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The Analytic Solutions of Nonlinear Generalized Pantograph Differential Equations of Higher Order Via Coupled Adomian-Homotopy Technique

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Abstract

In this study, an efficient novel technique is presented to obtain a more accurate analytical solution to nonlinear pantograph differential equations. This technique combines the Adomian decomposition method (ADM) with the homotopy analysis method concepts (HAM). The whole integral part of HAM is used instead of an integral part of ADM approach to get higher accurate results. The main advantage of this technique is that it gives a large and more extended convergent region of iterative approximate solutions for long time intervals that rapidly converge to the exact solution. Another advantage is capable of providing a continuous representation of the approximate solutions, which gives better information over whole time interval. Finally, selected examples are given to show the accuracy, efficiency and effectiveness of this technique. This technique can be addressed and applied to other non-linear problems.

Keywords: Adomian-homotopy technique, Homotopy method, Adomian approach.

الحل التحليلي لمعادلة بنتوكراف التفاضلية الغير خطية من رتب عليا بواسطة تقنية الادوميان مع الهوموتوبي المدمجة

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الخلاصة:

في هذا البحث استخدمنا تقنية جديدة لحل معادلة بنتوكراف التفاضلية الغير خطية ذات الرتب العليا بأستخدام طريقتي الادوميان مع الهوموتوبي مدمجة سوية للحصول على نتائج أكثر دقة . كما هو معروف للجميع طريقة الادوميان و الهوموتوبي تتميزان بالدقة والفعالية العالية. جزء التكامل المستخدم في الهوموتوبي سيتم أستخدامه جميعه بدل الجزء الخاص ب الادوميان وذلك للحصول على اكثر دقة وأعلى للنتائج. تتميز هذه الطريقة أنها تزيد و بالعملية التكرارية لفترة اكبر مع اكثر دقة للحل والوصول بشكل سريع للحل المظبوط . ميزة أخرى أنها لها القدرة بتزويدنا بدالة مستمرة فتعطينا القدرة على دراسة سلوك الحل لكل الفترة الزمنية المعطاة. وأخيرا تم عرض أمثلة للتحقق من دقة وفعالية هذه التقنية. يمكن أستخدام هذه التقنية لحل العديد من المسائل الغير خطية.

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1. Introduction

Many physics and engineering problems can be modelled by differential equations. However, closed-form solutions to such equations, especially for nonlinear ones, are difficult to obtain. In most cases, only approximate solutions (either analytical or numerical) can be expected [1-4]. Extensively, pantograph differential equations have been discussed in many fields of science and engineering in which these equations appear. The original name of the pantograph comes from the dynamics of a current collection system for an electric locomotive which was studied by Ockendon and Tayler [5]. These types of equations are raised in modelling different problems in sciences and engineering such as economy, biology control and other fields. For more details, we refer to several applications of these types of equations [5-7], [8,9]. Many authors studied the analytic solutions, numerical approach and some properties of these types of equations associated with the numerous applications [8,9], [10-16]. In [12], the authors conducted the approximate solution by implementing the Taylor method to a non-homogenous multipantograph equation with variable coefficients that is expanded for this type which is given in [14],[17],

$$s(r) = \alpha s(r) + \sum_{i=1}^{z} l_i(r) s(g_i r) + w(r), \qquad r \ge 0, \tag{1}$$

with initial condition $s(0) = \gamma$, $\alpha, \gamma \in \boldsymbol{R}$, with $l_i(s)$ and $w(r)$ are analytic functions ; $0 < g_i < 1$. Additionally, with properties of numerical and analytical solutions of Eq.(1) with $w(r) = 0$, and $l_i(r) = l_i$, are considered in [14]. A solution of the general pantograph equations associated with linear functional argument is numerically presented [8],[13]

$$s^{(p)}(r) = \sum_{v=0}^{V} \sum_{k=0}^{p-1} Q_{vk}(r) s^{(k)}(\sigma_v r + \alpha_v) + w(r), \tag{2}$$

with stationary conditions

$$\sum_{k=0}^{p-1} a_{ik} s^{(k)}(0) = \varsigma_i, \quad i = 0,1,\ldots,p-1, \tag{3}$$

where $Q_{vk}(r)$ and $w(r)$ are analytic functions. $\sigma_v, \alpha_v, \varsigma_i$ and a_{ik} are complex constants or real. For our work , we consider the two cases of differential equations:

Case 1.

$$\begin{cases} s'(r) = \alpha s(r) + w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), \ldots, s(\sigma_z(r))), \\ s(0) = s_0. \end{cases} \tag{4}$$

Case 2.

$$\begin{cases} s^{(p)}(r) = w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), \ldots, s(\sigma_z(r))) \\ \sum_{k=0}^{p-1} a_{ik} s^{(k)}(0) = \varsigma_i, i = 0,1,\ldots,p-1. \end{cases} \tag{5}$$

where w and σ_i, $i = 1,2,\ldots,z$, are analytic functions; a_{ik}, ς_i and α are constants or real complex.

3. Adomian-Homotopy Technique for Solving Equations 4 and 5

Firstly, we should review Adomian approach [2] to address these equations.
Consider the nonlinear differential equation

$$\begin{cases} s'(r) = \alpha s(r) + w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), \ldots, s(\sigma_z(r))), \\ s(0) = s_0. \end{cases} \tag{6}$$

We begin our analysis by rewriting equation (6) in operator form

$$Ls(r) = \alpha s(r) + w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), ..., s(\sigma_z(r))), \tag{7}$$

where $L = \frac{ds}{dr}$, we assume that L is invertible and the inverse operator L^{-1} is given by

$$L^{-1}(.) = \int_0^t (.)dx$$

So, applying L^{-1} to both sides of (7) and using the initial condition give

$$s(r) = \alpha L^{-1}[s(r)] + L^{-1}[w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), ..., s(\sigma_z(r)))] + s_0, \tag{8}$$

We next represent the linear term $\alpha s(r)$ by the decomposition series of components s_n, $n \geq 0$, and we equate the nonlinear term $w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), ..., s(\sigma_z(r)))$ by the series of Adomian polynomials A_n, $n \geq o$, to get

$$\sum_{n=0}^{\infty} s_n(r) = s_0 + L^{-1}\left(\sum_{n=0}^{\infty} s_n(r)\right) + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{9}$$

Now, the initial condition is s_0 and

$$s_{c+1} = L^{-1}(s_c) + L^{-1}(A_c),\ c \geq 0. \tag{10}$$

and so on. Based on these calculations, we get the solution in a series form. Similarly, the same procedure is applied to equation 5.

Secondly, we have to review the homotopy approach [1] for addressing these equations. Consider the nonlinear differential equation

$$\begin{cases} s'(r) = \alpha s(r) + w(r, s(r), s(\sigma_1(r)), s(\sigma_2(r)), ..., s(\sigma_z(r))), \\ s(0) = s_0. \end{cases} \tag{11}$$

By means of the homotopy analysis technique, the following linear operator may be chosen:

$$\mathrm{L}[\phi(t, p)] = \frac{\partial \phi(t, p)}{\partial t}$$

the nonlinear operator is defined as:

$$N[\phi(t, p)] = \frac{\partial \phi(t, p)}{\partial t} - \alpha\phi - \phi \tag{12}$$

where $\phi = \phi(t, p)$ is known function. Using definition of the zero-order deformation equation

$$(1-p)\mathrm{L}[\phi(t, p) - s(0)] = p\hbar H(t) N[\phi(t, p)]. \tag{13}$$

We can construct with $H(t) = 1$ and $\hbar = -1$, the following (Liao, 2003):

$$(1-p)\mathrm{L}[\phi(t, p)] - s(0)] + pN[\phi(t, p)] = 0 \tag{14}$$

with the initial condition:

$$\phi(0, p) = s_0$$

For $p = 0$ and $p = 1$, in equation (14), we have:

$$\mathrm{L}[\phi(t, 0) - s(0)] = 0 \tag{15}$$

which implies

$$\phi(t, 0) = s(0)$$

and also:

$$N[\phi(t, 1)] = 0, \tag{16}$$

respectively. By the definition of the nonlinear matrix operator (12), the matrix equation (16) is equivalent to the original matrix equation (11). Expanding the $\phi(t, p)$ in Taylor series with respect to the embedding parameter p, we get:

$$\phi(t,p)=\phi(t,0)+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\partial^m\phi(t,p)}{\partial p^m}\bigg|_{p=0}p^m$$

from equation (15), we have:

$$\phi(t,p)=s(0)+\sum_{m=1}^{\infty}s_m(t)\,p^m \tag{17}$$

$$s_m(t)=\frac{1}{m!}\frac{\partial^m\phi(t,p)}{\partial p^m}\bigg|_{p=0}$$

If the auxiliary linear operator, the initial condition $s(0)$, the auxiliary parameter $\hbar$ and the auxiliary function $H(t)$ are so properly chosen, such that the series (17) converges at $p=1$, then due to equation (16), the matrix series (17) becomes:

$$s(t)=s(0)+\sum_{m=1}^{\infty}s_m(t)$$

which satisfies the original matrix equation as proved by Liao (2003). Now, define the vector:

$$\vec{s}_n(t)=\{s_0(t),s_1(t),...,s_n(t)\}.$$

Differentiating matrix equation (14) m-times with respect to the embedding parameter p, and then setting $p=0$. Finally, dividing them by $m!$,we have the so-called mth-order deformation equation which is given as follows:

$$\mathrm{L}[s_m(t)-\chi_m s_{m-1}(t)]=\hbar H(t)R_m(\vec{s}_{m-1}) \tag{18}$$

where

$$R_m(s_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(t,p)]}{\partial p^{m-1}}\bigg|_{p=0},$$

$$\chi_m=\begin{cases}0, m\le 1,\\ 1, m>1.\end{cases}$$

$$R_m s_{m-1}=\frac{\partial s_{m-1}}{\partial t}-\alpha s_{m-1}-\sum_{i=0}^{m-1}R_m(s_{m-1})+(1-\chi_m)(C)$$

Now, the solution of the m^{th} order deformation matrix equation (18) after applying the inverse operator to both sides with a given initial condition, for $m\ge 1$, the result becomes:

$$s_m(t)=\chi_m s_{m-1}+\hbar\int_0^t\left(\dot{s}_{m-1}(r)-s_{m-1}(r)-\sum_{i=0}^{m-1}R_m(s_{m-1})+(1-\chi_m)(C)\right)dr$$

$$=\chi_m s_{m-1}+\hbar\left[s_{m-1}(t)-(1-\chi_m)s_{m-1}(0)\right]+\hbar\int_0^t\left(s_{m-1}(r)A-\sum_{i=0}^{m-1}R_m(s_{m-1})+(1-\chi_m)(C)\right)dr \tag{19}$$

Therefore, it should be emphasized that $s_m(t)$, $m\ge 1$ which can be easily obtained by computation software such as Mathematica or Maple. Similarly this procedure for equation 5. Consequently, the general solution is given by

$$s(t)=\sum_{k=0}^{M}s_k(t).$$

Now, in order to apply Adomian-Homotopy (ADM-HAM), the whole integral for ADM in equation (10) is replaced by the whole integral of HAM in equation (19). In this case, a more accurate solution, efficient, and more extended convergence area are obtained compared with that obtained by equation 10 as we see in the numerical examples.

4. Numerical examples

Example 4.1. Consider the pantograph equation of second order [13]

$$\begin{cases} s''(t) = \frac{3}{4}s(t) + s\left(\frac{t}{2}\right) - t^2 + 2, & 0 \le t \le 1, \\ s(0) = 0, s'(0) = 0, \end{cases} \tag{20}$$

With the exact solution $s(t) = t^2$, then by applying Adomian-Homotopy which is explained in section 3, firstly, it is solved by ADM:

$$y_{n+1}(x) = \int_0^x \int_0^x \left(\frac{3}{4} y_n(x) + y_n\left(\frac{x}{2}\right) \right) dx dx, \quad n \ge 0, \tag{21}$$

secondly, we solve the same equation by HAM:

$$y_n(x) = \int_0^x \int_0^x \left(y''_{n-1} - \frac{3}{4} y_{n-1}(x) - y_{n-1}\left(\frac{x}{2}\right) + (1 - \chi[n])(t^2 - 2) \right) dx dx, \quad n \ge 1, \tag{22}$$

Now, to apply the suggested technique ADM-HAM , we replace the whole part integral of equation (21) with the whole integral part of the equation (22), we get:

$$y_{n+1}(x) = \int_0^x \int_0^x \left(y''_n - \frac{3}{4} y_n(x) - y_n\left(\frac{x}{2}\right) + (1 - \chi[n])(t^2 - 2) \right) dx dx, \ n \ge 0.$$

Simply by Maple or Mathematica software, we can obtain seven iterates approximate solutions as in the following Table.

Table 1: Numerical results for Adomian-Homotopy of seventh iterate

t	*Adomian-Homotopy*	*Exact Solution*	*Absolute Error*
0.1	0.00999999907592	0.01	9.02408×10^{-9}
0.2	0.03999942316955	0.04	5.7683×10^{-7}
0.3	0.089993443010507	0.09	6.55699×10^{-6}
0.4	0.15996326429320	0.16	3.67357×10^{-5}
0.5	0.24986038183363	0.25	1.39618×10^{-4}
0.6	0.35958498497379	0.36	4.15015×10^{-4}
0.7	0.488959076732129	0.49	1.04092×10^{-3}
0.8	0.637694936724653	0.64	2.30506×10^{-3}
0.9	0.805359689305261	0.81	4.64031×10^{-3}
1.	0.991336850847586	1.	8.66315×10^{-3}

Example 4.2. Consider the pantograph equation of the third order [13] **.**

$$\begin{cases} s'''(t) = -s(t) - s(t - 0.3) + e^{-t+0.3}, & 0 \le t \le 1, \\ s(0) = 1, s'(0) = -1, s''(0) = 1. \end{cases} \tag{23}$$

With the exact solution $s(t) = e^{-t}$. , we implement Adomian-Homotopy which is explained in Section 3 and in Example1, we can obtain seven iterates approximate solution as given in Table 2.

Table 2: Numerical results for Adomian-Homotopy of seventh iterate

t	*Adomian-Homotopy*	*Exact Solution*	*Absolute Error*
0.1	0.90483986032836	0.90483741803596	2.44229×10^{-6}
0.2	0.818745867814137	0.818730753077982	1.51147×10^{-5}
0.3	0.740856480442197	0.740818220681718	3.82598×10^{-5}
0.4	0.670381440494918	0.670320046035639	6.13945×10^{-5}
0.5	0.606587287206899	0.606530659712633	5.66275×10^{-5}
0.6	0.548776641190715	0.548811636094027	3.49949×10^{-5}
0.7	0.496261732374315	0.49658530379141	3.23571×10^{-4}
0.8	0.448332551546541	0.449328964117222	9.96413×10^{-4}
0.9	0.404219877024632	0.406569659740599	2.34978×10^{-3}
1.	0.3630532665298557	0.367879441171442	4.82618×10^{-3}

We can observe from the above examples that the results of this technique are excellent although with few iterations. They showed that this technique with the fewest number of iterations can converge to the correct results and will get a more accurate solution with a more extended convergence area by increasing the number of iterations. This technique that is developed in this work enables to solve different strong problems and second Painleve equation is investigated in [18].

4. Conclusion

The Adomian-Homotopy technique has been successfully implemented of nonlinear generalized pantograph differential equations of higher order. Selected examples to investigate the validity of this technique. Compared with other approaches, the results that are given demonstrate that this technique is more accurate than the stated existing techniques and few iterations are enough to obtain a highly accurate solution which converges rapidly to the exact solution and extended the convergence region.

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