 + \xi_1(t)R_1Mp\\ \mu_2(t)R_2\vartheta_2 + \xi_2(t)R_2Nq\end{pmatrix} \tag{33}$$

holds, where the equations (3) and (5) have been derived to receive the following

$$E_1(t,x_0,y_0) = \int_0^t e^{A_1(t-s)}z(s)\,ds - \frac{e^{A_1t}-I}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}z(s)\,ds$$
$$-\frac{e^{A_1t}-I}{e^{A_1T}-I}\sum_{i=1}^{p} e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big)$$
$$+\sum_{0<\tau_i<t} e^{A_1(t-\tau_i)}I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big),$$
$$E_2(t,x_0,y_0) = \int_0^t e^{C_2(t-s)}w(s)\,ds - \frac{e^{C_2t}-I}{e^{C_2T}-I}\int_0^T e^{C_2(T-s)}w(s)\,ds$$
$$-\frac{e^{C_2t}-I}{e^{C_2T}-I}\sum_{i=1}^{q} e^{C_2(t-\tau_i)}J_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),v(\tau_i)\big)$$
$$+\sum_{0<\tau_i<t} e^{C_2(t-\tau_i)}J_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),v(\tau_i)\big)$$

for $0 \le t \le T$.

Proof. According to the equations (3), (4) and (7) with the inequalities and conditions (9), (10) and (19) we have

$\|E_1(t,x_0,y_0)\| \le$
$\left(\|I\| - \frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\right)\int_0^t e^{\|A_1\|(t-s)}\|z(t)\|\,ds + \frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\int_t^T e^{\|A_1\|(T-s)}\|z(t)\|\,ds +$
$\frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\sum_{i=1}^{p} e^{\|A_1\|\|t-\tau_i\|}\left\|I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big)\right\| +$
$\sum_{0<\tau_i<t}\left\|I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big)\right\|$

$$\le \frac{t\big(e^{\|A_1\|T}-2e^{\|A_1\|t}+\|I\|\big)+T\big(e^{\|A_1\|t}-\|I\|\big)}{e^{\|A_1\|T}-\|I\|}R_1\vartheta_1 + \frac{e^{\|A_1\|T}+e^{\|A_1\|t}-2\|I\|}{e^{\|A_1\|T}-\|I\|}R_1Mp.$$

Thus from (22) we get $\|E_1(t,x_0,y_0)\| \le R_1\mu_1(t)\vartheta_1 + R_1Mp\xi_1(t)$ and from the equations (5), (6) and (8) with the assumptions (9), (10) and (19) we obtain that $\|E_2(t,x_0,y_0)\| \le R_2\mu_2(t)\vartheta_2 + R_2Nq\xi_2(t)$. Thus, the equation (33) is achieved.

2. Approximate of periodic solutions of system (1).

The approximate periodic solution of the system (1) is suggested by the following theorem.

Theorem 2.1. Consider that the vector functions $f(t,x(t,x_0,y_0),y(t,x_0,y_0),u(t))$ and $g(t,x(t,x_0,y_0),y(t,x_0,y_0),v(t))$ are defined and continuous on (2) and fulfilled all inequalities (11)-(17), conditions (19)-(23) and (19)-(23). Then in t of period T, the sequences (23) and (25) converge uniformly to the limit functions $x^o(t,x_0,y_0)$ and $y^o(t,x_0,y_0)$ as $m \to \infty$. Then the functions (3) and (5) are unique solutions of system (1).

$$\begin{pmatrix}\|x^o(t,x_0,y_0)-x_0(t,x_0,y_0)\|\\ \|y^o(t,x_0,y_0)-y_0(t,x_0,y_0)\|\end{pmatrix} \le \Omega(T) \tag{34}$$

$$\begin{pmatrix}\|x^o(t,x_0,y_0)-x_m(t,x_0,y_0)\|\\ \|y^o(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le \Phi^m(T)\big(I-\Phi(T)\big)^{-1}\Omega(T) \tag{35}$$

where $\Omega(T) = \begin{pmatrix}\mu_1(T)R_1\vartheta_1 + \xi_1(T)R_1Mp\\ \mu_2(T)R_2\vartheta_2 + \xi_2(T)R_2Nq\end{pmatrix}$ and I is an identity matrix.

Proof. The sequences of functions $\{x_m(t,x_0,y_0)\}_{m=1}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=1}^{\infty}$ from (25)-(28) are defined and continuous on (2) that are periodic in t of period T. First of all, from the equation (25) and by induction with the assumptions (9) and (10) where $m=0$ we have

$$
\begin{aligned}
&\|x_1(t,x_0,y_0)-x_0\| = \\
&\left\|x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\left(z_0-\Delta_{f,0}(s,x_0,y_0,u_0)\right)ds+\sum\nolimits_{0<\tau_i<t} I_i(x_0,y_0,u_0)-x_0\right\| \\
&\le \left(\|I\|-\frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\right)\int_0^t e^{\|A_1\|(t-s)}\|z_0\|\,ds+\frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\int_t^T e^{\|A_1\|(T-s)}\|z_0\|\,ds \\
&\qquad +\left(\|I\|+\frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\right)\sum_{i=1}^{p} e^{\|A_1\|\|t-\tau_i\|}\|I_i(x_0,y_0,u_0)\| \\
&\le \frac{t\left(e^{\|A_1\|T}-2e^{\|A_1\|t}+\|I\|\right)+T\left(e^{\|A_1\|t}-\|I\|\right)}{e^{\|A_1\|T}-\|I\|}R_1\vartheta_1+\frac{e^{\|A_1\|T}+e^{\|A_1\|t}-2\|I\|}{e^{\|A_1\|T}-\|I\|}R_1Mp.
\end{aligned}
$$

Thus, for $m\ge 1$ and by (22) we obtain that $\|x_m(t,x_0,y_0)-x_0\|\le \mu_1(t)R_1\vartheta_1+\xi_1(t)R_1Mp$. Also, from the equation (27) and according to the same steps with (23) we obtain that $\|y_m(t,x_0,y_0)-y_0\|\le \mu_2(t)R_2\vartheta_2+\xi_2(t)R_2Nq$. Thus, for all $t\in[0,T]$ where $x_0\in G_f$ and $y_0\in G_g$ we get that $x_m(t,x_0,y_0)\in G_0$ and $y_m(t,x_0,y_0)\in G_1$.

Furthermore, we must to show that the sequences $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ are uniformly convergent on (2). Then from (25) when $m=1$ we have

$$
\begin{aligned}
&\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\| \\
&\le \frac{e^{\|A_1\|T}-e^{\|A_1\|t}}{e^{\|A_1\|T}-\|I\|}\int_0^t \left\|e^{A_1(t-s)}\right\|\|z_1(t)-z_0\|\,ds \\
&+\frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\int_t^T \left\|e^{A_1(t-s)}\right\|\|z_1(t)-z_0\|\,ds \\
&+\frac{e^{\|A_1\|t}-\|I\|}{e^{\|A_1\|T}-\|I\|}\sum_{i=1}^{p} e^{\|A_1\|\|t-\tau_i\|}\left\|I_i\left(x_1(\tau_i,x_0,y_0),y_1(\tau_i,x_0,y_0),u_1(\tau_i)\right)\right. \\
&\left.-I_i(x_0,y_0,u_0)\right\| \\
&+\sum_{0<\tau_i<t} e^{\|A_1\|\|t-\tau_i\|}\left\|I_i\left(x_1(\tau_i,x_0,y_0),y_1(\tau_i,x_0,y_0),u_1(\tau_i)\right)-I_i(x_0,y_0,u_0)\right\|.
\end{aligned}
$$

Thus from (22) we receive that

$$
\begin{aligned}
&\le \left(\frac{\left(e^{\|A_1\|T}-2e^{\|A_1\|t}-\|I\|\right)t+\left(e^{\|A_1\|t}-\|I\|\right)T}{e^{\|A_1\|T}-\|I\|}\right)R_1\|z_1(t)-z_0\| \\
&+\frac{e^{\|A_1\|T}+e^{\|A_1\|t}-2\|I\|}{e^{\|A_1\|T}-\|I\|}pR_1\left\|I_i\left(x_1(\tau_i,x_0,y_0),y_1(\tau_i,x_0,y_0),u_1(\tau_i)\right)\right. \\
&\left.-I_i(x_0,y_0,u_0)\right\|.
\end{aligned}
$$

And from (22), (29), (31) and (32) we obtain that

$$
\begin{aligned}
&\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\| \\
&\le \left(\mu_1(t)R_1\varphi_1(t)+\xi_1(t)pR_1\chi_1(t)\right)\|x_1(t,x_0,y_0)-x_0\| \\
&+\left(\mu_1(t)R_1\varphi_2(t)+\xi_1(t)pR_1\chi_2(t)\right)\|y_1(t,x_0,y_0)-y_0\|
\end{aligned}
$$

where, there exists $\|z_1(t)-z_0\|\le\varphi_1(t)\|x_1(t,x_0,y_0)-x_0\|+\varphi_2(t)\|y_1(t,x_0,y_0)-y_0\|$ and $\|I_i(x_1,y_1,u_1)-I_i(x_0,y_0,u_0)\|\le\chi_1\|x_1(t,x_0,y_0)-x_0\|+\chi_2\|y_1(t,x_0,y_0)-y_0\|$.

From (27) and by the same iterations from inequalities and conditions (11)-(17), (19), (23), (31) and (32) we have

$$\begin{aligned}\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\| &\le \big(\mu_2(t)R_2\varphi_3(t)+\xi_2(t)qR_2\chi_3(t)\big)\|x_1(t,x_0,y_0)-x_0\| \\ &+ \big(\mu_2(t)R_2\varphi_4(t)+\xi_2(t)qR_2\chi_4(t)\big)\|y_1(t,x_0,y_0)-y_0\|.\end{aligned}$$

where there exists $\|w_1(t)-w_0\| \le \varphi_3\|x_1(t,x_0,y_0)-x_0\|+\varphi_4\|y_1(t,x_0,y_0)-y_0\|$ and

$$\begin{aligned}\big\|J_i\big(x_1(t,x_0,y_0),y_1(t,x_0,y_0),v_1(t)\big)-J_i(x_0,y_0,v_0)\big\| &\le \chi_3\|x_1(t,x_0,y_0)-x_0\|+\chi_4\|y_1(t,x_0,y_0)-y_0\|.\end{aligned}$$

Since for $m>1$ and by induction, from (29) we demonstrate a vector form:

$$\begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le$$
$$\begin{pmatrix}\Phi_1(T) & \Phi_2(T) \\ \Phi_3(T) & \Phi_4(T)\end{pmatrix}\begin{pmatrix}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix}.$$

By iterations and from (29) with the maximal value of t we receive the form

$$\begin{pmatrix}\|x_{m+1}(T,x_0,y_0)-x_m(T,x_0,y_0)\| \\ \|y_{m+1}(T,x_0,y_0)-y_m(T,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\Phi_1(T) & \Phi_2(T) \\ \Phi_3(T) & \Phi_4(T)\end{pmatrix}^m \begin{pmatrix}\mu_1R_1\vartheta_1+\xi_1R_2Mp \\ \mu_2R_2\vartheta_2+\xi_2R_2Nq\end{pmatrix}. \qquad (36)$$

For any $k\ge 0$ we conclude that

$$\begin{aligned}\begin{pmatrix}\|x_{m+k}(T,x_0,y_0)-x_m(T,x_0,y_0)\| \\ \|y_{m+k}(T,x_0,y_0)-y_m(T,x_0,y_0)\|\end{pmatrix} &\le \begin{pmatrix}\|x_{m+k}(T,x_0,y_0)-x_{m+k-1}(T,x_0,y_0)\| \\ \|y_{m+k}(T,x_0,y_0)-y_{m+k-1}(T,x_0,y_0)\|\end{pmatrix} \\ &+\cdots+\begin{pmatrix}\|x_{m+1}(T,x_0,y_0)-x_m(T,x_0,y_0)\| \\ \|y_{m+1}(T,x_0,y_0)-y_m(T,x_0,y_0)\|\end{pmatrix}.\end{aligned}$$

Rewrite the vector structure as follows:

$$\Omega_{m+k}(\mathrm{T}) \le (I-\Phi(T))^{-1}\Phi^m(T)\Omega(T).$$

As a result of condition (30) we received that $\lim_{n\to\infty}\Phi^m(t)=0$. So that the sequences of functions $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ converge uniformly on (18).
Consider the limits $\lim_{m\to\infty} x_m(t,x_0,y_0)=x(t,x_0,y_0)$ and $\lim_{m\to\infty} y_m(t,x_0,y_0)=y(t,x_0,y_0)$. Thus, we investigate that inequalities (34) and (35) hold for all $m\ge 0$, since $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ are periodic solutions of period T in t.

3. Uniqueness of periodic solutions of system (1).

The following theorem states that system (1) has a unique solution.

Theorem 3.1. The periodic solutions of problem (1) are unique on (18) of period T in t under all conditions and assumptions of theorem 2.1.

Proof. Let $k(t,x_0,y_0)$ and $r(t,x_0,y_0)$ be another periodic solutions of system (1), that is

$$\begin{aligned}r(t,x_0,y_0) &= x_0e^{A_1t}+\int_0^t e^{A_1(t-s)}\Big(z_r(s)-\Delta_f\big(s,r(s,x_0,y_0),k(s,x_0,y_0),u_r(s)\big)\Big)ds \\ &+\sum_{0<\tau_i<t} e^{A_1(t-\tau_i)}I_i\big(r(\tau_i,x_0,y_0),k(\tau_i,x_0,y_0),u_r(\tau_i)\big) \qquad (37)\end{aligned}$$

with $r(0,x_0,y_0)=x_0$ and

$$\begin{aligned}k(t,x_0,y_0) &= y_0e^{C_2t}+\int_0^t e^{C_2(t-s)}\Big(w_r(s)-\Delta_g\big(s,r(s,x_0,y_0),k(s,x_0,y_0),v_k(s)\big)\Big)ds \\ &+\sum_{0<\tau_i<t} e^{C_2(t-\tau_i)}J_i\big(r(\tau_i,x_0,y_0),k(\tau_i,x_0,y_0),v_r(\tau_i)\big) \qquad (38)\end{aligned}$$

with $k(0,x_0,y_0)=y_0$ and $m=0,1,2,\dots$. Thus, we have

$$\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \le \frac{\left(e^{\|A_1\|T}-2e^{\|A_1\|t}+\|I\|\right)t+\left(e^{\|A_1\|t}-\|I\|\right)T}{e^{\|A_1\|T}-\|I\|}R_1\|z(t)-z_r(t)\| + \frac{e^{\|A_1\|T}+e^{\|A_1\|t}-2\|I\|}{e^{\|A_1\|T}-\|I\|}R_1 p\left\|I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big) - I_i\big(r(\tau_i,x_0,y_0),k(\tau_i,x_0,y_0),u_r(\tau_i)\big)\right\|$$

From the inequalities and conditions (12), (14), (16), (17), (19), (23), (31) and (32) we obtain that

$$\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \le \big(\mu_1(t)R_1\varphi_1(t)+\xi_1(t)pR_1\chi_1(t)\big)\|x(t,x_0,y_0)-r(t,x_0,y_0)\| + \big(\mu_1(t)R_1\varphi_2(t)+\xi_1(t)pR_1\chi_2(t)\big)\|y(t,x_0,y_0)-k(t,x_0,y_0)\| \quad (39)$$

To achieve the norm bellow, the same procedures are followed. Thus

$$\|y(t,x_0,y_0)-k(t,x_0,y_0)\| \le \big(\mu_2(t)R_2\varphi_3(t)+\xi_2(t)qR_2\chi_3(t)\big)\|x(t,x_0,y_0)-r(t,x_0,y_0)\| + \big(\mu_2(t)R_2\varphi_4(t)+\xi_2(t)qR_2\chi_4(t)\big)\|y(t,x_0,y_0)-k(t,x_0,y_0)\| \quad (40)$$

We receive a vector form, from (39) and (40) as follows:

$$\begin{bmatrix}\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-k(t,x_0,y_0)\|\end{bmatrix} \le \begin{bmatrix}\Phi_1(T) & \Phi_2(T) \\ \Phi_3(T) & \Phi_4(T)\end{bmatrix}\begin{bmatrix}\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-k(t,x_0,y_0)\|\end{bmatrix}.$$

Hence from condition (30), the greatest Eigen-value of $\Phi(T)$'s matrix is less than one, thus we deduce that $x(t,x_0,y_0)=r(t,x_0,y_0)$ and $y(t,x_0,y_0)=k(t,x_0,y_0)$. This implies that the system (1) has a unique solution.

4. Existence of periodic solutions of system (1).

The existence solutions of the system (1) in t of period T are uniquely linked with the existence of zeroes of functions $\Delta_f(t,x_0,y_0)\in G_{f1}\times G_{g1}\to R^1$ and $\Delta_g(t,x_0,y_0)\in G_{f2}\times G_{g2}\to R^1$ that are defined by (4) and (6) respectively. Therefore, the approximate solutions (4) and (6) provide the function sequences (26) and (28) respectively.

Theorem 4.1. Assuming that all of the conditions of theorem (2.1) are satisfied. Thus the following vector inequality holds.

$$\begin{bmatrix}\left\|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\right\| \\ \left\|\Delta_g(t,x_0,y_0)-\Delta_{g,m}(t,x_0,y_0)\right\|\end{bmatrix} \le \Upsilon(T)\Phi^{m+1}(T)\big(I-\Phi(T)\big)^{-1}\Omega(T), \quad (41)$$

where, $\Upsilon(T)=\big(\Upsilon_1(T),\Upsilon_2(T)\big)$, for all $m\ge 0$.

Proof. From the equations (4) and (26) we have

$$\left\|\Delta_f(t,x_0,y_0)-\Delta_{f,m}(t,x_0,y_0)\right\| = \left\|\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\big(z(s)-z_m(s)\big)\,ds + \frac{A_1}{e^{A_1T}-I}\sum_{i=1}^{p} e^{A_1(t-\tau_i)}\Big(I_i\big(x(\tau_i,x_0,y_0),y(\tau_i,x_0,y_0),u(\tau_i)\big) - I_i\big(x_m(\tau_i,x_0,y_0),y_m(\tau_i,x_0,y_0),u_m(\tau_i)\big)\Big)\right\|$$

$$\leq \frac{\|A_1\|}{e^{\|A_1\|T} - \|I\|} R_1 T \|z(t) - z_m(t)\| + \frac{\|A_1\|}{e^{\|A_1\|T} - \|I\|} \sum_{i=1}^{p} e^{\|A_1\|\|t-\tau_i\|} \big\| I_i\big(x(\tau_i, x_0, y_0), y(\tau_i, x_0, y_0), u(\tau_i)\big) - I_i\big(x_m(\tau_i, x_0, y_0), y_m(\tau_i, x_0, y_0), u_m(\tau_i)\big)\big\|$$

Then we get

$$\|\Delta_f(t, x_0, y_0) - \Delta_{f,m}(t, x_0, y_0)\| \leq (\Upsilon_1(t)\varphi_1(t)R_1T + \Upsilon_1(t)\chi_1(t)R_1p)\|x(t, x_0, y_0) - x_m(t, x_0, y_0)\| + (\Upsilon_1(t)\varphi_2(t)R_1T + \Upsilon_1(t)\chi_2(t)R_1p)\|y(t, x_0, y_0) - y_m(t, x_0, y_0)\| \quad (42)$$

Obtaining the equation (43) under the same steps and by the inequality and constraints (6) and (28) will achieve.

$$\|\Delta_g(t, x_0, y_0) - \Delta_{g,m}(t, x_0, y_0)\| \leq (\Upsilon_2(t)\varphi_3(t)R_2T + \Upsilon_2(t)\chi_3(t)R_2q)\|x(t, x_0, y_0) - x_m(t, x_0, y_0)\| + (\Upsilon_2(t)\varphi_4(t)R_2T + \Upsilon_2(t)\chi_4(t)R_2q)\|y(t, x_0, y_0) - y_m(t, x_0, y_0)\| \quad (43)$$

Rewrite (42) and (43) in a vector form as

$$\begin{bmatrix} \|\Delta_f(t, x_0, y_0) - \Delta_{f,m}(t, x_0, y_0)\| \\ \|\Delta_g(t, x_0, y_0) - \Delta_{g,m}(t, x_0, y_0)\| \end{bmatrix} \leq \begin{bmatrix} \Upsilon_1(t)\Phi_1(t) & \Upsilon_1(t)\Phi_2(t) \\ \Upsilon_2(t)\Phi_3(t) & \Upsilon_2(t)\Phi_4(t) \end{bmatrix} \begin{bmatrix} \|x(t, x_0, y_0) - x_m(t, x_0, y_0)\| \\ \|y(t, x_0, y_0) - y_m(t, x_0, y_0)\| \end{bmatrix}.$$

Therefore from (29) it achieved the equation (41). From the periodic functions $\Delta_f(t, x_0, y_0)$ and $\Delta_g(t, x_0, y_0)$ there exist isolated singular points such that $\Delta_f(t, x_0, y_0) = 0$ and $\Delta_g(t, x_0, y_0) = 0$. Therefore, the system (1) has periodic solutions $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$.

Theorem 4.2. Let the following conditions hold in $G_f \in E_n$ and $G_g \in E_m$ from system (1).

1. The operators $\Delta_{f,m}(t, x_0, y_0)$ and $\Delta_{g,m}(t, x_0, y_0)$ on (26) and (28) have an isolated singular point $\Delta_{f,m}(0, x^o, y^o) = 0$ and $\Delta_{g,m}(0, x^o, y^o) = 0$ for some real τ and the integer m.
2. These isolated singular points have nonzero indexes.
3. There exist closed convex domains $G_0 \in G_f$ and $G_1 \in G_g$ that contain singular points x^o and y^o, thus from the boundaries of G_0 and G_1 which denoted by Γ_{G_0} and Γ_{G_1} respectively. The following vector inequality holds:

$$\begin{bmatrix} \inf_{x \in \Gamma_{G_0}} \|\Delta_{f,m}(t, x_0, y_0)\| \\ \inf_{y \in \Gamma_{G_1}} \|\Delta_{g,m}(t, x_0, y_0)\| \end{bmatrix} \geq \Upsilon(T) \left\| \Phi^{m+1}(T)\big(I - \Phi(T)\big)^{-1} \Omega(T) \right\| \quad (44)$$

for $m \geq 1$.

Then system (1) has periodic solutions $x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$ on G_0 and G_1 respectively.

Proof. The indexes of the isolated singular points x^o and y^o of the operators $\Delta_{f,m}(t, x_0, y_0)$ and $\Delta_{g,m}(t, x_0, y_0)$ are equal to the characteristic of the vector fields corresponding to the operators $\Delta_{f,m}(t, x_0, y_0)$ and $\Delta_{g,m}(t, x_0, y_0)$ on an enough smaller spheres S^n and S^m ,respectively with the centres at x^o and y^o. Since G_0 and G_1 have no singular points different from x^o and y^o and are homoeomorphic to the unit spheres E_n and E_m, then the characteristic of two vector fields which contain the maps $\Delta_{f,m}(t,x_0,y_0)$ and $\Delta_{g,m}(t,x_0,y_0)$ are equal to that of the same fields on Γ_{G_0} and Γ_{G_1}.

Next we have to show that the fields of operators $\Delta_{f,m}(t,x_0,y_0)$ and $\Delta_f(t,x_0,y_0)$ on Γ_{G_0}, also the fields of $\Delta_{g,m}(t,x_0,y_0)$ and $\Delta_g(t,x_0,y_0)$ on Γ_{G_1} are homotopic. Then for $m \geq 1$ and the parameter $0 \leq \theta \leq 1$ we follow the fact that the family of the vector fields

$$\left.\begin{aligned} V_f(\theta,x_0,y_0) &= \Delta_{f,m}(t,x_0,y_0) + \theta\left(\Delta_f(t,x_0,y_0) - \Delta_{f,m}(t,x_0,y_0)\right) \\ V_g(\theta,x_0,y_0) &= \Delta_{g,m}(t,x_0,y_0) + \theta\left(\Delta_g(t,x_0,y_0) - \Delta_{g,m}(t,x_0,y_0)\right) \end{aligned}\right\} \tag{45}$$

are continuous everywhere on Γ_{G_0} and Γ_{G_1} respectively. Since from the parameter θ and the equation (45) we have $V_f(0,x_0,y_0) = \Delta_{f,m}(t,x_0,y_0)$ and $V_f(1,x_0,y_0) = \Delta_f(t,x_0,y_0)$. Also from equation (45) we get $V_g(0,x_0,y_0) = \Delta_{g,m}(t,x_0,y_0)$ and $V_g(1,x_0,y_0) = \Delta_g(t,x_0,y_0)$, that does not disappear anywhere on the boundaries Γ_{G_0} and Γ_{G_1}, then from the equations (4), (6), (26) and (28) and by induction we receive the vector form as follows

$$\begin{bmatrix} \left\|\Delta_f(t,x_0,y_0) - \Delta_{f,m}(t,x_0,y_0)\right\| \\ \left\|\Delta_g(t,x_0,y_0) - \Delta_{g,m}(t,x_0,y_0)\right\| \end{bmatrix} = \Upsilon(T)\left\|\Phi^{m+1}(T)\big(I-\Phi(T)\big)^{-1}\Omega(T)\right\|.$$

This implies that the infimum value of the operators $\Delta_{f,m}(t,x_0,y_0)$ and $\Delta_{g,m}(t,x_0,y_0)$ in (44) holds and as a result, the inequalities

$$\left.\begin{aligned} \left\|V_f(\theta,x_0,y_0)\right\| &\geq \left\|\Delta_{f,m}(t,x_0,y_0)\right\| - \left\|\Delta_{f,m}(t,x_0,y_0) - \Delta_f(t,x_0,y_0)\right\| > 0 \\ \left\|V_g(\theta,x_0,y_0)\right\| &\geq \left\|\Delta_{g,m}(t,x_0,y_0)\right\| - \left\|\Delta_{g.m}(t,x_0,y_0) - \Delta_g(t,x_0,y_0)\right\| > 0 \end{aligned}\right\} \tag{46}$$

hold in the boundaries Γ_{G_0} and Γ_{G_1}.

Obviously, the characteristics of homotopic vector fields $V_f(\theta,x_0,y_0)$ and $V_g(\theta,x_0,y_0)$ are equal to one another on compact space. Therefore the characteristics of the fields of $\Delta_f(t,x_0,y_0)$ and $\Delta_g(t,x_0,y_0)$ on Γ_{G_0} and Γ_{G_1}respectively are equal to the index of the singular points x^o and y^o of the fields of $\Delta_{f,m}(t,x_0,y_0)$ and $\Delta_{g,m}(t,x_0,y_0)$ and hence they are different from zero. Then $\Delta_f(0,x_0^o,y_0^o) = 0$ and $\Delta_g(0,x_0^o,y_0^o) = 0$, so there exist such points x^o and y^o whose $\Delta_f - \text{costant}$ and $\Delta_g - \text{costant}$ are zero. Therefore, the solutions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ that passing through these points are periodic.

Theorem 4.3. Assume that system (1) has a period T and it is specified on the intervals $a \leq x \leq b$ and $c \leq y \leq$ d. Then for all $m \geq 1$ the vector function sequences $\Delta_{f,m}(t,x_0,y_0)$ and $\Delta_{g,m}(t,x_0,y_0)$ which are defined in (26) and (28) satisfy the following inequalities:

$$\left.\begin{aligned} \min_{\substack{a_1+e_1\leq x\leq b_1-e_1 \\ c_1+e_2\leq y\leq \mathrm{d}_1-e_2}} \Delta_{f,m}(t,x_0,y_0) &\leq -\eta_{1m} \\ \max_{\substack{a_1+e_1\leq x\leq b_1-e_1 \\ c_1+e_2\leq y\leq \mathrm{d}_1-e_2}} \Delta_{f,m}(t,x_0,y_0) &\geq \eta_{1m} \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} \min_{\substack{a_1+e_1\leq x\leq b_1-e_1 \\ c_1+e_2\leq y\leq \mathrm{d}_1-e_2}} \Delta_{g,m}(t,x_0,y_0) &\leq -\eta_{2m} \\ \max_{\substack{a_1+e_1\leq x\leq b_1-e_1 \\ c_1+e_2\leq y\leq \mathrm{d}_1-e_2}} \Delta_{g,m}(t,x_0,y_0) &\geq \eta_{2m} \end{aligned}\right\} \tag{48}$$

Thus, the system (1) has periodic solutions $x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$ such that

$$x_0 \in [a_1 + \mu_1(t)R_1\vartheta_1 + \xi_1(t)MR_1p, b_1 - \mu_1(t)R_1\vartheta_1 + \xi_1(t)MR_1p] \quad (49)$$
$$y_0 \in [c_1 + \mu_2(t)R_2\vartheta_2 + \xi_2(t)NR_2q, \mathrm{d}_1 - \mu_2(t)R_2\vartheta_2 + \xi_2(t)NR_2q]. \quad (50)$$

Proof. Consider the points $x_1(t, x_0, y_0)$ and $x_2(t, x_0, y_0)$ are defined in interval (49) and the points $y_1(t, x_0, y_0)$ and $y_2(t, x_0, y_0)$ are defined in interval (50) such that

$$\left.\begin{aligned} \Delta_{f,m}(t, x_0, y_0) &= \min_{\substack{a_1+e_1 \le x \le b_1 - e_1 \\ c_1 + e_2 \le y \le \mathrm{d}_1 - e_2}} \Delta_{f,m}(t, x_0, y_0) \\ \Delta_{f,m}(t, x_0, y_0) &= \max_{\substack{a_1+e_1 \le x \le b_1 - e_1 \\ c_1 + e_2 \le y \le \mathrm{d}_1 - e_2}} \Delta_{f,m}(t, x_0, y_0) \end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned} \Delta_{g,m}(t, x_0, y_0) &= \min_{\substack{a_1+e_1 \le x \le b_1 - e_1 \\ c_1 + e_2 \le y \le \mathrm{d}_1 - e_2}} \Delta_{g,m}(t, x_0, y_0) \\ \Delta_{g,m}(t, x_0, y_0) &= \max_{\substack{a_1+e_1 \le x \le b_1 - e_1 \\ c_1 + e_2 \le y \le \mathrm{d}_1 - e_2}} \Delta_{g,m}(t, x_0, y_0) \end{aligned}\right\} \quad (52)$$

From the inequalities of system (41), we received that

$$\begin{bmatrix} \Delta_f(t, x_0, y_0) \\ \Delta_f(t, x_0, y_0) \end{bmatrix} = \begin{bmatrix} \Delta_{f,m}(t, x_0, y_0) + \left(\Delta_f(t, x_0, y_0) - \Delta_{f,m}(t, x_0, y_0)\right) < 0 \\ \Delta_{f,m}(t, x_0, y_0) + \left(\Delta_f(t, x_0, y_0) - \Delta_{f,m}(t, x_0, y_0)\right) > 0 \end{bmatrix} \quad (53)$$

$$\begin{bmatrix} \Delta_g(t, x_0, y_0) \\ \Delta_g(t, x_0, y_0) \end{bmatrix} = \begin{bmatrix} \Delta_{g,m}(t, x_0, y_0) + \left(\Delta_g(t, x_0, y_0) - \Delta_{g,m}(t, x_0, y_0)\right) < 0 \\ \Delta_{g,m}(t, x_0, y_0) + \left(\Delta_g(t, x_0, y_0) - \Delta_{g,m}(t, x_0, y_0)\right) > 0 \end{bmatrix} \quad (54)$$

and from the continuity of functions $\Delta_f(t, x_0, y_0)$ and $\Delta_g(t, x_0, y_0)$ also the inequalities (53) and (54) there exist an isolated singular points $(x^o, y^o) = (x_0, y_0)$ where $x^o \in [x_1, x_2]$ and $y^o \in [y_1, y_2]$. Also $\Delta_f(t, x_0, y_0)$ and $\Delta_g(t, x_0, y_0)$ are equal to zero. Thus, the system (1) has periodic solutions $x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$ over (47) and (48).

Remark 4.1. The theorem 4.3 has been proved when x_0 and y_0 are scalar singular points that should be isolated.

Theorem 4.4. Assume that the vector functions $f(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t))$ and $g(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t))$ on (2) are defined and continuous, then they are periodic in t of period T, bounded and fulfilled all of the theorem's assumptions 2.1 and conditions of theorem (2.1). Thus, the functions $f\left(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right)$ and $g\left(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\right)$ on (2) are odd functions, that is

$$\left.\begin{aligned} f\left(-t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right) &= -f\left(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right) \\ g\left(-t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right) &= -g\left(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right) \end{aligned}\right\}. \quad (55)$$

Then the solutions $x(t, x_0, y_0)$ and $y(t, x_0, y_0)$ of (1) for which $x_0 \in G_f$ and $y_0 \in G_g$ are periodic in t of period T.

Proof. Let $\{x_m(t, x_0, y_0)\}_{m=1}^{\infty}$ and $\{y_m(t, x_0, y_0)\}_{m=1}^{\infty}$ be the sequences of functions that are defined in (25) and (27). Accordingly, $f\left(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right)$ and $g\left(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\right)$ on (2) are odd functions, then $\overline{f\left(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\right)} = 0$ and $\overline{g\left(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\right)} = 0$.

Hence, for $m = 0$ we obtain that

$$x_1(t,x_0,y_0) = x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)} \Big(z_0(s) - \Delta_{f,0}\big(s, x_0(s,x_0,y_0), y_0(s,x_0,y_0), u_0(s)\big)\Big) ds$$
$$+ \sum_{0<\tau_i<t} e^{A_1(t-\tau_i)} I_i\big(x_0(\tau_i,x_0,y_0), y_0(\tau_i,x_0,y_0), u_0(\tau_i)\big) = x_1(t+T,x_0,y_0),$$

$$y_1(t,x_0,y_0) = y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)} \Big(w_0(s) - \Delta_{g,0}\big(s, x_0(s,x_0,y_0), y_0(s,x_0,y_0), v_0(s)\big)\Big) ds$$
$$+ \sum_{0<\tau_i<t} e^{C_2(t-\tau_i)} J_i\big(x_0(\tau_i,x_0,y_0), y_0(\tau_i,x_0,y_0), v_0(\tau_i)\big) = y_1(t+T,x_0,y_0).$$

That is $x_1(t,x_0,y_0)$ and $y_1(t,x_0,y_0)$ are periodic of period T in t. Furthermore, $\|x_1(t,x_0,y_0) - x_0\| \le \mu_1(t)R_1\vartheta_1 + \xi_1(t)MR_1 p$ and $\|y_1(t,x_0,y_0) - y_0\| \le \mu_2(t)R_2\vartheta_2 + \xi_2(t)NR_2 q$. That is the functions $x_1(t,x_0,y_0) \in G_f$ and $y_1(t,x_0,y_0) \in G_g$.
Finally, we obtain that $x_1(t,x_0,y_0) = x_1(-t,x_0,y_0)$ and $y_1(t,x_0,y_0) = y_1(-t,x_0,y_0)$ as the integral of an odd function. Beside that, it is clear by induction and for $m \ge 1$ the functions $x_m(t,x_0,y_0)$ and $y_m(t,x_0,y_0)$ are defined and periodic of T's period in t. As a result, the equation (55) to be correct.

Theorem 4.5. Consider the system (1) is known and continuous on (2), where $G_f \in G_0$ and $G_g \in G_1$. Then, for G_f and G_g to obtain points where the constants $\Delta_f(0,x_0,y_0)$ and $\Delta_g(0,x_0,y_0)$ are zero, it is required that for all $m's$ and the following vector inequality must hold for any $x_1 \in G_0$ and $y_1 \in G_1$:

$$\left.\begin{aligned} \begin{bmatrix} \|\Delta_{f,m}(0,x_0,y_0)\| \\ \|\Delta_{g,m}(0,x_0,y_0)\| \end{bmatrix} \le \begin{bmatrix} x_0\|A_1\| + R_1T\vartheta_1\Upsilon_1(t) + pR_1M\Upsilon_1(t) \\ y_0\|C_2\| + R_2T\vartheta_2\Upsilon_2(t) + qR_2N\Upsilon_2(t) \end{bmatrix} \\ +\Upsilon(T)\Phi^{m+1}(T)\big(I - \Phi(T)\big)^{-1}\Omega(T). \end{aligned}\right\} \tag{56}$$

Proof. From the equations (4) and (26) we have
$\|\Delta_{f,m}(0,x_0,y_0)\| \le \|\Delta_f(0,x_0,y_0)\| + \|\Delta_{f,m}(0,x_0,y_0) - \Delta_f(0,x_0,y_0)\|$
where $\|\Delta_f(0,x_0,y_0)\| \le x_0\|A_1\| + R_1T\vartheta_1\Upsilon_1(t) + pR_1M\Upsilon_1(t)$ and
$\|\Delta_{f,m}(0,x_0,y_0) - \Delta_f(0,x_0,y_0)\| \le (\Upsilon_1(t)\varphi_1(t)R_1T + \Upsilon_1(t)\chi_1(t)R_1p)\|x_m(t,x_0,y_0) - x(t,x_0,y_0)\| + (\Upsilon_1(t)\varphi_2(t)R_1T + \Upsilon_1(t)\chi_2(t)R_1p)\|y_m(t,x_0,y_0) - y(t,x_0,y_0)\|$.
Therefore,

$$\begin{aligned} &\|\Delta_{f,m}(0,x_0,y_0)\| \\ &\quad \le x_0\|A_1\| + R_1T\vartheta_1\Upsilon_1(t) + pR_1M\Upsilon_1(t) \\ &\quad + (\Upsilon_1(t)\varphi_1(t)R_1T + \Upsilon_1(t)\chi_1(t)R_1p)\|x_m(t,x_0,y_0) - x(t,x_0,y_0)\| \\ &\quad + (\Upsilon_1(t)\varphi_2(t)R_1T + \Upsilon_1(t)\chi_2(t)R_1p)\|y_m(t,x_0,y_0) - y(t,x_0,y_0)\|. \end{aligned}$$

And from the equations (6) and (28) we obtain that

$$\begin{aligned} &\|\Delta_{g,m}(0,x_0,y_0)\| \\ &\quad \le y_0\|C_2\| + R_2T\vartheta_2\Upsilon_2(t) + qR_2N\Upsilon_2(t) \\ &\quad + (\Upsilon_2(t)\varphi_3(t)R_2T + \Upsilon_2(t)\chi_3(t)R_2q)\|x_m(t,x_0,y_0) - x(t,x_0,y_0)\| \\ &\quad + (\Upsilon_2(t)\varphi_4(t)R_2T + \Upsilon_2(t)\chi_4(t)R_2q)\|y_m(t,x_0,y_0) - y(t,x_0,y_0)\|. \end{aligned}$$

Therefore, the vector form (56) is then found to be true.

5. Stability of periodic solutions of system (1).

The stability of periodic solutions is investigated by the following.

Theorem 5.1. Let the functions $\Delta_f(0,x_0,y_0) \in G_f \times G_g \to R$ and $\Delta_g(0,x_0,y_0) \in G_f \times G_g \to R$ are defined from (4) and (6) respectively, where functions $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ are the limit functions of the function's sequences that are given in (26) and (28). Then from the functions $\Delta_f(0,x_0^1,y_0^1)$, $\Delta_f(0,x_0^2,y_0^2)$, $\Delta_g(0,x_0^1,y_0^1)$ and $\Delta_g(0,x_0^2,y_0^2)$, also $x(0,x_0^1,y_0^1)$, $x(0,x_0^2,y_0^2)$, $y(0,x_0^1,y_0^1)$ and $y(0,x_0^2,y_0^2)$ we have to verify that the following vector inequalities hold:

$$\begin{bmatrix}\|\Delta_f(0,x_0^1,y_0^1)-\Delta_f(0,x_0^2,y_0^2)\| \\ \|\Delta_g(0,x_0^1,y_0^1)-\Delta_g(0,x_0^2,y_0^2)\|\end{bmatrix} \le \begin{bmatrix}\|x_0^1-x_0^2\| \\ \|y_0^1-y_0^2\|\end{bmatrix} + \begin{bmatrix}\Phi_1(t) & \Phi_2(t) \\ \Phi_3(t) & \Phi_4(t)\end{bmatrix}\begin{bmatrix}\|x_0^1(t,x_0^1,y_0^1)-x_0^2(t,x_0^2,y_0^2)\| \\ \|y_0^1(t,x_0^1,y_0^1)-y_0^2(t,x_0^2,y_0^2)\|\end{bmatrix} \quad (57)$$

$$\begin{bmatrix}\|x(0,x_0^1,y_0^1)-x(0,x_0^2,y_0^2)\| \\ \|y(0,x_0^1,y_0^1)-y(0,x_0^2,y_0^2)\|\end{bmatrix} \le \begin{bmatrix}\|x_0^1-x_0^2\| \\ \|y_0^1-y_0^2\|\end{bmatrix} + \mu(t)\begin{bmatrix}\Phi_1(t) & \Phi_2(t) \\ \Phi_3(t) & \Phi_4(t)\end{bmatrix}\begin{bmatrix}\|x(0,x_0^1,y_0^1)-x(0,x_0^2,y_0^2)\| \\ \|y(0,x_0^1,y_0^1)-y(0,x_0^2,y_0^2)\|\end{bmatrix} \quad (58)$$

Then the system (1) has a stable solution.

Proof. From equation (4) and all of theorem (2.1)'s assumptions we conclude the following:

$$\begin{aligned}&\left\|\Delta_f(0,x_0^1,y_0^1)-\Delta_f(0,x_0^2,y_0^2)\right\| = \\ &\Bigg\|A_1x_0^1+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}z^1(s)\,ds + \\ &\frac{A_1}{e^{A_1T}-I}\sum_{i=1}^p e^{A_1(t-\tau_i)}I_i\big(x^1(\tau_i,x_0,y_0),y^1(\tau_i,x_0,y_0),u^1(\tau_i)\big) - \\ &\Big(A_1x_0^2+\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}z^2(s)\,ds + \\ &\frac{A_1}{e^{A_1T}-I}\sum_{i=1}^p e^{A_1(t-\tau_i)}I_i\big(x^2(\tau_i,x_0,y_0),y^2(\tau_i,x_0,y_0),u^2(\tau_i)\big)\Big)\Bigg\| \\ &\le \|A_1\|\|x_0^1-x_0^2\| + (\Upsilon_1\varphi_1R_1T+\Upsilon_1\chi_1R_1p)\|x_0^1(t,x_0,y_0)-x_0^2(t,x_0,y_0)\| \\ &\qquad + (\Upsilon_1\varphi_2R_1T+\Upsilon_1\chi_2R_1p)\|y_0^1(t,x_0,y_0)-y_0^2(t,x_0,y_0)\|.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}&\left\|\Delta_f(0,x_0^1,y_0^1)-\Delta_f(0,x_0^2,y_0^2)\right\| \\ &\le \|A_1\|\|x_0^1-x_0^2\| \\ &+ (\Upsilon_1(t)\varphi_1(t)R_1T+\Upsilon_1(t)\chi_1(t)R_1p)\|x_0^1(t,x_0,y_0)-x_0^2(t,x_0,y_0)\| \\ &+ (\Upsilon_1(t)\varphi_2(t)R_1T+\Upsilon_1(t)\chi_2(t)R_1p)\|y_0^1(t,x_0,y_0)-y_0^2(t,x_0,y_0)\|. \quad (59)\end{aligned}$$

Using the identical inequalities and conditions and by (6) we obtain the following conclusion:

$$\begin{aligned}&\left\|\Delta_g(0,x_0^1,y_0^1)-\Delta_g(0,x_0^2,y_0^2)\right\| \\ &\le \|C_2\|\|y_0^1-y_0^2\| \\ &+ (\Upsilon_2(t)\varphi_3(t)R_2T+\Upsilon_2(t)\chi_3(t)R_2q)\|x_0^1(t,x_0,y_0)-x_0^2(t,x_0,y_0)\| \\ &+ (\Upsilon_2(t)\varphi_4(t)R_2T+\Upsilon_2(t)\chi_4(t)R_2q)\|y_0^1(t,x_0,y_0)-y_0^2(t,x_0,y_0)\|. \quad (60)\end{aligned}$$

Obtaining the vector form (57) from above inequalities where $x(0,x_0^1,y_0^1)$, $x(0,x_0^2,y_0^2)$, $y(0,x_0^1,y_0^1)$ and $y(0,x_0^2,y_0^2)$ are solutions of (1).

Next, from the solutions $x(0,x_0^1,y_0^1)$, $x(0,x_0^2,y_0^2)$, $y(0,x_0^1,y_0^1)$ and $y(0,x_0^2,y_0^2)$ we get

$$\begin{aligned}&\|x(0,x_0^1,y_0^1),-x(0,x_0^2,y_0^2)\| \le \|x_0^1-x_0^2\|\|e^{A_1t}\| + \Bigg\|\int_0^t e^{A_1(t-s)}\big(z_1^1(s)-z_1^2(s)\big)\,ds - \\ &\int_0^t e^{A_1(t-s)}\frac{A_1}{e^{A_1T}-I}\int_0^T e^{A_1(T-s)}\big(z_1^1(s)-z_1^2(s)\big)\,ds\,ds + \\ &\int_0^t e^{A_1(t-s)}\Bigg(\frac{{A_1}^2}{T(e^{A_1T}-TA_1-I)}\Bigg(-\int_0^T\int_0^T\Big(F\big(s,x_0^1(s),y_0^1(s),u_0^1(s)\big) - \\ &F\big(s,x_0^2(s),y_0^2(s),u_0^2(s)\big)\Big)\,dt\,dt + \big(u_0^1(T)-u_0^2(T)\big) - \frac{T}{A_1}(e^{A_1T}-I)(x_0^1-x_0^2)\Bigg)\Bigg)\,ds\Bigg\|\end{aligned}$$

$\leq$
$\|x_0^1 - x_0^2\| + \frac{e^{\|A_1\|T} - e^{\|A_1\|t}}{e^{\|A_1\|T} - \|I\|} \int_0^t \left\|e^{A_1(t-s)}\right\| \|z^1(t) - z^2(t)\| \, ds +$
$\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|} \int_t^T \left\|e^{A_1(t-s)}\right\| \|z^1(t) - z^2(t)\| \, ds +$
$\frac{e^{\|A_1\|t} - \|I\|}{e^{\|A_1\|T} - \|I\|} \sum_{i=1}^{p} \left\|e^{A_1(t-\tau_i)}\right\| \left\| I_i\big(x^1(\tau_i, x_0, y_0), y^1(\tau_i, x_0, y_0), u^1(\tau_i)\big) -\right.$
$\left. I_i\big(x^2(\tau_i, x_0, y_0), y^2(\tau_i, x_0, y_0), u^2(\tau_i)\big)\right\| +$
$\sum_{0<\tau_i<t} \left\|e^{A_1(t-\tau_i)}\right\| \left\| I_i\big(x^1(\tau_i, x_0, y_0), y^1(\tau_i, x_0, y_0), u^1(\tau_i)\big) -\right.$
$\left. I_i\big(x^2(\tau_i, x_0, y_0), y^2(\tau_i, x_0, y_0), u^2(\tau_i)\big)\right\|$

Thus, we receive

$$\begin{aligned}
\|x(0, x_0^1, y_0^1) - x(0, x_0^2, y_0^2)\| \\
&\leq \|x_0^1 - x_0^2\| \\
&+ \big(\mu_1(t) R_1 \varphi_1(t) + \xi_1(t) p R_1 \chi_1(t)\big) \|x^1(t, x_0, y_0) - x^2(t, x_0, y_0)\| \\
&+ \big(\mu_1(t) R_1 \varphi_2(t) + \xi_1(t) p R_1 \chi_2(t)\big) \|y^1(t, x_0, y_0) - y^2(t, x_0, y_0)\| \qquad (61)
\end{aligned}$$

The same iterations are used to obtain the following results.

$$\begin{aligned}
\|y(0, x_0^1, y_0^1) - y(0, x_0^2, y_0^2)\| \\
&\leq \|y_0^1 - y_0^2\| \\
&+ \big(\mu_2(t) R_2 \varphi_3(t) + \xi_2(t) q R_2 \chi_3(t)\big) \|x^1(t, x_0, y_0) - x^2(t, x_0, y_0)\| \\
&+ \big(\mu_2(t) R_2 \varphi_4(t) + \xi_2(t) q R_2 \chi_4(t)\big) \|y^1(t, x_0, y_0) - y^2(t, x_0, y_0)\| \qquad (62)
\end{aligned}$$

We rewrite (61) and (62) in a vector form to get (58). Thus, the system (1) must have a stable solution as a result of (31) and (32).

6. Another method for solution of system (1)(Banach fixed point theorem)

The existence and uniqueness of solutions of system (1) have been investigated as:

Theorem 6.1. Suppose that $f\big(t, x(t, x_0, y_0), y(t, x_0, y_0), u(t)\big)$ and $g\big(t, x(t, x_0, y_0), y(t, x_0, y_0), v(t)\big)$ be vector functions that are defined by (2) and continuous on (18) and periodic in t of period T and satisfy the inequalities and conditions in theorem (2.1). Then the system (1) has a unique periodic solution that follows Banach's fixed point theorem.

Proof. Let $(S, \|.\|)$ is a Banach space on $C[0, T]$ and the mapping T^* is defined on S as follows:-

$$\begin{aligned}
T^* x(t, x_0, y_0) &= x_0 e^{A_1 t} + \int_0^t e^{A_1(t-s)} \Big(z(s) - \Delta_f\big(s, x(s, x_0, y_0), y(s, x_0, y_0), u(s)\big)\Big) ds \\
&+ \sum_{0<\tau_i<t} e^{A_1(t-\tau_i)} I_i\big(x(\tau_i, x_0, y_0), y(\tau_i, x_0, y_0), u(\tau_i)\big), \qquad (63)
\end{aligned}$$

with $x(0, x_0, y_0) = x_0$, $m = 0,1,2, \ldots$, and

$$\begin{aligned}
T^* y(t, x_0, y_0) &= y_0 e^{C_2 t} + \int_0^t e^{C_2(t-s)} \Big(w(s) - \Delta_g\big(s, x(s, x_0, y_0), y(s, x_0, y_0), v(s)\big)\Big) ds \\
&+ \sum_{0<\tau_i<t} e^{C_2(t-\tau_i)} J_i\big(x(\tau_i, x_0, y_0), y(\tau_i, x_0, y_0), v(\tau_i)\big) \qquad (64)
\end{aligned}$$

with $y(0, x_0, y_0) = y_0$ and $m = 0,1,2, \ldots.$

Since $(S, \|.\|)$ define a mapping T^* on S such as (63) and (64), then $f\big(t, x(t, x_0, y_0), y(t, x_0, y_0)\big) \in C[0, T]$ and $t \in S$. Thus, the integral

$$\int_0^t e^{A_1(t-s)}\left(z(s)-\Delta_f\left(s,x(s,x_0,y_0),y(s,x_0,y_0),u(s)\right)\right)ds$$

is continuous on the interval $[0,T]$. Also $x_0e^{A_1t}\in S$, then $T^*x(t,x_0,y_0)\in S$, thus $T^*: C[0,T]\to C[0,T]$.

Next, we claim that T^* is a contraction mapping on S. Let $x(t,x_0,y_0)$, $r(t,x_0,y_0)$, $y(t,x_0,y_0)$ and $k(t,x_0,y_0)$ are belong to S, then

$$\|T^*x(t,x_0,y_0)-T^*r(t,x_0,y_0)\| = \max_{t\in[0,T]}\{|T^*x(t,x_0,y_0)-T^*r(t,x_0,y_0)|\},$$

$$\|T^*y(t,x_0,y_0)-T^*k(t,x_0,y_0)\| = \max_{t\in[0,T]}\{|T^*y(t,x_0,y_0)-T^*k(t,x_0,y_0)|\}.$$

Thus,

$$\begin{aligned}\|T^*x\,(t,x_0,y_0)-T^*r(t,x_0,y_0)\| &\le \Phi_1(T)\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\ &+\Phi_2(T)\|y(t,x_0,y_0)-k(t,x_0,y_0)\|,\end{aligned} \tag{65}$$

$$\begin{aligned}\|T^*y(t,x_0,y_0)-T^*k(t,x_0,y_0)\| &\le \Phi_3(T)\|x(t,x_0,y_0)-r(t,x_0,y_0)\| \\ &+\Phi_4(T)\|y(t,x_0,y_0)-k(t,x_0,y_0)\|.\end{aligned} \tag{66}$$

As a result of (65) and (66) the following vector structure was obtained

$$\begin{bmatrix}\|T^*x\,(t,x_0,y_0)-T^*r(t,x_0,y_0)\|\\ \|T^*y\,(t,x_0,y_0)-T^*k(t,x_0,y_0)\|\end{bmatrix}\le\begin{bmatrix}\Phi_1(T) & \Phi_2(T)\\ \Phi_3(T) & \Phi_4(T)\end{bmatrix}\begin{bmatrix}\|x\,(t,x_0,y_0)-r(t,x_0,y_0)\|\\ \|y\,(t,x_0,y_0)-k(t,x_0,y_0)\|\end{bmatrix}.$$

So that T^* is a mapping cause of condition (30). And from the Banach fixed point theorem there are fixed points $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$, where $T^*x(t,x_0,y_0)=x(t,x_0,y_0)$ and $T^*y(t,x_0,y_0)=y(t,x_0,y_0)$. Consequently, the equations (25) and (27) are unique solutions of system (1).

Conclusions

Numerical-analytic methods are considered for the investigation of the existence and the approximate construction of periodic solutions for nonlinear systems of multiple integro-differential equations that containing symmetric matrices, which are subjected to impulsive action. Theorems on the existence, uniqueness and stability of the solutions are established under some necessary and sufficient conductions on compact space and piecewise continuous functions. This study is based on the Hölder condition, in which the ordering α, β and γ are real numbers between 0 and 1.

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