The Operator \( S(a, b; \theta_x) \) for the Polynomials \( Z_n(x, y, a, b; q) \)

Husam L. Saad* , Faiz A. Reshem
Department of Mathematics, College of Science, Basrah University, Basrah, Iraq

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Abstract
In this work, we give an identity that leads to establishing the operator \( S(a, b; \theta_x) \). Also, we introduce the polynomials \( Z_n(x, y, a, b; q) \). In addition, we provide Operator proof for the generating function with its extension and the Rogers formula for \( Z_n(x, y, a, b; q) \). The generating function with its extension and the Rogers formula for the bivariate Rogers-Szegö polynomials \( h_n(x, y|q) \) are deduced. The Rogers formula for \( Z_n(x, y, a, b; q) \) allows to obtain the inverse linearization formula for \( Z_n(x, y, a, b; q) \), which allows to deduce the inverse linearization formula for \( h_n(x, y|q) \). A solution to a \( q \)-difference equation is introduced and the solution is expressed in terms of the operators \( S(a, b; \theta_x) \). The \( q \)-difference method is used to recover an identity of the operator \( S(a, b; \theta_x) \) and the generating function for the polynomials \( Z_n(x, y, a, b; q) \).

Keywords: The bivariate Rogers-Szegö polynomials, Generating function, Rogers formula, Inverse linearisation formula, \( q \)-difference equation.

1. Introduction
The definitions and notations for the basic hypergeometric series [1] are adopted as follows:
Let \( 0 < q < 1 \). The definition of the \( q \)-shifted factorial is:

\[ *Email: hus6274@hotmail.com \]
The definition

\[(a, q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).\]

is also given.

The researchers employed the following notation for the multiple \(q\)-shifted factorials:

\[(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n(a_2; q)_n \cdots (a_m; q)_n, \quad n = 1, 2, 3, \ldots\]

The definition of the basic hypergeometric series \(\phi_s\) is:

\[\phi_s \left( \frac{a_1, a_2, \cdots, a_r}{b_1, b_2, \cdots, b_s}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n(a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n(b_1; q)_n(b_2; q)_n \cdots (b_s; q)_n} \left( (-1)^n q^{(n)} \right)^{1+s-r} x^n.\]

The \(q\)-binomial coefficients is defined by:

\[\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.\]

The Cauchy identity is given by:

\[\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1.\]  \hspace{1cm} (1.1)

The particular cases of Cauchy identity were identified by Euler:

\[\sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \quad |x| < 1.\]  \hspace{1cm} (1.2)

\[\sum_{k=0}^{\infty} \frac{(-1)^k q^{(k)} x^k}{(q; q)_k} = (x; q)_\infty.\]  \hspace{1cm} (1.3)

Euler’s identity (1.3) can be expressed in a finite form as [2]

\[x; q)_n = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{(k)} x^k.\]  \hspace{1cm} (1.4)

The following identities will commonly occur in this paper [1]:

\[(q/a; q)_k = (-a)^{-k} q^{(k+1)} 2 (aq^{-k}; q)_\infty/(a; q)_\infty.\]  \hspace{1cm} (1.5)

The Cauchy polynomials are defined as follows [3,4]:

\[P_k(x; y) = (x - y)(x - yq) \cdots (x - yq^{k-1}) = (y/x; q)_k x^k,\]

with the generating function:

\[\sum_{k=0}^{\infty} P_k(x; y) \frac{t^k}{(q; q)_k} = \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1.\]  \hspace{1cm} (1.6)

Another version of the Cauchy polynomials is given as follows:

\[P_n(x, y) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{(k)} y^k x^{n-k}.\]

The bivariate Rogers-Szegö polynomials were presented by Chen et al. [5] in 2003,

\[h_n(x, y|q) = \sum_{k=0}^{n} \binom{n}{k}_q P_k(x, y),\]

with this generating function

\[\sum_{n=0}^{\infty} h_n(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(t, xt; q)_\infty}, \quad \max(|t|, |xt|) < 1.\]  \hspace{1cm} (1.7)
The Rogers formula for $h_n(x, y|q)$ was derived by Chen et al. [6]

$$
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(ys; q)_{\infty}}{(s, xs, xt; q)_{\infty}} \phi_1 \left( \frac{ys}{q}, t \right),
$$

with the condition that $\max\{|t|, |xt|, |s|, |xs|\} < 1.$

The $q^{-1}$-Rogers-Szegö polynomials [7] is defined by:

$$
h_n(a, b|q^{-1}) = \sum_{k=0}^{n} \frac{n!}{k!} q_{c^{k-n}} a^k b^{n-k}.
$$

The $q$-differential operator $\theta$ is defined by [8]:

$$
\theta(f(a)) = f(aq^{-1}) - f(a).
$$

The Leibniz rule for $\theta$ is [9]:

$$
\theta^n(f(a)g(a)) = \sum_{k=0}^{n} \binom{n}{k} \theta^k(f(a)) \theta^{n-k}(g(aq^{-k})).
$$

We can easily verify the identities:

$$
\theta^k_{x}\{P_n(y, x)\} = (-1)^k \frac{(q; q)_n}{(q; q)_{n-k}} P_{n-k}(y, x).
$$

$$
\theta^k_{x}\{(xt; q)_{\infty}\} = (-1)^k t^k (xt; q)_{\infty}.
$$

The $q$-exponential operator $E(b\theta)$ was defined by Chen and Liu [9] in 1998 as follows:

$$
E(b\theta) = \sum_{k=0}^{\infty} \left( \frac{q^{[k]}(b\theta)}{(q; q)_k} \right).
$$

In 2007, the Cauchy companion operator $E(a, b; \theta)$ was presented by Fang [10] as follows:

$$
E(a, b; \theta) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (-b\theta)^n.
$$

The Cauchy operator was defined by Chen and Gu [11] in 2008 as follows:

$$
T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_q)^n.
$$

In 2010, the solutions of $q$-difference equation are obtained, and the solution is expressed in terms of the operator $E(b\theta)$, by Liu [7], who derived Mehler’s formula for the $q^{-1}$-Rogers-Szegö polynomials $h_n(a, b|q^{-1})$.

In 2010, the solutions of $q$-difference equations were obtained by Zhu [12], who expressed the solutions in terms of the operator $T(-\frac{1}{a}; ab; \theta)$.

In 2010, the solutions of $q$-difference equations were obtained by Abdul Hussein [13] who expressed the solutions in terms of the operator $E(a, b; \theta).$ The generating function, the Mehler formula and the Rogers formula for the Al-Salam-Carlits polynomials $U_n(x, y, a; q)$ are also proved.

Our paper is structured as follows: Section 2 contains an identity, which leads to creating the operator $S(a, b; \theta_x).$ We also present the polynomials $Z_n(x, y, a, b; q).$ We used the operator $S(a, b; \theta_x)$ to represent the polynomials $Z_n(x, y, a, b; q).$ In section 3, we present an operator proof for the generating function as well as its extension for the polynomials $Z_n(x, y, a, b; q).$ The generating function and its extension for the polynomials $h_n(x, y|q)$ are then deduced. Section 4 presents an operator proof of the Rogers formula for the polynomials.
The Rogers formula for the polynomials $h_n(x, y | q)$ is then recovered. The Rogers formula for $Z_n(x, y, a, b; q)$ allows to derive the inverse linearization formula for $Z_n(x, y, a, b; q)$, from which we can get the inverse linearization formula for $h_n(x, y | q)$. Section 5 introduces and solves a $q$-difference equation. The operator $S(a, b; \theta_x)$ is then used to describe the solution. This approach is used to confirm identity of the operator $S(a, b; \theta_x)$ and the generating function for the polynomials $Z_n(x, y, a, b; q)$.

### 2. The Operator $S(a, b; \theta_x)$ for the Polynomials $Z_n(x, y, a, b; q)$

An identity is provided in this section. This identity is the inspiration to introduce the operator $S(a, b; \theta_x)$. Furthermore, the polynomials $Z_n(x, y, a, b; q)$ were presented.

**Theorem 2.1** We have
\[
\sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} (-1)^k q^{-\binom{k}{2}} (abt)^k = \frac{1}{(bt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq)_k}{(q, q/\sqrt{bt}; q)_k}.
\]  

**Proof.** By using (1.4), the result is obtained as follows:
\[
\sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} (-1)^k q^{-\binom{k}{2}} (abt)^k = \sum_{k=0}^{\infty} \frac{\sum_{i=0}^{k} [k](a)_i (1/a)^i}{(q; q)_k} (-1)^k q^{-\binom{k}{2}} (abt)^k
\]
\[
= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} (-1)^k q^{-\binom{k}{2}} (abt)^k \frac{(-1)^i q^{-\binom{i}{2}} (1/a)^i (q; q)_i (q; q)_{k-i}}{(q; q)_k}
\]
\[
= \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2}} (abt)^k \sum_{i=0}^{\infty} \frac{(btq^{-k})^i}{(q; q)_i}
\]
\[
= \sum_{k=0}^{\infty} (-1)^k q^{-\binom{k}{2}} (abt)^k \frac{1}{(btq^{-k}; q)_{\infty}}
\]
\[
= \frac{1}{(bt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(-1)^k q^{-\binom{k}{2}} (abt)^k}{(q, q/\sqrt{bt}; q)_k (btq^{-k}q^{-\binom{k}{2}})}
\]
\[
= \frac{1}{(bt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq)_k}{(q, q/\sqrt{bt}; q)_k}.
\]

Now, let $\theta$ be defined as in (1.9). Taking inspiration from identity (2.1), the following operator is now presented:
\[
S(a, b, \theta_x) = \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} (-1)^k (ab\theta_x)^k,
\]  

and then the following polynomials are introduced:
\[
Z_n(x, y, a, b; q) = \sum_{k=0}^{n} \binom{n}{k} b^k (aq^{1-k}, q)_k P_{n-k}(y, x).
\]
Setting \(a = 0, b = 1\) and exchanging \(x\) and \(y\) in \(Z_n(x, y, a, b; q)\), we get the bivariate Rogers-Szegö polynomials \(h_n(x, y|q)\), signifying that the bivariate Rogers-Szegö polynomials \(h_n(x, y|q)\) is a particular case of the polynomials \(Z_n(x, y, a, b; q)\).

From (1.11) and (2.2), the following representation for the polynomials \(Z_n(x, y, a, b; q)\) is obtained:

\[
\mathcal{S}(a, b; \theta_x)(P_n(y, x)) = Z_n(x, y, a, b; q). \tag{2.3}
\]

By using (1.12), it is easy to prove that

\[
\mathcal{S}(a; b; \theta_x)\{\langle xt; q\rangle_{\infty}\} = \frac{(xt; q)_{\infty}}{(bt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}. \tag{2.4}
\]

### 3. The generating function for \(Z_n(x, y, a, b; q)\)

In this section, the operator proof for the generating function and its extension for the polynomials \(Z_n(x, y, a, b; q)\) are provided. The generating function and its extension for the polynomials \(h_n(x, y|q)\) are then deduced.

**Theorem 3.1 (The generating function for \(Z_n(x, y, a, b; q)\)).** We have

\[
\sum_{n=0}^{\infty} Z_n(x, y, a, b; q) \frac{t^n}{(q; q)_n} = \frac{(xt; q)_{\infty}}{(yt, bt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}. \tag{3.1}
\]

Provided that \(\max\{|yt|, |bt|\} < 1\).

**Proof.**

\[
\sum_{n=0}^{\infty} Z_n(x, y, a, b; q) \frac{t^n}{(q; q)_n} = \sum_{n=0}^{\infty} \mathcal{S}(a, b; \theta_x)(P_n(y, x)) \frac{t^n}{(q; q)_n} \quad \text{(by using (2.3))}
\]

\[
= \frac{1}{(yt; q)_{\infty}} \mathcal{S}(a, b; \theta_x)\{(xt; q)_{\infty}\} \quad \text{(by using (1.6))}
\]

\[
= \frac{(xt; q)_{\infty}}{(yt, bt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}. \quad \text{(by using (2.4))}
\]

Setting \(a = 0, b = 1\) and exchanging \(x\) and \(y\) in the generating function for the polynomials \(Z_n(x, y, a, b; q)\), we obtain the generating function of the bivariate Rogers-Szegö polynomials \(h_n(x, y|q)\) (1.7).

**Lemma 3.2** Let \(\mathcal{S}(a, b, \theta_x)\) be defined as in (2.2). Then

\[
\mathcal{S}(a, b; \theta_x)(P_n(y, x)(xs; q)_{\infty}) = \frac{(xs; q)_{\infty}}{(bs; q)_{\infty}} \sum_{m=k}^{n} \left[ \frac{1}{m} \right] \frac{(1/a, q/xs; q)_m}{(q/bs; q)_m} q^{-(\frac{m}{2})} (-ax)^{m} P_{n-m}(y, x)
\]

\[
\times \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bsq^{-m}; q)_k}, \quad |bs| < 1. \tag{3.2}
\]

**Proof.**

\[
\mathcal{S}(a, b; \theta_x)(P_n(y, x)(xs; q)_{\infty})
\]

\[
= \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} q^{-(\frac{k}{2})} (ab)^k \theta_x^k \{P_n(y, x)(xs; q)_{\infty}\} \quad \text{(by using (2.2))}
\]

\[
= \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} q^{-(\frac{k}{2})} (ab)^k \sum_{m=0}^{k} \left[ \frac{k}{m} \right] \theta_x^m \{P_n(y, x)\} \theta_x^{k-m}(xsq^{-m}; q)_{\infty} \quad \text{(by using (1.10))}
\]
Lemma 3.3 (Extension of the generating function for $Z_n(x, y, a, b; q)$). We have

$$\sum_{n=0}^{\infty} Z_{n+k}(x, y, a, b; q) \frac{t^n}{(q; q)_n} = \frac{(xtq^k; q)_\infty}{(yt, btq^k; q)_\infty} \sum_{m=0}^{k} \frac{[m] (1/a, q/xtq^k; q)_m}{(q/htq^k; q)_m} (-1)^m q^{-m(2)}(ax)^m \times P_{k-m}(y, x) \sum_{n=0}^{\infty} \frac{(aq^{1-m})^n}{(q, q/bsq^{-m}; q)_n}, \quad \max(|yt|, |br|) < 1. \quad (3.3)$$
Proof.

\[
\sum_{n=0}^{\infty} Z_{n+k}(x, y, a, b; q) \frac{t^n}{(q; q)_n}
= \sum_{n=0}^{\infty} S(a, b; \theta_x)\left\{P_{n+k}(y, x)\right\} \frac{t^n}{(q; q)_n} \quad \text{(by using (2.3))}
\]

\[
= S(a, b; \theta_x) \left\{\sum_{n=0}^{\infty} P_k(y, x)P_n(y, xq^k) \frac{t^n}{(q; q)_n}\right\}
\]

\[
= S(a, b; \theta_x) \left\{P_k(y, x) \frac{(xtq^k; q)_\infty}{(yt; q)_\infty}\right\} \quad \text{(by using (1.6))}
\]

\[
= \frac{1}{(yt; q)_\infty} S(a, b; \theta_x)\left\{P_k(y, x)(xtq^k; q)_\infty\right\}
\]

\[
= \frac{(xtq^k; q)_\infty}{(yt, btq^k; q)_\infty} \sum_{n=0}^{\infty} \left[\frac{k}{(q/btq^k; q)_\infty} \frac{(1/a, q/xtq^k; q)_m}{(q/btq^k; q)_m} (-1)^m q^{-\binom{m}{2}} (ax)^m p_{k-m}(y, x) \right]
\times \sum_{n=0}^{\infty} \frac{(aq^{1-m})^n}{(q, q/btq^{k-m}; q)_n}. \quad \text{(by using (3.2))}
\]

Setting \(a = 0, b = 1\) and exchanging \(x\) and \(y\) in the extension of generating function for the polynomials \(Z_n(x, y, a, b; q)\) (3.3), an extension of the generating function of the bivariate Rogers-Szegö polynomials \(h_n(x, y|q)\) is obtained:

\[
\sum_{n=0}^{\infty} h_{n+k}(x, y|q) \frac{t^n}{(q; q)_n} = \frac{(ytq^k; q)_\infty}{(xt, tq^k; q)_\infty} \sum_{m=0}^{k} \frac{k}{m} \frac{(q/ytq^k; q)_m}{(q/tq^k; q)_m} y^m p_{k-m}(x, y),
\]

Provided that \(\max\{|xt|, |yt|\} < 1\).

4. The Rogers formula for \(Z_n(x; y; a; b; q)\)

This section provides the operator proof of the Rogers formula for the polynomials \(Z_n\), and the inverse linearization formula for \(Z_n(x, y, a, b; q)\) is obtained. The Rogers formula and the inverse linearization formula for \(h_n(x, y|q)\) are then recovered \(h_n(x, y|q)\).

**Theorem 4.1** (The Rogers formula for \(Z_n(x; y; a; b; q)\)). We have

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x, y, a, b; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m}
= \frac{(xs; q)_\infty}{(ys, bs; q)_\infty} \sum_{m=0}^{\infty} \frac{(1/a; q)_m}{(q; q)_m} (-1)^m q^{-\binom{m}{2}} (abt)^m \sum_{n=0}^{\infty} \frac{(x/y, bs; q)_n}{(q, xs; q)_n} (yt)^n
\times \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bsq^{n-k}; q)_k}, \quad \max\{|ys|, |bs|\} < 1.
\]

**Proof.**
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x, y, a, b; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{S}(a, b; \theta_x) \{ P_{n+m}(y, x) \} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \quad \text{(by using (2.3))} \]

\[ = \mathcal{S}(a, b; \theta_x) \left\{ \sum_{n=0}^{\infty} P_n(y, x) \frac{t^n}{(q; q)_n} \sum_{m=0}^{\infty} P_m(y, x q^n) \frac{s^m}{(q; q)_m} \right\} \]

\[ = \mathcal{S}(a, b; \theta_x) \left\{ \sum_{n=0}^{\infty} P_n(y, x) \frac{t^n}{(q; q)_n} (x q^n; q)_{\infty} \right\} \quad \text{(by using (1.6))} \]

\[ = \frac{1}{(ys; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{(q; q)_n} \mathcal{S}(a, b; \theta_x) \{ P_n(y, x) (x q^n; q)_{\infty} \} \]

\[ = \frac{1}{(ys; q)_{\infty}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{t^n}{(q; q)_n} (x q^n; q)_{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{(1/a, q/x q^n; q)_m}{(q/bq^n, q)_m} q^{-m} (-ax)^m P_{n-m}(y, x) \]

\[ \times \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bq^{n-m}; q)_k} \quad \text{(by using (3.2))} \]

\[ = \frac{1}{(ys; q)_{\infty}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{t^n (1/a; q)_{m} q^{-m} (-ax)^m (x q^n; q)_{\infty} (q/x q^n; q)_{\infty}}{(q/q)_m (q/q)_{n-m} (bsq^{n-m}; q)_{\infty} (q/bq^n; q)_{\infty}} P_{n-m}(y, x) \]

\[ \times \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bq^{n-m}; q)_k} \quad \text{(by using (1.5))} \]

\[ = \frac{1}{(ys; q)_{\infty}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{t^n (1/a; q)_{m} q^{-m} (-ax)^m (x q^n; q)_{\infty} (b/x)^n P_{n-m}(y, x)}{(q/q)_m (q/q)_{n-m} (bsq^{n-m}; q)_{\infty} (q/bq^n; q)_{\infty}} \]

\[ \times \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bq^{n-m}; q)_k} \]

\[ = \frac{1}{(ys; q)_{\infty}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{t^{n+m} (1/a; q)_{m} q^{-m} (-ab)^m (x q^n; q)_{\infty} P_n(y, x) \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bq^n; q)_k}}{(q/q)_m (q/q)_n (bsq^{n-m}; q)_{\infty} (q/bq^n; q)_{\infty}} \]

\[ = \frac{(x q^n; q)_{\infty}}{(ys, bs; q)_{\infty}} \sum_{m=0}^{\infty} \frac{(1/a; q)_{m} (-1)^m q^{-m} (abt)^m}{(q/q)_m} \sum_{n=0}^{\infty} \frac{x/y, bs; q)_n (yt)^n}{(q, q, x q^n; q)_n} \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bq^n; q)_k}. \]

Set \( a = 0, b = 1 \) and exchange \( x \) and \( y \) in the Rogers formula for the polynomials \( Z_n(x, y, a, b; q) \) (4.1), the Rogers formula for the bivariate Rogers-Szegö polynomials \( h_n(x, y; q) \) (1.8) is recovered.

The Rogers formula for \( Z_n(x, y, a, b; q) \) (4.1) can be written differently, this allows to obtain the inverse linearisation formula of \( Z_n(x, y, a, b; q) \) as follows:
Lemma 4.2 For $n, m \geq 0$, we have
\[ Z_{n+m}(x, y, a, b; q) = \sum_{i=0}^{\infty} \left( \frac{1}{a; q} \right) (-1)^i q^{-i/2} (ab)^i P_{n-i}(y, x) Z_m(x q^{n-i}, y, a q^{-i}, b q^{n-i}; q). \] (4.2)

Proof. Rewrite the Rogers formula (4.1) as:
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x, y, a, b; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \]
\[ = \frac{(x; q)_\infty}{(y; q)_\infty} \sum_{i=0}^{\infty} \frac{(1/a; q)_i}{(q; q)_i} (-1)^i q^{-i/2} (ab)^i \sum_{n=0}^{\infty} \frac{(x/y; q)_n}{(q; q)_n} \frac{(yt)^n}{(q; q)_n} \sum_{k=0}^{\infty} \frac{(aq^{1-i})^k}{(q; bsq^n; q)_k} \]
\[ = \sum_{i=0}^{\infty} \frac{(1/a; q)_i}{(q; q)_i} (-1)^i q^{-i/2} (ab)^i \sum_{n=0}^{\infty} \frac{(x/y; q)_n}{(q; q)_n} \frac{(yt)^n}{(q; q)_n} \sum_{k=0}^{\infty} \frac{(aq^{1-i})^k}{(q; bsq^n; q)_k} \]
\[ = \sum_{i=0}^{\infty} \frac{(1/a; q)_i}{(q; q)_i} (-1)^i q^{-i/2} (ab)^i \sum_{n=0}^{\infty} P_n(y, x) \frac{t^{n+i}}{(q; q)_n} \sum_{k=0}^{\infty} Z_k(x q^n, y, a q^{-i}, b q^n; q) \frac{s^k}{(q; q)_k}. \]
(by using (3.3))

Equating the coefficients of $t^n s^m$ on both sides, we get the required result.

Set $a = 0$, $b = 1$ and exchange $x$ and $y$ in the inverse linearisation formula for $Z_n(x, y, a, b; q)$ (4.2), the inverse linearisation formula for the bivariate Rogers-Szegö polynomials for polynomials $h_n(x, y|q)$ is obtained:
\[ h_{n+m}(x, y|q) = \sum_{i=0}^{n} \sum_{j=0}^{m} \left( \begin{array}{c} n \\ i \\ \end{array} \right) \left( \begin{array}{c} m \\ j \\ \end{array} \right) q^{i(m-j)} P_{i+j}(x, y). \]

5. The $q$-Difference Equation and the $S(a, b; \theta_c)$ operator
In this section, a $q$-difference equation is presented and solved. The solution is then expressed in terms of the operator $S(a, b; \theta_c)$. With this method, an operator identity for the operator $S(a, b; \theta_c)$ and the generating function for the polynomials $Z_n(x, y, a, b; q)$ were verified.

Proposition 5.1 Let $f(a, b, c)$ be an analytic function of three variables in a neighborhood of $(a, b, c) = (0, 0, 0) \in \mathbb{C}^3$ satisfying the $q$-difference equation
\[ cq^{-1} f(a, b, c) - f(a, bq, c) = b \left\{ af \left( a, \frac{b}{q}, \frac{c}{q} \right) - f \left( a, b, c \right) + f(a, b, c) af(a, b/q, c) \right\}. \] (5.1)

Then we have
\[ f(a, b, c) = S(a, b; \theta_c) \{ f(a, 0, c) \}, \] (5.2)

where $S(a, b; \theta_c)$ acts on the parameter $c$.

Proof. Let
\[ f(a, b, c) = \sum_{n=0}^{\infty} A_n(a, c) b^n, \] (5.3)

where $A_n$ is independent of $b$. Substituting the (5.3) into (5.1) the following is obtained
\[ cq^{-1} \left( \sum_{n=0}^{\infty} A_n(a, c) b^n - \sum_{n=0}^{\infty} A_n(a, c) (bq)^n \right) \]
\[
\sum\left(\sum\left(\sum\right)\right)
\]

which can be rewritten as
\[
\sum_{n=0}^{\infty} c q^{-1}(1 - q^n)A_n(a, c) b^n
\]

\[
= \sum_{n=0}^{\infty} \{- (1 - a q^{-n})A_n(a, c q^{-1}) + (1 - a q^{-n})A_n(a, c)\} b^{n+1}
\]

\[
= \sum_{n=0}^{\infty} - (1 - a q^{-n})\{A_n(a, c q^{-1}) - A_n(a, c)\} b^{n+1}.
\]

If the coefficients of \(b^n\) on both sides are equated, the result is:
\[
A_n(a, c) = \frac{(1 - a q^{-1-n})}{(1 - q^n)} \left\{ A_{n-1}(a, c q^{-1}) - A_{n-1}(a, c) \right\}
\]

\[
= \frac{a q^{-1-n}(1 - q^{-n-1}/a)}{(1 - q^n)} \theta_c \{A_{n-1}(a, c)\}.
\]

In an iterative process, the following is obtained:
\[
A_n(a, c) = \frac{a^n q^{-\binom{n}{2}}(1/a; q)_n}{(q; q)_n} \theta_c^n \{A_0(a, c)\}. \tag{5.4}
\]

By set \(b = 0\) in (5.3), we obtain
\[
f(a, 0, c) = A_0(a, c). \tag{5.5}
\]

If (5.5) is substituted into (5.4) and then the result is substituted into (5.3), the following is obtained:
\[
f(a, b, c) = \sum_{n=0}^{\infty} A_n(a, c) b^n
\]

\[
= \sum_{n=0}^{\infty} q^{-\binom{n}{2}}(1/a; q)_n (ab)^n \theta_c^n \{f(a, 0, c)\}
\]

\[
= \mathcal{S}(a, b; \theta_c) \{f(a, 0, c)\}.
\]

This completes the proof.

**Theorem 5.2** We have
\[
\mathcal{S}(a, b; \theta_c)\{(ct; q)\}_\infty = \frac{(ct; q)_\infty}{(bt; q)_\infty} \sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}
\]

provided that \(|bt| < 1\).

*Proof.* The Proposition 5.1 is used in order to prove this theorem. Rewriting the \(q\)-difference equation (5.1) the result is as follows:
\[
c q^{-1}[f(a, b, c) - f(a, bq, c)]
\]

\[
- b \left\{ a f\left(\frac{b}{q}, c\right) - f\left(\frac{a}{q}, c\right) + f(a, b, c) - a f\left(\frac{b}{q}, c\right) \right\} = 0 \tag{5.6}
\]

Let
Now, we can use the \( q \)-Gospers algorithm \([14,15,16]\) to verify that \( f(a,b,c) \) satisfies the \( q \)-difference equation (5.6). Setting \( b = 0 \) in (5.7), the result is:

\[
f(a,0,c) = (ct; q)_\infty.
\]

(5.8)

If (5.7) and (5.8) are substituted into (5.2), the result will be as follows:

\[
(ct; q)_\infty \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} (-1)^k q^{-\frac{k}{2}} (abt)^k = \$\{a,b,\theta_c\}(ct; q)_\infty.
\]

By using (2.1), we get the required result.

Now the \( q \)-difference equation (5.1) can be rewritten as:

\[
(xq^{-1} - b)f(a,b,x) - xq^{-1} f(a,bq,x) = abf\left(\frac{b}{q},\frac{x}{q}\right) + bf\left(\frac{b}{q},\frac{x}{q}\right) + abf\left(\frac{b}{q},x\right) = 0.
\]

(5.9)

**Theorem 5.3** Let \( f(a,b,x) \) be a three variables analytic function in a neighborhood of \( (a,b,x) = (0,0,0) \in \mathbb{C}^3 \) satisfying the \( q \)-difference equation (5.9) and \( f(a,0,x) \) has the following series expansion

\[
f(a,0,x) = \sum_{n=0}^{\infty} A_n P_n(y,x),
\]

where \( A_n \) is independent of \( x \), then

\[
f(a,b,x) = \sum_{n=0}^{\infty} A_n Z_n(x,y,a,b; q).
\]

(5.10)

**Proof.** Setting \( c = x \) in equation (5.2), this is obtained:

\[
f(a,b,x) = \$\{a,b; \theta_x\}\{f(a,0,x)\}
\]

\[
= \$\{a,b; \theta_x\}\left(\sum_{n=0}^{\infty} A_n P_n(y,x)\right)
\]

\[
= \sum_{n=0}^{\infty} A_n \$\{a,b; \theta_x\}\{P_n(y,x)\}
\]

\[
= \sum_{n=0}^{\infty} A_n Z_n(x,y,a,b; q).
\]

Next, the generating function for polynomials \( Z_n(x,y,a,b; q) \) is reproved with the use of the \( q \)-difference equation method.

**Theorem 5.4** (The generating function for \( Z_n(x,y,a,b; q)\)). We have

\[
\sum_{n=0}^{\infty} Z_n(x,y,a,b; q) \left(\frac{t^n}{(q; q)_n}\right) = \frac{(xt; q)_\infty}{(yt, bt; q)_\infty} \sum_{k=0}^{\infty} \frac{(a q)^k}{(q, q/ bt; q)_k}, \quad \max\{|yt|, |bt|\} < 1.
\]

**Proof.** Let \( f(a,b,x) \) be the right-hand side of the equation above:

\[
f(a,b,x) = \frac{(xt; q)_\infty}{(yt, bt; q)_\infty} \sum_{k=0}^{\infty} \frac{(a q)^k}{(q, q/ bt; q)_k}
\]
Employing the same technique used in Theorem 5.2, it can be demonstrated that the \( q \)-difference equation (5.9) is satisfied by (5.11). Setting \( b = 0 \) in (5.11), then we get

\[
f(a, 0, x) = \frac{(xt; q)_{\infty}}{(yt; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} (-1)^k q^{-\binom{k}{2}} (abt)^k.
\]

(by using (2.1)) \hspace{1cm} (5.11)

With Theorem 5.3, the following results is obtained:

\[
A_n = \frac{t^n}{(q; q)_n}.
\]

By using (5.10), we get the required result.

Conclusions

The polynomials \( Z_n(x, y, a, b; q) \) has been introduced and identity is given to establish the operator \( \mathcal{S}(a, b; \theta_q) \). Also, the operator proof for the generating function with its extension and the Rogers formula for \( Z_n(x, y, a, b; q) \) is provided. In addition, we introduce a solution to a \( q \)-difference equation then it is expressed in terms of the operator. We also use the \( q \)-difference method to recover an identity of the operator \( \mathcal{S}(a, b; \theta_q) \) and the generating function for the polynom\[ials \( Z_n(x, y, a, b; q) \). Finally, many results and outcomes are given.

References

