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Face Antimagic Labeling for Double Duplication of Barycentric and Middle Graphs

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Abstract

This paper proves the existence of face antimagic labeling for $DD_{VV}\big(C_n(C_n)\big)$, $n \geq 4, DD_{VV}(C_m \odot 2P_n), m \geq 4, n \geq 3, DD_{VV}(M(C_n), n \geq 4$ and $DD_{VV}(G^+)$.

Keywords: Labeling, Graph, Function, Double duplication, Face antimagic

Mathematics Subject Classification: 05C78.

1. Introduction

Let $G = (V, E, F)$ be a finite connected plane graph without loops and multiple edges, whereV, E and F are its vertex set, edge set and face set, respectively. Labeling of type $(0, 1, 0)$ assigns labels from the set $\{1, 2, \dots, |E(G)|\}$ to the edges of a graph G. The weight of a face under this labeling is the sum of the labels of the edges surrounding that face.

Labeling of type$(0, 1, 1)$ assigns labels from the set $\{1, 2, \dots, |E(G) + F(G)|\}$to the edges and faces of a graph G. The weight of a face under this labeling is the sum of the labels of the edges surrounding that face and also the label of the same face.

In [1], Baca defined a labeling of a plane graph G which is called $(a,\ d)$ $-$antimagic if for every positive integer s, the set of S $-$sided face weight is $W_s = \{\, a_s,\ \ a_s + d, \dots,\ a_s + (|F(G)| - 1)\}$for some integer a_s and $d \geq 0$. We allow different sets W_s for different 's'.The concept of the $(a,\ d)$ antimagic labeling of the plane graphs is defined in [2], where it was also proved that the problem D_n has d $-$antimagic labeling of type $(1, 1, 1)$for $d \in \{2, 3, 4, 6\}$ and $n \equiv 3(mod\ 4)$.

Definition 1.3 [3]: The double duplication of a vertex by an edge of a graph is defined as a duplication of a vertex v_k by an edge $e = v_k'v_k''$ in a graph G produces a graph G' in which $N(v_k')=\{v_k, v_k''\}$ and $N(v_k'') = \{v_k, v_k'\}$. Again duplication of vertices v_k, v_k' and v_k'' by edges $e'=u_kw_k$, $e''= u_k'w_k'$ and $e'''= u_k''w_k''$,respectively in G' produces a new graph G'' such that $N(u_k) = \{w_k,\ \ v_k\}$, $N(w_k) = \{u_k,\ \ v_k\}$, $N(u_k') = \{w_k',v_k'\}$, $N(w_k') = \{\ u_k',v_k'\}$, $N(u_k'') = \{\ w_k'',v_k''\}$, $N(w_k'') = \{\ u_k'',v_k''\}$. The double duplication of all vertices by edges, respectively of a graph G is denoted by $DD_{VV}(G)$.

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Definition 1.4[4]: Let *G* = (*V*, *E*) be a graph. If every edge of graph *G* is subdivided, then the resulting graph is called the barycentric subdivision of graph *G*. In other words, the barycentric subdivision is the graph obtained by inserting a vertex of degree 2 into every edge of the original graph.

Definition 1.5 [4]: Consider barycentric subdivision of cycle C_n and join each newly inserted vertices of incident edges by an edge. It is denoted by $C_n(C_n)$ as it looks like C_n inscribed in C_n.

Definition 1.6[5]: Bi-armed crown $C_n \odot 2P_m$is a graph obtained from a cycle C_n by identifying the pendant vertices of two vertex disjoint paths of the same length $m-1$ at each vertex of the cycle.

Definition 1.7 [6]: The middle graph of a connected graph *G* denoted by *M*(*G*) is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if

(i) They are adjacent edges of *G*, or

(ii) One is a vertex of *G* and the other is an edge incident with it.

Definition 1.8: The graph G^+ is obtained by joining exactly one pendant edge to every vertex of a graph G.

2. MAIN RESULTS

In this section, the existence of face antimagic labeling for some special graphs is discussed.

Theorem 2.1: The graph $DD_{VV}(C_n(C_n)), n \geq 4$ is face antimagic labeling of types$(0, 1, 0)$ and $(0, 1, 1)$.

Proof

Let $G(V, E, F)$ denote $C_n(C_n), n \geq 4$ with vertex set $V = \{v_i | 1 \leq i \leq 2n\}$,
$E = \{v_i v_{i+1} | 1 \leq i \leq 2n-1\} \cup \{v_1 v_{2n}\}$ and
$F = \{v_2 v_4 v_6 \ldots v_{2n} v_2\} \cup \{v_{2i} v_{2i+1} v_{2i+2} | 1 \leq i \leq n-1\} \cup \{v_1 v_2 v_{2n}\}$. Let $G'(V', E', F')$be the graph obtained from $G$by duplication of all vertices by edges of G with $V' = \{a_i b_i | 1 \leq i \leq 2n\} \cup V$, $E' = \{v_i a_i, v_i \mathrm{b}_i, a_i b_i | 1 \leq i \leq 2n\} \cup E$ and $F' = \{v_i a_i b_i | 1 \leq i \leq 2n\} \cup F$.

Let $G''(V'', E'', F'')$ be the graph that obtained from G' by duplication of all vertices by edges of G' with $V'' = \{c_i, d_i, e_i, f_i, g_i, h_i | 1 \leq i \leq 2n\} \cup V'$, $E'' = \{a_i c_i, a_i d_i, c_i d_i, b_i e_i, b_i f_i, e_i f_i, g_i h_i, v_i g_i, v_i h_i | 1 \leq i \leq 2n\} \cup E'$and
$F'' = \{a_i c_i d_i,\ b_i e_i f_i, g_i v_i h_i | 1 \leq i \leq 2n\} \cup F'$.

-Type (i)$(0,1,0)$

Define a mapping $\Gamma: E'' \rightarrow \{1, 2, 3, \ldots, 27n\}$ as follows:

For $1 \leq i \leq 2n$,

$\Gamma(v_i a_i) = 3i - 2$; $\Gamma(v_i b_i) = 3i - 1$; $\Gamma(a_i b_i) = 3i$; $\Gamma(a_i c_i) = 6n + 3i - 2$;
$\Gamma(a_i d_i) = 6n + 3i - 1$; $\Gamma(c_i d_i) = 6n + 3i$; $\Gamma(b_i e_i) = 12n + 3i - 2$;
$\Gamma(e_i f_i) = 12n + 3i$; $\Gamma(b_i f_i) = 12n + 3i - 1$; $\Gamma(g_i v_i) = 18n + 3i - 2$;
$\Gamma(v_i h_i) = 18n + 3i - 1$; $\Gamma(g_i h_i) = 18n + 3i$.

For $1 \leq i \leq n-1$,

$\Gamma(v_{2i} v_{1+2i}) = 24n + 3i - 2$, $\Gamma(v_{2i+1} v_{2i+2}) = 24n + 3i - 1$, $\Gamma(v_{2i} v_{2i+2}) = 24n + 3i$,
$\Gamma(v_1 v_2) = 27n - 2$; $\Gamma(v_2 v_{2n}) = 27\mathrm{n}$; $\Gamma(v_1 v_{2n}) = 27\mathrm{n} - 1$.

The following are the labeling types of face antimagic labeling. f_{ij} denotes the faces of the graph G, $f_i{}'$denotes the faces formed after the first duplication and $f_{ij}{}''$ denotes the faces formed after the second duplication.

The calculated face weights are as follows:

$\beta(f_i') = \Gamma(v_i a_i) + \Gamma(a_i b_i) + \Gamma(v_i b_i) = 9i - 3, 1 \leq i \leq 2n$
$\beta(f_{i1}'') = \Gamma(a_i c_i) + \Gamma(a_i d_i) + \Gamma(c_i d_i) = 18\mathrm{n} + 9i - 3, 1 \leq i \leq 2n$
$\beta(f_{i2}'') = \Gamma(b_i e_i) + \Gamma(b_i f_i) + \Gamma(e_i f_i) = 36n + 9i - 3, 1 \leq i \leq 2n$
$\beta(f_{i3}'') = \Gamma(g_i v_i) + \Gamma(v_i h_i) + \Gamma(g_i h_i) = 54n + 9i - 3, 1 \leq i \leq 2n$

$\beta(f_{i1}) = \Gamma(v_{2i}v_{2i+2}) + \Gamma(v_{2i+1}v_{2i+2}) + \Gamma(v_{2i}v_{1+2i}) = 72n + 9i - 3, 1 \leq i \leq n - 1$ and $\beta(f_{n1}) = \Gamma(v_1v_2) + \Gamma(v_2v_{2n}) + \Gamma(v_1v_{2n}) = 81n - 3$ which forms an arithmetic progression $\{6, 6 + (1 \times 9),\ 6 + (2 \times 9), \ldots, 6 + (9n - 1)9\}$ and the face weight of C_n is $\frac{(51n+3)n}{2}$.

-Type (ii) $(0, 1, 1)$

Define $\Gamma: E'' \cup F'' \to \{1, 2, 3, \ldots, 36n + 2\}$ as follows:

$\Gamma(f_i') = 27n + i, 1 \leq i \leq 2n$; $\Gamma(f_{i1}'') = 29n + i, 1 \leq i \leq 2n$; $\Gamma(f_{i2}'') = 31n + i, 1 \leq i \leq 2n$; $\Gamma(f_{i3}'') = 33n + i, 1 \leq i \leq 2n$; $\Gamma(f_{i1}) = 35n + i, 1 \leq i \leq n - 1$; $\Gamma(f_{n1}) = 36n$

The labeling for the n sided face is $36n + 1$ and the labeling for the external face is $36n + 2$. The calculated face weights are as follows:

$\beta(f_i') = \Gamma(v_ia_i) + \Gamma(a_ib_i) + \Gamma(v_ib_i) + \Gamma(f_i') = 27n + 10i - 3; 1 \leq i \leq 2n$

$\beta(f_{i1}'') = \Gamma(a_ic_i) + \Gamma(a_id_i) + \Gamma(c_id_i) + \Gamma(f_{i1}'') = 47n + 10i - 3; 1 \leq i \leq 2n$

$\beta(f_{i2}'') = \Gamma(b_ie_i) + \Gamma(b_if_i) + \Gamma(e_if_i) + \Gamma(f_{i2}'') = 67n + 10i - 3; 1 \leq i \leq 2n$

$\beta(f_{i3}'') = \Gamma(g_iv_i) + \Gamma(v_ih_i) + \Gamma(g_ih_i) + \Gamma(f_{i3}'') = 87n + 10i - 3; 1 \leq i \leq 2n$

$\beta(f_{i1}) = \Gamma(v_{2i}v_{2i+2}) + \Gamma(v_{2i+1}v_{2i+2}) + \Gamma(v_{2i}v_{1+2i}) + \Gamma(f_{i1}) = 107n + 10i - 3; 1 \leq i \leq n - 1$, $\beta(f_{n1}) = \Gamma(v_1v_2) + \Gamma(v_2v_{2n}) + \Gamma(v_1v_{2n}) + \Gamma(f_{n1}) = 117n - 3$ which forms an arithmetic progression$\{27n + 7, 27n + 7 + 1 \times 10,\ 27n + 7 + 2 \times 10, \ldots, 27n + 7 + (9n - 1)10\}$ and the face weight of n sided cycle is $\frac{1}{2}[51n^2 + 75n + 2]$.

Theorem 2.2: The graph $DD_{VV}(C_m \odot 2P_n), m \geq 4, n \geq 3$ of types $(0, 1, 0)$ and $(0, 1, 1)$ *is* face antimagic.

Proof:

Let $G = (C_m \odot 2P_n)$ $m \geq 3, n \geq 4$ be a graph with $V = \{v_i | 1 \leq i \leq mn\} \cup \{w_i | 1 \leq i \leq mn - m\}$,

$E = \{v_iv_{i+1} | 1 \leq i \leq mn - 1 \textit{ except for } n, 2n, 3n, \ldots, (m - 1)n\} \cup \{w_iw_{i+1} | 1 \leq i \leq m(n - 1) - 1 \textit{ except for } (n - 1), (2(n - 1), (3(n - 1), \ldots, (m - 1)(n - 1)\} \cup \{v_{ni-n+1}v_{in+1} | 1 \leq i \leq m - 1\} \cup \{v_{nm-n+1}v_1\} \cup \{v_{ni-n+1}w_{(i-1)(n-1)+1} | 1 \leq i \leq m\}$ and $F = (v_1v_{n+1}\ v_{2n+1\ldots..}v_{(m-1)n+1}v_1)$.

Let $G'(V', E', F')$ be the graph obtained from G by duplication of all vertices by edges of G with

$V' = \{a_i, b_i | 1 \leq i \leq 2mn - m\}$,

$E' = \{v_ia_i, v_ib_i, a_ib_i | 1 \leq i \leq mn\} \cup \{w_ia_{mn+i}w_ib_{mn+i}, a_{mn+i}b_{mn+i} | 1 \leq i \leq mn - m\} \cup E$ and $F' = \{v_ia_ib_i | 1 \leq i \leq mn\} \cup \{w_ia_{mn+i}b_{mn+i} | 1 \leq i \leq mn - m\} \cup F$

Let $G''(V'', E'', F'')$ be the graph obtained from G' by the duplication of all vertices by edges of G' with $V' = \{c_i, d_i, e_i, f_i, g_i, h_i | 1 \leq i \leq 2mn - m\} \cup V'$,

$E'' = \{c_id_i, e_if_i, \quad c_ia_i,\ b_if_i,\ d_ia_i,\ b_ie_i | 1 \leq i \leq 2mn - m\} \cup \{g_iv_i, v_ih_i, g_ih_i | 1 \leq i \leq mn\} \cup \{w_ig_{mn+i}, w_ih_{mn+i}, g_{mn+i}h_{mn+i} | 1 \leq i \leq mn - m\} \cup E'$ and

$F'' = \{a_ic_id_i, b_ie_if_i | 1 \leq i \leq 2mn - m\} \cup \{g_iv_ih_i | 1 \leq i \leq mn\}\} \cup \{w_ih_{mn+i}g_{mn+i} | 1 \leq i \leq mn - m\}$

Type (i) - $(0, 1, 0)$

Define $\Gamma: E' \to \{1, 2, 3, \ldots, 26mn - 13m\}$ as follows:

For $1 \leq i \leq mn$,

$\Gamma(v_ia_i) = 3i - 2$, $\Gamma(v_ib_i) = 3i - 1$, $\Gamma(a_ib_i) = 3i$.

For $1 \leq i \leq mn - m$,

$\Gamma(w_ia_{mn+i}) = 3mn + 3i - 2$; $\Gamma(w_ib_{mn+i}) = 3mn + 3i - 1$;

$\Gamma(b_{i+mn}a_{mn+i}) = 3mn + 3i$.

For $1 \leq i \leq 2mn - m$,

$\Gamma(c_ia_i) = 6mn - 3m + 3i - 2$; $\Gamma(d_ia_i) = 6mn - 3m + 3i - 1$;

$\Gamma(c_i d_i) = 6mn - 3m + 3i$; $\Gamma(b_i e_i) = 12mn - 6m + 3i - 2$;
$\Gamma(b_i f_i) = 12mn - 6m + 3i - 1$; $\Gamma(f_i e_i) = 12mn - 6m + 3i$.
For $1 \leq i \leq mn$,
$\Gamma(g_i v_i) = 18mn - 9m + 3i - 2$; $\Gamma(h_i v_i) = 18mn - 9m + 3i - 1$;
$\Gamma(g_i h_i) = 18mn - 9m + 3i$.
For $1 \leq i \leq mn - m$,
$\Gamma(w_i g_{mn+i}) = 21mn - 9m + 3i - 2$; $\Gamma(w_i h_{mn+i}) = 21mn - 9m + 3i - 1$;
$\Gamma(h_{mn+i} g_{mn+i}) = 21mn - 9m + 3i$;
$\Gamma(v_i v_{i+1}) = \{24mn - 12m + i, 1 \leq i \leq mn - 1\ except\ for\ n, 2n, 3n, \dots, (m-1)n$
$\Gamma(v_{ni-n+1} v_{ni+1}) = 24mn - 12m + ni, 1 \leq i \leq m - 1$ $\Gamma(v_1 v_{nm+1-n}) = 25mn - 12m$
$\Gamma(w_i w_{i+1}) = 25mn - 12m + i, 1 \leq i \leq m(n-1) - 1$ $except$ for $(n-1), (2(n-1), (3(n-1), \dots, (m-1)(n-1)$
$\Gamma\left(v_{1+ni-n} w_{1+(i-1)(n-1)}\right) = 25mn - 12m + (n-1)i; 1 \leq i \leq m$.
The calculated face weights are as follows*:*
$\beta(f'_{i1}) = \Gamma(v_i a_i) + \Gamma(a_i b_i) + \Gamma(v_i b_i) = 9i - 3;\ 1 \leq i \leq mn$
$\beta(f'_{i2}) = \Gamma(w_i a_{mn+i}) + \Gamma(w_i b_{mn+i}) + \Gamma(b_{i+mn} a_{mn+i}) = 9mn + 9i - 3; 1 \leq i \leq mn - m$
$\beta(f''_{i1}) = \Gamma(c_i a_i) + \Gamma(d_i a_i) + \Gamma(c_i d_i) = 18mn - 9m + 9i - 3; 1 \leq i \leq 2mn - m$
$\beta\left(f''_{i2}\right) = \Gamma(b_i e_i) + \Gamma(b_i f_i) + \Gamma(f_i e_i) = 36mn - 18m + 9i - 3;\ 1 \leq i \leq 2mn - m$
$\beta\left(f''_{i3}\right) = \Gamma(v_i g_i) + \Gamma(v_i h_i) + \Gamma(g_i h_i) = 54mn - 27m + 9i - 3;\ 1 \leq i \leq mn$
$\beta\left(f''_{i4}\right) = \Gamma(w_i g_{mn+i}) + \Gamma(w_i h_{mn+i}) + \Gamma(h_{mn+i} g_{mn+i}) = 63mn - 27m + 9i - 3;\ 1 \leq i \leq mn - m$ which forms an arithmetic progression $\{6, 6 + (1 \times 9), 6 + (2 \times 9), \dots, 6 + [4(2mn - m) - 1]9\}$.
The face weight of m sided cycle is $\frac{m}{2}\left[49\,mn - 24m + n\right]$.
Type (ii) – $(0, 1, 1)$
Define $\Gamma: E'' \cup F'' \rightarrow \{1, 2, 3, \dots\ 34mn - 17m + 2\}$ as follows. $\Gamma(f'_{i1}) = 26mn - 13m + i$; $1 \leq i \leq mn$; $\Gamma(f'_{i2}) = 27\,mn - 13m + i; 1 \leq i \leq mn - m$
$\Gamma(f''_{i1}) = 28\,mn - 14m + i; 1 \leq i \leq 2mn - m$;
$\Gamma(f''_{i2}) = 30mn - 15m + i; 1 \leq i \leq 2mn - m$;
$\Gamma(f''_{i3}) = 32mn - 16m + i;\ 1 \leq i \leq mn$;
$\Gamma(f''_{i4}) = 33mn - 16m + i;\ 1 \leq i \leq mn - m$
The face labeling of m sided face is $34mn - 17m + 1$and the labeling for the external face is $34mn - 17m + 2$.
The calculated face weights are as follows
$\beta(f'_{i1}) = \Gamma(v_i a_i) + \Gamma(a_i b_i) + \Gamma(v_i b_i) + \Gamma(f'_{i1}) = 26mn - 13m + 10i - 3;\ 1 \leq i \leq mn$
$\beta(f'_{i2}) = \Gamma(w_i a_{mn+i}) + \Gamma(w_i b_{mn+i}) + \Gamma(b_{i+mn} a_{mn+i}) + \Gamma(f'_{i2}) = 36mn - 13m + 10i - 3$; $1 \leq i \leq mn - m$
$\beta(f''_{i1}) = \Gamma(c_i a_i) + \Gamma(d_i a_i) + \Gamma(c_i d_i) + \Gamma(f''_{i1}) = 56mn - 23m + 10i - 3; 1 \leq i \leq 2mn - m$
$\beta(f''_{i2}) = \Gamma(b_i e_i) + \Gamma(b_i f_i) + \Gamma(f_i e_i) + \Gamma(f''_{i2}) = 66mn - 33m + 10i - 3; 1 \leq i \leq 2mn - m$
$\beta(f''_{i3}) = \Gamma(v_i g_i) + \Gamma(v_i h_i) + \Gamma(g_i h_i) + \Gamma(f''_{i3}) = 86mn - 43m + 10i - 3; 1 \leq i \leq mn$
$\beta(f''_{i4}) = \Gamma(w_i g_{mn+i}) + \Gamma(w_i h_{mn+i}) + \Gamma(h_{mn+i} g_{mn+i}) + \Gamma(f''_{i4}) = 96mn - 43m + 10i - 3;\ 1 \leq i \leq mn - m$, which forms an arithmetic progression $\{26mn - m + 7, 26mn - m + 7 + (1 \times 10),$
$26mn - m + 7 + (2 \times 10), \dots, 26mn - m + 7 + [4(2mn - m) - 1]10\}$ and the face weight of $m$sided face is $\frac{m}{2}[49mn - 24m + n] + 34mn - 17m + 1$.

Theorem 2.3: The graph $DD_{VV}(M(C_n)), n \geq 4$ of types $(0, 1, 0)$ and $(0, 1, 1)$is face antimagic labeling.

Proof:
Let $G(V,E,F)$ be the graph $M(C_n), n \geq 4$ with $V = \{u_i, v_i | 1 \leq i \leq n\}$,
$E = \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\} \cup \{u_i v_i | 1 \leq i \leq n\} \cup \{u_{i+1} v_i | 1 \leq i \leq n-1\} \cup \{v_n u_1\}$
and $F = \{v_i v_{i+1} u_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1 u_1\} \cup \{v_1 v_2 v_3 \dots v_n\}$.
Let $G'(V',E',F')$ be the graph obtained from G by duplication of all vertices by edges of G with $V' = \{a_i, b_i, a_i', b_i' | 1 \leq i \leq n\} \cup V$, $E' = \{v_i a_i, v_i b_i, a_i b_i, u_i a_i', u_i b_i', a_i' b_i' | 1 \leq i \leq n\} \cup E$ and $F' = \{v_i a_i b_i, u_i a_i' b_i' | 1 \leq i \leq n\} \cup F$.
Let $G''(V'', E'', F'')$ be the graph obtained from G' by the duplication of all vertices by edges of G' with,
$V'' = \{c_i, d_i, e_i, f_i, g_i, h_i, c_i', d_i', e_i', f_i', g_i', h_i' | 1 \leq i \leq n\} \cup V'$,
$E'' = \{c_i d_i, e_i f_i, c_i a_i, b_i f_i, d_i a_i, b_i e_i, g_i v_i, v_i h_i, g_i h_i, c_i' d_i', e_i' f_i', c_i' a_i', b_i' f_i', d_i' a_i', b'_i e_i', g'_i u_i', u_i' h_i', g'_i h'_i | 1 \leq i \leq n\} \cup E'$ and
$F'' = \{c_i a_i d_i, c_i' a_i' d_i', v_i g_i h_i, v_i' g_i' h_i', b_i e_i f_i, b_i' e_i' f_i' \mid 1 \leq i \leq n\} \cup F'$.
Type (i) $(0,1,0)$
Define $\alpha: E'' \to \{1, 2, \dots, 27n\}$ as follows:
For $1 \leq i \leq n$,
$\alpha(v_i a_i) = 3i - 2$; $\alpha(v_i b_i) = 3i - 1$; $\alpha(a_i b_i) = 3i$;
$\alpha(u_i a_i') = 3n + 3i - 2$; $\alpha(u_i b_i') = 3n + 3i - 1$; $\alpha(a_i' b_i') = 3n + 3i$;
$\alpha(a_i c_i) = 6n + 3i - 2$; $\alpha(a_i d_i) = 6n + 3i - 1$; $\alpha(c_i d_i) = 6n + 3i$;
$\alpha(a'_i c_i') = 9n + 3i - 2$; $\alpha(a'_i d'_i) = 9n + 3i - 1$; $\alpha(c_i' d_i') = 9n + 3i$;
$\alpha(b_i e_i) = 12n + 3i - 2$; $\alpha(e_i f_i) = 12n + 3i$; $\alpha(b_i f_i) = 12n + 3i - 1$;
$\alpha(b'_i e_i') = 15n + 3i - 2$; $\alpha(b'_i f_i') = 15n + 3i - 1$; $\alpha(e'_i f_i') = 15n + 3i$;
$\alpha(g_i v_i) = 18n + 3i - 2$; $\alpha(h_i v_i) = 18n + 3i - 1$; $\alpha(g_i h_i) = 18n + 3i$;
$\alpha(g_i' u_i) = 21n + 3i - 2$; $\alpha(h'_i u_i) = 21n + 3i - 1$; $\alpha(g_i' h_i') = 21n + 3i$;
$\alpha(v_{i+1} u_{i+1}) = 24n + 3i - 1$; $1 \leq i \leq n - 1$;
$\alpha(v_i u_{1+i}) = 24n + 3i - 2$; $1 \leq i \leq n - 1$; $\alpha(v_i v_{1+i}) = 24n + 3i$; $1 \leq i \leq n - 1$;
$\alpha(v_1 u_1) = 27n - 1$; $\alpha(v_n u_1) = 27n - 2$; $\alpha(v_n v_1) = 27n$.
For $1 \leq i \leq n$,
$\beta(f_{i1}') = \alpha(v_i a_i) + \alpha(a_i b_i) + \alpha(v_i b_i) = 9i - 3$
$\beta(f_{i2}') = \alpha(u_i a_i') + +\alpha(u_i b_i') + \alpha(a'_i b'_i) = 9n + 9i - 3$
$\beta(f_{i1}'') = \alpha(a_i c_i) + \alpha(a_i d_i) + \alpha(c_i d_i) = 18n + 9i - 3$
$\beta(f_{i2}'') = \alpha(a_i' c'_i) + \alpha(a'_i d_i') + \alpha(c'_i d_i') = 27n + 9i - 3$
$\beta(f_{i3}'') = \alpha(b_i e_i) + \alpha(b_i f_i) + \alpha(e_i f_i) = 36n + 9i - 3$
$\beta(f_{i4}'') = \alpha(b'_i e_i') + \alpha(b'_i f_i') + \alpha(e_i' f_i') = 45n + 9i - 3$
$\beta(f_{i5}'') = \alpha(v_i g_i) + \alpha(v_i h_i) + \alpha(g_i h_i) = 54n + 9i - 3$
$\beta(f_{i6}'') = \alpha(u_i g_i') + \alpha(u_i h'_i) + \alpha(g_i' h'_i) = 63n + 9i - 3$
$\beta(f_{i1}) = \alpha(v_i u_{i+1}) + \alpha(v_{i+1} u_{i+1}) + \alpha(v_i v_{1+i}) = 72n + 9i - 3; 1 \leq i \leq n - 1$
$\beta(f_{n1}) = \alpha(v_1 u_1) + \alpha(v_n u_1) + \alpha(v_n v_1) = 81n - 3$ which forms an arithmetic progression $\{6, 6 + (1 \times 9), 6 + (2 \times 9), \dots, 6 + (9n - 1)9\}$ and the face weight of n sided face is $(51n + 3)\frac{n}{2}$.
Type (ii) – $(0,1,1)$
Define $\alpha: E'' \to \{1, 2, 3, \dots, 36n + 2\}$ as follows:
$\alpha(f_{i1}') = 27n + i$; $1 \leq i \leq n$; $\alpha(f_{i2}') = 28n + i$; $1 \leq i \leq n$
$\alpha(f_{i1}'') = 29n + i$; $1 \leq i \leq n$; $\alpha(f_{i2}'') = 30n + i$; $1 \leq i \leq n$
$\alpha(f_{i3}'') = 31n + i$; $1 \leq i \leq n$; $\alpha(f_{i4}'') = 32n + i$; $1 \leq i \leq n$
$\alpha(f_{i5}'') = 33n + i$; $1 \leq i \leq n$ $\alpha(f_{i6}'') = 34n + i$; $1 \leq i \leq n$;
$\alpha(f_{i1}) = 35n + i$; $1 \leq i \leq n - 1$; $\alpha(f_{n1}) = 36n$

The label of n sided face is $36n + 1$ and the label for the external face is $36n + 2$.
$\beta(f'_{i1}) = \alpha(v_i a_i) + \alpha(a_i b_i) + \alpha(v_i b_i) + \alpha(f'_{i1}) = 27n + 10i - 3; 1 \le i \le n$
$\beta(f'_{i2}) = \alpha(u_i a_i') + \alpha(a'_i b'_i) + \alpha(u_i b'_i) + \alpha(f'_{i2}) = 37n + 10i - 3;\ 1 \le i \le n$
$\beta(f''_{i1}) = \alpha(a_i c_i) + \alpha(a_i d_i) + \alpha(c_i d_i) + \alpha(f''_{i1}) = 47n + 10i - 3;\ 1 \le i \le n$
$\beta(f''_{i2}) = \alpha(a_i' c_i') + \alpha(a_i' d_i') + \alpha(c_i' d_i') + \alpha(f''_{i2}) = 57n + 10i - 3; 1 \le i \le n$
$\beta(f''_{i3}) = \alpha(b_i e_i) + \alpha(b_i f_i) + \alpha(e_i f_i) + \alpha(f''_{i3}) = 67n + 10i - 3; 1 \le i \le n$
$\beta(f''_{i4}) = \alpha(b'_i e'_i) + \alpha(b'_i f'_i) + \alpha(e'_i f_i') + \alpha(f''_{i4}) = 77n + 10i - 3;\ 1 \le i \le n$
$\beta(f''_{i5}) = \alpha(v_i g_i) + \alpha(v_i h_i) + \alpha(g_i h_i) + \alpha(f''_{i5}) = 87n + 10i - 3; 1 \le i \le n$
$\beta(f''_{i6}) = \alpha(u_i g'_i) + \alpha(u_i h'_i) + \alpha(g'_i h'_i) + \alpha(f''_{i6}) = 97n + 10i - 3; 1 \le i \le n$
$\beta(f_{i1}) = \alpha(v_i u_{i+1}) + \alpha(v_{i+1} u_{i+1}) + \alpha(v_i v_{1+i}) + \alpha(f_{i1})$
$= 107n + 10i - 3; 1 \le i \le n - 1$
$\beta(f_{n1}) = \alpha(v_1 u_1) + \alpha(v_n u_1) + \alpha(v_n v_1) + \alpha(f_{n1}) = 117n - 3$
which forms an arithmetic progression$\{27n + 7,\ 27n + 7 + (1 \times 10),\ 27n + 7 + (2 \times 10), \dots, 27n + 7 + (9n - 1)10\}$ and the face weight of *n* sided cycle is $\frac{1}{2}[51n^2 + 75n + 2]$.

Theorem 2.4: If the graph G^+ except for $3-$ sided faces is $(a,\ d)-$face antimagicof types $(0,1,0)$ and $(0,1,1)$, then $DD_{VV}(G^+)$ is also a $(a,\ d)-$ face antimagic labeling of types $(0,1,0)$ and $(0,1,1)$.

Proof: Let G be the graph with vertex set $V = V = \{v_i | 1 \le i \le m\}$, edge set $E = \{e_i | 1 \le i \le n\}$and $F = \{f_i | 1 \le i \le r\}$.
Let G^+ be the graph obtained from G by attaching a pendant edge to each vertex of G with vertex set $V^+ = \{v_i | 1 \le i \le 2m\}$ edge set $E^+ = \{e_i | 1 \le i \le n\} \cup \{v_i v_{i+m} | 1 \le i \le m\}$ and face set $F^+ = \{f_i | 1 \le i \le r\}$.Let $G'(V', E', F')$be the graph obtained from $G^+$by duplication of all vertices by edges of $G^+$with$V' = \{a_i, b_i | 1 \le i \le 2m\} \cup V$, $E' = \{v_i a_i,\ v_i b_i, a_i b_i | 1 \le i \le 2m\} \cup E$ and $F' = \{f'_{i1} : v_i a_i b_i | 1 \le i \le 2m\} \cup F$. Let $G^{"}(V^{"}, E^{"}, F^{"})$be the graph obtained from $G'$by duplication of all vertices by edges of $G'$with. $V^{"} = \{a, c_i, d_i, e_i, f_i, g_i, h_i | 1 \le i \le 2m\} \cup V'$,
$E^{"} = \{v_i a_i, v_i b_i, a_i b_i,\ a_i c_i,\ a_i d_i, c_i d_i, b_i e_i, b_i f_i, e_i f_i, g_i h_i, v_i g_i, v_i h_i | 1 \le i \le 2m\} \cup E'$ and $F^{"} = \{\ f_{i1}": a_i c_i d_i,\ f"_{i2}: b_i e_i f_i,\ f_{i3}": v_i g_i h_i | 1 \le i \le 2m\} \cup F'$.
Define $\alpha: E'' \to \{1, 2, 3, \dots, 25m\}$ as follows:
$\alpha(v_i a_i) = 3i - 2;\ 1 \le i \le 2m$; $\alpha(v_i b_i) = 3i - 1;\ 1 \le i \le 2m$;
$\alpha(a_i b_i) = 3i; 1 \le i \le 2m$; $\alpha(a_i c_i) = 6m + 3i - 2; 1 \le i \le 2m$;
$\alpha(a_i d_i) = 6m + 3i - 1; 1 \le i \le 2m$; $\alpha(c_i d_i) = 6m + 3i; 1 \le i \le 2m$;
$\alpha(b_i e_i) = 12m + 3i - 2; 1 \le i \le 2m$; $\alpha(b_i f_i) = 12m + 3i - 1; 1 \le i \le 2m$
$\alpha(e_i f_i) = 12m + 3i; 1 \le i \le 2m$; $\alpha(v_i g_i) = 18m + 3i - 2; 1 \le i \le 2m$;
$\alpha(v_i h_i) = 18m + 3i - 1; 1 \le i \le 2m$; $\alpha(g_i h_i) = 18m + 3i; 1 \le i \le 2m$.
$\alpha(v_i v_{i+m}) = 24m + i; 1 \le i \le m$
We calculate the face weights as follows:
For$1 \le i \le 2m$,
$\beta(f'_{i1}) = \alpha(v_i a_i) + \alpha(a_i b_i) + \alpha(v_i b_i) = 9i - 3$
$\beta(f''_{i1}) = \alpha(a_i c_i) + \alpha(a_i d_i) + \alpha(c_i d_i) = 18m + 9i - 3$
$\beta(f''_{i2}) = \alpha(b_i e_i) + \alpha(b_i f_i) + \alpha(e_i f_i) = 36m + 9i - 3$
$\beta(f''_{i3}) = \alpha(v_i g_i) + \alpha(v_i h_i) + \alpha(g_i h_i) = 54m + 9i - 3$
which is forming an arithmetic progression $\{6, 6 + (1 \times 9), 6 + (2 \times 9), \dots, 6 + (8m - 1)9\}$.
Type (ii) (0, 1, 1)
Define $\alpha: E'' \cup F'' \to \{1, 2, \dots, 33m + 1\}$ as follows:
$\alpha(f'_{i1}) = 25m + i;\ 1 \le i \le 2m$; $\alpha(f'_{i2}) = 27m + i;\ 1 \le i \le 2m$
$\alpha(f'_{i3}) = 29m + i;\ 1 \le i \le 2m$;
$\alpha(f'_{i4}) = 31m + i; 1 \le i \le 2m$

and the labeling for the external face is $33m + 1$.
The face weights are calculated as follows:
For $1 \leq i \leq 2m$,
$\beta(f_{i1}") = \alpha(a_i c_i) + \alpha(a_i d_i) + \alpha(c_i d_i) + \alpha(f'_{i2}) = 45m + 10i - 3$
$\beta(f'_{i1}) = \alpha(v_i a_i) + \alpha(a_i b_i) + \alpha(v_i b_i) + \alpha(f'_{i1}) = 25m + 10i - 3$
$\beta(f_{i2}") = \alpha(b_i e_i) + \alpha(b_i f_i) + \alpha(e_i f_i) + \alpha(f'_{i3}) = 65m + 10i - 3$
$\beta(f_{i3}") = \alpha(v_i g_i) + \alpha(v_i h_i) + \alpha(g_i h_i) + \alpha(f'_{i4}) = 85m + 10i - 3$
which is forming an arithmetic progression $\{45m + 7,\ 45m + 7 + (1 \times 10),\ 45m + 7 + (2 \times 10), \ldots, 45m + 7 + (8m - 1)10\}$.

Conclusion

The (a, d) antimagic labeling for the biarmed crown graphs, the barycentric subdivision of a cycle graph and middle graph of a cycle are proved. If the graph G^+except for 3 - sided faces is $(a, d) -$ face antimagicof types $(0, 1, 0)$ and $(0, 1, 1)$ then $DD_{VV}(G^+)$ is also of the same a $(a, d) -$ face antimagic. Recently, the face antimagic labeling of double duplication graphs are applied to encrypt and decrypt secret messages and numbers in communication networks.

References

[1] Martin Baca, Ljiljana Brankovic Marcela Lascsakova, Oudone Phanalasy, Andrea Seminicova-Fenovcikova, On d-antimagic labeling of plane graphs, *Electronic Journal of Graph Theory and Applications* Vol.1, no.1, 28-39, 2013.

[2] M. Baca and M. Miller, On d-antimagic labeling of type (1, 1, 1) for prisms, *J. Combin. Math. Combin. Comput*, Vol.44, 199 – 207, 2003.

[3] L. Shobana and B. Roopa, On face magic labeling of double duplication of some families of graphs, *International Journal of Pure and Applied Mathematics*, Vol.114, no.6, 49 – 60, 2017.

[4] S. K. Vaidya, K. K. Kanani, Some Cordial Graphs in the Context of Barycentric Subdivision, *J. Contemp. Math. Sciences*, Vol. 4, no. 30, 1479 –1492, 2009.

[5] P. Jeyanthi, D. Ramya, P. Thangavelu, On Super mean graphs, *AKCE J . Graphs. Combin*., Vol.6, no. 1, pp.103-112, 2009.

[6] B. Thenmozhi, Dr. R. Prabha, Power Domination of Middle graph of path, cycle and star, *International Journal of Pure and Applied Mathematics* Vol. 114,no. 5, 13-19, 2017.

[7] J. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, #DS6, 2020.

[8] M. Baca, Face antimagic labeling of convex polytypes, *Utilitas, Math*, Vol.55, 221 – 226, 1999.