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The Operator $\mathbb{S}(a, b; \theta_x)$ for the Polynomials $Z_n(x, y, a, b; q)$

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Abstract

In this work, we give an identity that leads to establishing the operator $\mathbb{S}(a, b; \theta_x)$. Also, we introduce the polynomials $Z_n(x, y, a, b; q)$. In addition, we provide Operator proof for the generating function with its extension and the Rogers formula for $Z_n(x, y, a, b; q)$. The generating function with its extension and the Rogers formula for the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ are deduced. The Rogers formula for $Z_n(x, y, a, b; q)$ allows to obtain the inverse linearization formula for $Z_n(x, y, a, b; q)$, which allows to deduce the inverse linearization formula for $h_n(x, y|q)$. A solution to a q-difference equation is introduced and the solution is expressed in terms of the operators $\mathbb{S}(a, b; \theta_x)$. The q-difference method is used to recover an identity of the operator $\mathbb{S}(a, b; \theta_x)$ and the generating function for the polynomials $Z_n(x, y, a, b; q)$.

Keywords: The bivariate Rogers-Szegö polynomials, Generating function, Rogers formula, Inverse linearisation formula, q-difference equation.

تطبيقات المؤثر ${}_r\Phi_s$ في التكاملات-q

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الخلاصة

نعطي متطابقة تقودنا إلى إنشاء المؤثر $\mathbb{S}(a, b; \theta_x)$. أيضًا، نعرف متعددات الحدود $Z_n(x, y, a, b; q)$. نعطي برهان المؤثر للدالة المولدة وتوسيعها وصيغة روجرز لـ $Z_n(x, y, a, b; q)$. بعد ذلك يتم استنتاج الدالة المولدة وتوسيعها وصيغة روجرز لمتعددات حدود روجرز– زيجو ثنائية المتغير $h_n(x, y|q)$. تمكننا صيغة روجرز لـ $Z_n(x, y, a, b; q)$ باشتقاق الصيغة الخطية العكسية لـ $Z_n(x, y, a, b; q)$، والتي تمكنا من خلالها ايجاد الصيغة الخطية العكسية لـ $h_n(x, y|q)$. يتم تقديم حل لمعادلة الفروقات-q ويتم التعبير عن الحل بدلالة المؤثر $\mathbb{S}(a, b; \theta_x)$. تم استخدام طريقة الفروقات-q لاعادة برهان متطابقة للمؤثر $\mathbb{S}(a, b; \theta_x)$ والدالة المولدة لمتعددات الحدود $Z_n(x, y, a, b; q)$.

1. Introduction

The definitions and notations for the basic hypergeometric series [1] are adopted as follows:

Let $0 < q < 1$. The definition of the q-shifted factorial is:

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$$(a;q)_n = \begin{cases} 1 & if \quad n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & if \quad n=1,2,3,\dots \end{cases}$$

The definition

$$(a,q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

is also given.

The researchers employed the following notation for the multiple q-shifted factorials:

$$(a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \quad n=1,2,3,\dots$$

The definition of the basic hypergeometric series ${}_r\phi_s$ is:

$${}_r\phi_s\left(\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_s \end{matrix};q,x\right) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n.$$

The q-binomial coefficients is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

The Cauchy identity is given by:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1. \tag{1.1}$$

The particular cases of Cauchy identity were identified by Euler:

$$\sum_{n=0}^{\infty}\frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x|<1. \tag{1.2}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}x^k}{(q;q)_k} = (x;q)_\infty. \tag{1.3}$$

Euler's identity (1.3) can be expressed in a finite form as [2]

$$(x;q)_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}x^k. \tag{1.4}$$

The following identities will commonly occur in this paper [1]:

$$(q/a;q)_k = (-a)^{-k}q^{\binom{k+1}{2}_2}(aq^{-k};q)_\infty/(a;q)_\infty. \tag{1.5}$$

The Cauchy polynomials are defined as follows [3,4]:

$$P_k(x;y) = (x-y)(x-qy)\cdots(x-yq^{k-1}) = (y/x;q)_k x^k,$$

with the generating function:

$$\sum_{k=0}^{\infty}P_k(x,y)\frac{t^k}{(q,q)_k} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt|<1. \tag{1.6}$$

Another version of the Cauchy polynomials is given as follows:

$$P_n(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}y^k x^{n-k}.$$

The bivariate Rogers-Szegö polynomials were presented by Chen *et al*. [5] in 2003,

$$h_n(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}P_k(x,y),$$

with this generating function

$$\sum_{n=0}^{\infty}h_n(x,y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}, \quad \max\{|t|,|xt|\}<1. \tag{1.7}$$

The Rogers formula for $h_n(x,y|q)$ was derived by Chen et al. [6]

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t\right), \tag{1.8}$$

with the condition that $\max\{|t|,|xt|,|s|,|xs|\} < 1$.

The q^{-1}-Rogers-Szegö polynomials [7] is defined by:

$$h_n(a,b|q^{-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2-kn}\, a^k\, b^{n-k}.$$

The q-differential operator θ is defined by [8]:

$$\theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}. \tag{1.9}$$

The Leibniz rule for θ is [9]:

$$\theta^n\{f(a)g(a)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{1.10}$$

We can easily verify the identities:

$$\theta_x^k\{P_n(y,x)\} = (-1)^k \frac{(q;q)_n}{(q;q)_{n-k}} P_{n-k}(y,x). \tag{1.11}$$

$$\theta_x^k\{(xt;q)_\infty\} = (-1)^k t^k (xt;q)_\infty. \tag{1.12}$$

The q-exponential operator $E(b\theta)$ was defined by Chen and Liu [9] in 1998 as follows:

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}(b\theta)^k}{(q;q)_k}.$$

In 2007, the Cauchy companion operator $E(a,b;\theta)$ was presented by Fang [10] as follows:

$$E(a,b;\theta) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(-b\theta)^n.$$

The Cauchy operator was defined by Chen and Gu [11] in 2008 as follows:

$$T(a,b;D_q) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}(bD_q)^n.$$

In 2010, the solutions of q-difference equation are obtained, and the solution is expressed in terms of the operator $E(b\theta)$, by Liu [7], who derived Mehler's formula for the q^{-1}-Rogers-Szegö polynomials $h_n(a,b|q^{-1})$.

In 2010, the solutions of q-difference equations were obtained by Zhu [12], who expressed the solutions in terms of the operator $T(-\frac{1}{a}, ab; \theta)$.

In 2010, the solutions of q-difference equations were obtained by Abdul Hussein [13] who expressed the solutions in terms of the operator $E(a,b;\theta)$. The generating function, the Mehler formula and the Rogers formula for the Al-Salam-Carlits polynomials $U_n(x,y,a;q)$ are also proved.

Our paper is structured as follows: Section 2 contains an identity, which leads to creating the operator $\mathbb{S}(a,b;\theta_x)$. We also present the polynomials $Z_n(x,y,a,b;q)$. We used the operator $\mathbb{S}(a,b;\theta_x)$ to represent the polynomials $Z_n(x,y,a,b;q)$. In section 3, we present an operator proof for the generating function as well as its extension for the polynomials $Z_n(x,y,a,b;q)$. The generating function and its extension for the polynomials $h_n(x,y|q)$ are then deduced. Section 4 presents an operator proof of the Rogers formula for the polynomials

$Z_n(x,y,a,b;q)$. The Rogers formula for the polynomials $h_n(x,y|q)$ is then recovered. The Rogers formula for $Z_n(x,y,a,b;q)$ allows to derive the inverse linearization formula for $Z_n(x,y,a,b;q)$, from which we can get the inverse linearization formula for $h_n(x,y|q)$. Section 5 introduces and solves a q-difference equation. The operator $\mathbb{S}(a,b;\theta_x)$ is then used to describe the solution. This approach is used to confirm identity of the operator $\mathbb{S}(a,b;\theta_x)$ and the generating function for the polynomials $Z_n(x,y,a,b;q)$.

2. The Operator $\mathbb{S}(a,b;\theta_x)$ for the Polynomials $Z_n(x,y,a,b;q)$

An identity is provided in this section. This identity is the inspiration to introduce the operator $\mathbb{S}(a,b;\theta_x)$. Furthermore, the polynomials $Z_n(x,y,a,b;q)$ were presented.

Theorem 2.1 We have

$$\sum_{k=0}^{\infty}\frac{(1/a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(abt)^k=\frac{1}{(bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q,q/bt;q)_k}. \tag{2.1}$$

Proof. By using (1.4), the result is obtained as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(1/a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(abt)^k\\
&=\sum_{k=0}^{\infty}\frac{\sum_{i=0}^{k}\begin{bmatrix}k\\ i\end{bmatrix}(-1)^i q^{\binom{i}{2}}(1/a)^i}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(abt)^k\\
&=\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^k q^{-\binom{k}{2}}(abt)^k\frac{(-1)^i q^{\binom{i}{2}}\ (1/a)^i}{(q;q)_i(q;q)_{k-i}}\\
&=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{k+i}q^{-\binom{k}{2}-\binom{i}{2}-ik}(abt)^{k+i}\frac{(-1)^i q^{\binom{i}{2}}(1/a)^i}{(q;q)_i(q;q)_k}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}(abt)^k}{(q;q)_k}\sum_{i=0}^{\infty}\frac{(btq^{-k})^i}{(q;q)_i}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}(abt)^k}{(q;q)_k}\frac{1}{(btq^{-k};q)_\infty}\qquad\text{(by using (1.2))}\\
&=\frac{1}{(bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{-\binom{k}{2}}(abt)^k}{(q,q/bt;q)_k(-bt/q)^k q^{-\binom{k}{2}}}\qquad\text{(by using (1.5))}\\
&=\frac{1}{(bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q,q/bt;q)_k}.
\end{aligned}$$

Now, let θ be defined as in (1.9). Taking inspiration from identity (2.1), the following operator is now presented:

$$\mathbb{S}(a,b,\theta_x)=\sum_{k=0}^{\infty}\frac{(1/a;q)_k}{(q;q)_k}q^{-\binom{k}{2}}(ab\theta_x)^k, \tag{2.2}$$

and then the following polynomials are introduced:

$$Z_n(x,y,a,b;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}b^k(aq^{1-k},q)_k P_{n-k}(y,x).$$

Setting $a = 0$, $b = 1$ and exchanging x and y in $Z_n(x, y, a, b; q)$, we get the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$, signifying that the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ is a particular case of the polynomials $Z_n(x, y, a, b; q)$.

From (1.11) and (2.2), the following representation for the polynomials $Z_n(x, y, a, b; q)$ is obtained:

$$\mathbb{S}(a, b; \theta_x)\{P_n(y, x)\} = Z_n(x, y, a, b; q). \tag{2.3}$$

By using (1.12), it is easy to prove that

$$\mathbb{S}(a, b; \theta_x)\{(xt; q)_\infty\} = \frac{(xt; q)_\infty}{(bt; q)_\infty}\sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}. \tag{2.4}$$

3. The generating function for $Z_n(x, y, a, b; q)$

In this section, the operator proof for the generating function and its extension for the polynomials $Z_n(x, y, a, b; q)$ are provided. The generating function and its extension for the polynomials $h_n(x, y|q)$ are then deduced.

Theorem 3.1 (The generating function for $Z_n(x, y, a, b; q)$)**.** We have

$$\sum_{n=0}^{\infty} Z_n(x, y, a, b; q)\frac{t^n}{(q; q)_n} = \frac{(xt; q)_\infty}{(yt, bt; q)_\infty}\sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}, \tag{3.1}$$

Provided that $\max\{|yt|, |bt|\} < 1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty} Z_n(x, y, a, b; q)\frac{t^n}{(q; q)_n}\\
&= \sum_{n=0}^{\infty} \mathbb{S}(a, b; \theta_x)\{P_n(y, x)\}\frac{t^n}{(q; q)_n} \quad \text{(by using (2.3))}\\
&= \frac{1}{(yt; q)_\infty}\mathbb{S}(a, b; \theta_x)\{(xt; q)_\infty\} \quad \text{(by using (1.6))}\\
&= \frac{(xt; q)_\infty}{(yt, bt; q)_\infty}\sum_{k=0}^{\infty} \frac{(aq)^k}{(q, q/bt; q)_k}. \quad \text{(by using (2.4))}
\end{aligned}$$

Setting $a = 0$, $b = 1$ and exchanging x and y in the generating function for the polynomials $Z_n(x, y, a, b; q)$, we obtain the generating function of the bivariate Rogers-Szegö polynomials $h_n(x, y|q)$ (1.7).

Lemma 3.2 Let $\mathbb{S}(a, b, \theta_x)$ be defined as in (2.2). Then

$$\begin{aligned}
\mathbb{S}(a, b; \theta_x)\{P_n(y, x)(xs; q)_\infty\} &= \frac{(xs; q)_\infty}{(bs; q)_\infty}\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{(1/a, q/xs; q)_m}{(q/bs; q)_m} q^{-\binom{m}{2}}(-ax)^m P_{n-m}(y, x)\\
&\times \sum_{k=0}^{\infty} \frac{(aq^{1-m})^k}{(q, q/bsq^{-m}; q)_k}, \quad |bs| < 1.
\end{aligned} \tag{3.2}$$

Proof.

$$\begin{aligned}
&\mathbb{S}(a, b; \theta_x)\{P_n(y, x)(xs; q)_\infty\}\\
&= \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} q^{-\binom{k}{2}}(ab)^k \theta_x^k\{P_n(y, x)(xs; q)_\infty\} \quad \text{(by using (2.2))}\\
&= \sum_{k=0}^{\infty} \frac{(1/a; q)_k}{(q; q)_k} q^{-\binom{k}{2}}(ab)^k \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix} \theta_x^m\{P_n(y, x)\}\theta_x^{k-m}\{(xsq^{-m}; q)_\infty\} \quad \text{(by using (1.10))}
\end{aligned}$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(\frac{1}{a};q)_{k+m}\frac{q^{-\binom{k+m}{2}}(ab)^{k+m}}{(q;q)_m(q;q)_k}\theta_x^m\{P_n(y,x)\}\theta_x^k\{(xsq^{-m};q)_\infty\}$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1/a;q)_m(q^m/a;q)_k q^{-\binom{k}{2}}q^{-\binom{m}{2}}q^{-mk}(ab)^{k+m}}{(q;q)_m(q;q)_k}(-1)^m\frac{(q;q)_n}{(q;q)_{n-m}}P_{n-m}(y,x)$$
$$\times(-sq^{-m})^k(xsq^{-m};q)_\infty \quad \big(\text{by using } (1.11),(1.12)\big)$$

$$= \sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^m(1/a;q)_m q^{-\binom{m}{2}}(ab)^m(xsq^{-m};q)_\infty P_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(1/q^{-m}a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(absq^{-2m})^k$$

$$= (xs;q)_\infty\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^m(1/a,q/xs;q)_m q^{-\binom{m}{2}}(ab)^m(-xs)^m q^{-\binom{m}{2}-m}P_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(1/q^{-m}a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(absq^{-2m})^k \quad \big(\text{by using } (1.5)\big)$$

$$= (xs;q)_\infty\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(1/a,q/xs;q)_m q^{-m^2}(abxs)^m P_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(1/q^{-m}a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(absq^{-2m})^k$$

$$= (xs;q)_\infty\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(1/a,q/xs;q)_m q^{-m^2}(abxs)^m P_{n-m}(y,x)$$
$$\times\frac{1}{(bsq^{-m};q)_\infty}\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{-m};q)_k} \quad \big(\text{by using } (2.1)\big)$$

$$= \frac{(xs;q)_\infty}{(bs;q)_\infty}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(1/a,q/xs;q)_m}{(q/bs;q)_m(-1)^m q^{-\binom{m}{2}-m}(bs)^m}q^{-m^2}(abxs)^m P_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{-m};q)_k} \quad \big(\text{by using } (1.5)\big)$$

$$= \frac{(xs;q)_\infty}{(bs;q)_\infty}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(1/a,q/xs;q)_m}{(q/bs;q)_m}q^{-\binom{m}{2}}(-ax)^m P_{n-m}(y,x)\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{-m};q)_k}.$$

Lemma 3.3 (Extension of the generating function for $Z_n(x,y,a,b;q)$). We have

$$\sum_{n=0}^{\infty}Z_{n+k}(x,y,a,b;q)\frac{t^n}{(q;q)_n}$$
$$= \frac{(xtq^k;q)_\infty}{(yt,btq^k;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}\frac{(1/a,q/xtq^k;q)_m}{(q/btq^k;q)_m}(-1)^m q^{-\binom{m}{2}}(ax)^m$$
$$\times P_{k-m}(y,x)\sum_{n=0}^{\infty}\frac{(aq^{1-m})^n}{(q,q/btq^{k-m};q)_n}, \quad \max\{|yt|,|bt|\}<1. \tag{3.3}$$

Proof.

$$\sum_{n=0}^{\infty} Z_{n+k}(x,y,a,b;q)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty} \mathbb{S}(a,b;\theta_x)\{P_{n+k}(y,x)\}\frac{t^n}{(q;q)_n} \quad \big(\text{by using } (2.3)\big)$$
$$=\mathbb{S}(a,b;\theta_x)\left\{\sum_{n=0}^{\infty} P_{n+k}(y,x)\frac{t^n}{(q;q)_n}\right\}$$
$$=\mathbb{S}(a,b;\theta_x)\left\{\sum_{n=0}^{\infty} P_k(y,x)P_n(y,xq^k)\frac{t^n}{(q;q)_n}\right\}$$
$$=\mathbb{S}(a,b;\theta_x)\left\{P_k(y,x)\frac{(xtq^k;q)_\infty}{(yt;q)_\infty}\right\} \quad \big(\text{by using } (1.6)\big)$$
$$=\frac{1}{(yt;q)_\infty}\mathbb{S}(a,b;\theta_x)\{P_k(y,x)(xtq^k;q)_\infty\}$$
$$=\frac{(xtq^k;q)_\infty}{(yt,btq^k;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}\frac{(1/a,q/xtq^k;q)_m}{(q/btq^k;q)_m}(-1)^m q^{-\binom{m}{2}}(ax)^m P_{k-m}(y,x)$$
$$\times\sum_{n=0}^{\infty}\frac{(aq^{1-m})^n}{(q,q/btq^{k-m};q)_n}. \quad (\text{by using } (3.2))$$

Setting $a=0$, $b=1$ and exchanging x and y in the extension of generating function for the polynomials $Z_n(x,y,a,b;q)$ (3.3), an extension of the generating function of the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ is obtained:

$$\sum_{n=0}^{\infty} h_{n+k}(x,y|q)\frac{t^n}{(q;q)_n}=\frac{(ytq^k;q)_\infty}{(xt,tq^k;q)_\infty}\sum_{m=0}^{k}\begin{bmatrix}k\\m\end{bmatrix}\frac{(q/ytq^k;q)_m}{(q/tq^k;q)_m}y^m P_{k-m}(x,y),$$

Provided that $\max\{|xt|,|t|\}<1$.

4. The Rogers formula for $Z_n(x;y;a;b;q)$

This section provides the operator proof of the Rogers formula for the polynomials Z_n, and the inverse linearization formula for $Z_n(x,y,a,b;q)$ is obtained. The Rogers formula and the inverse linearization formula for $h_n(x,y|q)$ are then recovered $h_n(x,y|q)$.

Theorem 4.1 (The Rogers formula for Z_n(x; y; a; b; q)). We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Z_{n+m}(x,y,a,b;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(xs;q)_\infty}{(ys,bs;q)_\infty}\sum_{m=0}^{\infty}\frac{(1/a;q)_m}{(q;q)_m}(-1)^m q^{-\binom{m}{2}}(abt)^m\sum_{n=0}^{\infty}\frac{(x/y,bs;q)_n}{(q,xs;q)_n}(yt)^n$$
$$\times\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^n;q)_k}, \qquad \max\{|ys|,|bs|\}<1. \tag{4.1}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Z_{n+m}(x,y,a,b;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{S}(a,b;\theta_x)\{P_{n+m}(y,x)\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\quad \text{(by using (2.3))}$$
$$=\mathbb{S}(a,b;\theta_x)\left\{\sum_{n=0}^{\infty}P_n(y,x)\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty}P_m(y,xq^n)\frac{s^m}{(q;q)_m}\right\}$$
$$=\mathbb{S}(a,b;\theta_x)\left\{\sum_{n=0}^{\infty}P_n(y,x)\frac{t^n}{(q;q)_n}\frac{(xsq^n;q)_\infty}{(ys;q)_\infty}\right\}\quad \text{(by using (1.6))}$$
$$=\frac{1}{(ys;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\mathbb{S}(a,b;\theta_x)\{P_n(y,x)(xsq^n;q)_\infty\}$$
$$=\frac{1}{(ys;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{(xsq^n;q)_\infty}{(bsq^n;q)_\infty}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(1/a,q/xsq^n;q)_m}{(q/bsq^n;q)_m}q^{-\binom{m}{2}}(-ax)^mP_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{n-m};q)_k}\quad \text{(by using (3.2))}$$
$$=\frac{1}{(ys;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{t^n(1/a;q)_mq^{-\binom{m}{2}}(-ax)^m}{(q;q)_m(q;q)_{n-m}}\frac{(xsq^n;q)_\infty(q/xsq^n;q)_m}{(bsq^n;q)_\infty(q/bsq^n;q)_m}P_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{n-m};q)_k}$$
$$=\frac{1}{(ys;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{t^n(1/a;q)_mq^{-\binom{m}{2}}(-ax)^m}{(q;q)_m(q;q)_{n-m}}\frac{(xsq^{n-m};q)_\infty(-xsq^n)^{-m}q^{\binom{m+1}{2}}}{(bsq^{n-m};q)_\infty(-bsq^n)^{-m}q^{\binom{m+1}{2}}}P_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{n-m};q)_k}\quad \text{(by using (1.5))}$$
$$=\frac{1}{(ys;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{t^n(1/a;q)_mq^{-\binom{m}{2}}(-ax)^m}{(q;q)_m(q;q)_{n-m}}\frac{(xsq^{n-m};q)_\infty}{(bsq^{n-m};q)_\infty}(b/x)^mP_{n-m}(y,x)$$
$$\times\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^{n-m};q)_k}$$
$$=\frac{1}{(ys;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^{n+m}(1/a;q)_mq^{-\binom{m}{2}}(-ab)^m}{(q;q)_m(q;q)_n}\frac{(xsq^n;q)_\infty}{(bsq^n;q)_\infty}P_n(y,x)\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^n;q)_k}$$
$$=\frac{(xs;q)_\infty}{(ys,bs;q)_\infty}\sum_{m=0}^{\infty}\frac{(1/a;q)_m(-1)^mq^{-\binom{m}{2}}(abt)^m}{(q;q)_m}\sum_{n=0}^{\infty}\frac{(x/y,bs;q)_n}{(q,xs;q)_n}(yt)^n\sum_{k=0}^{\infty}\frac{(aq^{1-m})^k}{(q,q/bsq^n;q)_k}.$$

Set $a=0$, $b=1$ and exchang x and y in the Rogers formula for the polynomials $Z_n(x,y,a,b;q)$ (4.1), the Rogers formula for the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$ (1.8) is recovered.

The Rogers formula for $Z_n(x,y,a,b;q)$ (4.1) can be written differently, this allows to obtain the inverse linearisation formula of $Z_n(x,y,a,b;q)$ as follows:

Lemma 4.2 *For $n, m \geq 0$, we have*

$$Z_{n+m}(x,y,a,b;q) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (1/a;q)_i (-1)^i q^{-\binom{i}{2}} (ab)^i P_{n-i}(y,x) Z_m(xq^{n-i}, y, aq^{-i}, bq^{n-i}; q). \quad (4.2)$$

Proof. Rewrite the Rogers formula (4.1) as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Z_{n+m}(x,y,a,b;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \frac{(xs;q)_\infty}{(ys,bs;q)_\infty}\sum_{i=0}^{\infty}\frac{(1/a;q)_i}{(q;q)_i}(-1)^i q^{-\binom{i}{2}}(abt)^i \sum_{n=0}^{\infty}\frac{(x/y,bs;q)_n}{(q,xs;q)_n}(yt)^n \sum_{k=0}^{\infty}\frac{(aq^{1-i})^k}{(q,q/bsq^n;q)_k}$$
$$= \sum_{i=0}^{\infty}\frac{(1/a;q)_i}{(q;q)_i}(-1)^i q^{-\binom{i}{2}}(abt)^i \sum_{n=0}^{\infty}\frac{(x/y;q)_n}{(q;q)_n}(yt)^n \frac{(xsq^n;q)_\infty}{(ys,bsq^n;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq^{1-i})^k}{(q,q/bsq^n;q)_k}$$
$$= \sum_{i=0}^{\infty}\frac{(1/a;q)_i}{(q;q)_i}(-1)^i q^{-\binom{i}{2}}(ab)^i \sum_{n=0}^{\infty} P_n(y,x)\frac{t^{n+i}}{(q;q)_n}\sum_{k=0}^{\infty} Z_k(xq^n, y, aq^{-i}, bq^n; q)\frac{s^k}{(q;q)_k}.$$

(by using (3.3))

Equating the coefficients of $t^n s^m$ on both sides, we get the required result.

Set $a = 0$, $b = 1$ and exchang x and y in the inverse linearisation formula for $Z_n(x,y,a,b;q)$ (4.2), the inverse linearisation formula for the bivariate Rogers-Szegö polynomials for polynomials $h_n(x,y|q)$ is obtained:

$$h_{n+m}(x,y|q) = \sum_{i=0}^{n}\sum_{j=0}^{m}\begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} m \\ j \end{bmatrix} q^{i(m-j)} P_{i+j}(x,y).$$

5. The q-Difference Equation and the $\mathbb{S}(a, b; \theta_x)$ operator

In this section, a q-difference equation is presented and solved. The solution is then expressed in terms of the operator $\mathbb{S}(a,b;\theta_c)$. With this method, an operator identity for the operator $\mathbb{S}(a,b;\theta_c)$ and the generating function for the polynomials $Z_n(x,y,a,b;q)$ were verified.

Proposition 5.1 Let $f(a,b,c)$ be an analytic function of three variables in a neighborhood of $(a,b,c) = (0,0,0) \in \mathbb{C}^3$ satisfying the q-difference equation

$$cq^{-1}\{f(a,b,c) - f(a,bq,c)\} = b\left\{af\left(a,\frac{b}{q},\frac{c}{q}\right) - f\left(a,b,\frac{c}{q}\right) + f(a,b,c)af(a,b/q,c)\right\}. \quad (5.1)$$

Then we have

$$f(a,b,c) = \mathbb{S}(a,b;\theta_c)\{f(a,0,c)\}, \quad (5.2)$$

where $\mathbb{S}(a,b;\theta_c)$ acts on the parameter c.

Proof. Let

$$f(a,b,c) = \sum_{n=0}^{\infty} A_n(a,c)b^n, \quad (5.3)$$

where A_n is independent of b. Substituting the (5.3) into (5.1) the following is obtained

$$cq^{-1}\left\{\sum_{n=0}^{\infty} A_n(a,c)b^n - \sum_{n=0}^{\infty} A_n(a,c)(bq)^n\right\}$$

$$= b\left\{a\sum_{n=0}^{\infty} A_n(a,c/q)(bq^{-1})^n - \sum_{n=0}^{\infty} A_n\left(a,\frac{c}{q}\right)b^n + \sum_{n=0}^{\infty} A_n(a,c)b^n - a\sum_{n=0}^{\infty} A_n(a,c)(bq^{-1})^n\right\}.$$

which can be rewritten as

$$\sum_{n=0}^{\infty} cq^{-1}(1-q^n)A_n(a,c)b^n$$
$$=\sum_{n=0}^{\infty}\{-(1-aq^{-n})A_n(a,cq^{-1})+(1-aq^{-n})A_n(a,c)\}b^{n+1}$$
$$=\sum_{n=0}^{\infty}-(1-aq^{-n})\{A_n(a,cq^{-1})-A_n(a,c)\}b^{n+1}.$$

If the coefficients of b^n on both sides are equated, the result is:

$$A_n(a,c) = -\frac{(1-aq^{1-n})}{(1-q^n)}\left\{\frac{A_{n-1}(a,cq^{-1})-A_{n-1}(a,c)}{cq^{-1}}\right\}$$
$$=\frac{aq^{1-n}(1-q^{n-1}/a)}{(1-q^n)}\theta_c\{A_{n-1}(a,c)\}.$$

In an iterative process, the following is obtained:

$$A_n(a,c) = \frac{a^n q^{-\binom{n}{2}}(1/a;q)_n}{(q;q)_n}\theta_c^n\{A_0(a,c)\}. \tag{5.4}$$

By set $b = 0$ in (5.3), we obtain

$$f(a,0,c) = A_0(a,c). \tag{5.5}$$

If (5.5) is substituted into (5.4) and then the result is substituted into (5.3), the following is obtained:

$$f(a,b,c) = \sum_{n=0}^{\infty} A_n(a,c)b^n$$
$$=\sum_{n=0}^{\infty}\frac{q^{-\binom{n}{2}}(1/a;q)_n}{(q;q)_n}(ab)^n\theta_c^n\{f(a,0,c)\}$$
$$=\mathbb{S}(a,b;\theta_c)\{f(a,0,c)\}.$$

This completes the proof.

Theorem 5.2 We have

$$\mathbb{S}(a,b;\theta_c)\{(ct;q)_\infty\} = \frac{(ct;q)_\infty}{(bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q,q/bt;q)_k},$$

provided that $|bt| < 1$.

Proof. The Proposition 5.1 is used in order to prove this theorem. Rewriting the q-difference equation (5.1) the result is as follows:

$$cq^{-1}\{f(a,b,c)-f(a,bq,c)\}$$
$$-b\left\{af\left(a,\frac{b}{q},\frac{c}{q}\right)-f\left(a,b,\frac{c}{q}\right)+f(a,b,c)-af\left(a,\frac{b}{q},c\right)\right\}=0 \tag{5.6}$$

Let

$$f(a,b,c)=\frac{(ct;q)_\infty}{(bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q,q/bt;q)_k}$$
$$=(ct;q)_\infty\sum_{k=0}^{\infty}\frac{(1/a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(abt)^k \tag{5.7}$$

Now, we can use the q-Gospers algorithm [14,15,16] to verify that $f(a,b,c)$ satisfies the q-difference equation (5.6). Setting $b=0$ in (5.7), the result is:

$$f(a,0,c)=(ct;q)_\infty. \tag{5.8}$$

If (5.7) and (5.8) are substituted into (5.2), the result will be as follows:

$$(ct;q)_\infty\sum_{k=0}^{\infty}\frac{(1/a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(abt)^k=\mathbb{S}(a,b,\theta_c)\{(ct;q)_\infty\}.$$

By using (2.1), we get the required result.

Now the q-difference equation (5.1) can be rewritten as :

$$(xq^{-1}-b)f(a,b,x)-xq^{-1}f(a,bq,x)-abf\left(a,\frac{b}{q},\frac{x}{q}\right)+bf\left(a,b,\frac{x}{q}\right)+abf\left(a,\frac{b}{q},x\right)$$
$$=0. \tag{5.9}$$

Theorem 5.3 Let $f(a,b,x)$ be a three variables analytic function in a neighborhood of $(a,b,x)=(0,0,0)\in\mathbb{C}^3$ satisfying the q-difference equation (5.9) and $f(a,0,x)$ has the following series expansion

$$f(a,0,x)=\sum_{n=0}^{\infty}A_nP_n(y,x),$$

where A_n is independent of x, then

$$f(a,b,x)=\sum_{n=0}^{\infty}A_nZ_n(x,y,a,b;q). \tag{5.10}$$

Proof. Setting $c=x$ in equation (5.2), this is obtained:

$$f(a,b,x)=\mathbb{S}(a,b;\theta_x)\{f(a,0,x)\}$$
$$=\mathbb{S}(a,b;\theta_x)\left\{\sum_{n=0}^{\infty}A_nP_n(y,x)\right\}$$
$$=\sum_{n=0}^{\infty}A_n\mathbb{S}(a,b;\theta_x)\{P_n(y,x)\}$$
$$=\sum_{n=0}^{\infty}A_nZ_n(x,y,a,b;q).$$

Next, the generating function for polynomials $Z_n(x,y,a,b;q)$ is reproved with the use of the q-difference equation method.

Theorem 5.4 (The generating function for $Z_n(x,y,a,b;q)$). We have

$$\sum_{n=0}^{\infty}Z_n(x,y,a,b;q)\frac{t^n}{(q;q)_n}=\frac{(xt;q)_\infty}{(yt,bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q,q/bt;q)_k},\qquad \max\{|yt|,|bt|\}<1.$$

Proof. Let $f(a,b,x)$ be the right-hand side of the equation above:

$$f(a,b,x)=\frac{(xt;q)_\infty}{(yt,bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(aq)^k}{(q,q/bt;q)_k}$$

$$=\frac{(xt;q)_\infty}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{(1/a;q)_k}{(q;q)_k}(-1)^k q^{-\binom{k}{2}}(abt)^k. \quad \big(\text{by using (2.1)}\big) \qquad (5.11)$$

Employing the same technique used in Theorem 5.2, it can be demonstrated that the q-difference equation (5.9) is satisfied by (5.11). Setting $b=0$ in (5.11), then we get

$$f(a,0,x)=\frac{(xt,q)_\infty}{(yt;q)_\infty}$$

$$=\sum_{n=0}^{\infty}P_n(y,x)\frac{t^n}{(q;q)_n}.$$

With Theorem 5.3, the following results is obtained:

$$A_n=\frac{t^n}{(q;q)_n}.$$

By using (5.10), we get the required result.

Conclusions

The polynomials $Z_n(x,y,a,b;q)$ has been introduced and identity is given to establish the operator $\mathbb{S}(a,b;\theta_x)$. Also, the operator proof for the generating function with its extension and the Rogers formula for $Z_n(x,y,a,b;q)$ is provided. In addition, we introduce a solution to a q-difference equation then it is expressed in terms of the operator. We also use the q-difference method to recover an identity of the operator $\mathbb{S}(a,b;\theta_x)$ and the generating function for the polynomials $Z_n(x,y,a,b;q)$. Finally, many results and outcomes are given.

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