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The development of the unified version mixing maps between arbitrary sets

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Abstract

The concepts of nonlinear mixed summable families and maps for the spaces that only non-void sets are developed. Several characterizations of the corresponding concepts are achieved and the proof for a general Pietsch Domination-type theorem is established. Furthermore, this work has presented plenty of composition and inclusion results between different classes of mappings in the abstract settings. Finally, a generalized notation of mixing maps and their characteristics are extended to a more general setting.

Keywords: Composition theorem; Hausdorff topological space; Nonlinear mixing maps; Pietsch Domination theorem; Set theory.

تحديث النسخة الموحدة من الدوال المختلطة بين مجموعات غير اختيارية

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الخلاصة:

تم تطوير مفاهيم الغير خطية للعائلات الجمعية المختلطة والدوال للمجموعات الغير اختيارية. بالاضافة الى انه تم تحقيق العديد من الخصائص المهمة لهذه المفاهيم. تم اثبات برهان النظرية العامة "Pietsch Domination-type" بالإضافة الى ذلك، هذا العمل قدم الكثير من نتائج التركيب بين مجموعات مختلفة من الدوال في الحالات المجردة. أخيرًا، تم توسيع المجموعة العامة للدوال المختلطة وخصائصهم الى حاله أكثر عمومية.

1. Notations and Preliminaries

In the beginning, some notations are introduced that will be used throughout this article. Let I be an index set. The symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ represent the sets of natural numbers, real numbers, and positive real numbers, respectively. The letter $\mathbb{K}$ denotes the field of real or complex numbers. Let A, B, C, C_1 and D be non-void sets and $\mathcal{H}$ be a non-void family of mappings from A into B. Let E, F and G be Banach spaces and the closed unit ball of a Banach space E is denoted by B_E. The dual space of E is denoted by E^*. The letters X and Y stand for pointed metric spaces. Suppose T be a map from X into Y, then T can be defined to be Lipschitz if there is a nonnegative constant C such that $d_Y(Tx_1, Tx_2) \leq C\ d_X(x_1, x_2)$, for all x_1, x_2 in X, where C is the Lipschitz constant of $T(\mathrm{L}ip(T))$. In addition, let the space $X^{\#}$ be the Lipschitz dual of X that is the Banach space of real-valued maps defined on X send the

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space point 0 to 0 with the Lipschitz norm $\mathrm{L}ip(\cdot)$.Let K and W be compact Hausdorff topological spaces. The symbols $\boldsymbol{W}(W)$, $\boldsymbol{W}(B_{E^*})$ and $\boldsymbol{W}(B_{X^\#})$ stand for the set of all Borel probability measures defined on W, B_{E^*} and $B_{X^\#}$, respectively. The value of a at the element x is denoted by $\langle x, a\rangle$.

2. Introduction

The usual mathematical problems include nonlinear operators, occasionally influential on arbitrary sets with few (or none) algebraic structures, hence the extension of linear mechanisms to the nonlinear setting, besides its essential mathematical interest, is an important duty for potential applications. The full general version of maps (with no structure of the spaces included) would certainly be interesting for potential applications. Botelho *et al.* [1] defined the concept of **R**–**S**-abstract p-summing map as follows. Let $0 < p < \infty$. A mapping $T \in \mathcal{H}$ is said to be **R**–**S**-abstract p-summing if there is a constant $\delta > 0$ such that

$$\left[\sum_{j=1}^{m} \boldsymbol{S}(T, c_j, b_j)^p\right]^{\frac{1}{p}} \leq \delta \cdot \sup_{\varphi\in K}\left[\sum_{j=1}^{m} \boldsymbol{R}(\varphi, c_j, b_j)^p\right]^{\frac{1}{p}} \tag{1}$$

for all $c_1, \cdots, c_m \in C$, $b_1, \cdots, b_m \in G$ and $m \in \mathbb{N}$. The infimum of such constants δ is denoted by $\pi_{RS,p}(T)$. They established a quite general Pietsch Domination-type Theorem under certain hypotheses on **R** and **S** as follows.

1- For each $T \in \mathcal{H}$, there is $c_0 \in C$ such that $\boldsymbol{R}(\varphi, c_0, b) = \boldsymbol{S}(T, c_0, b) = 0$ for every $\varphi \in K$ and $b \in G$.

2- The mapping $\boldsymbol{R}_{c,b}: K \longrightarrow [0, \infty)$ is defined by $\boldsymbol{R}_{c,b}(\varphi) = \boldsymbol{R}(c, b, \varphi)$ which is continuous for every $c \in C$ and $b \in G$.

3- It holds that $\boldsymbol{R}(\varphi, c, \eta\ b) \leq \eta \cdot \boldsymbol{R}(\varphi, c, b)$ and $\eta \cdot \boldsymbol{S}(T, c, b) \leq \boldsymbol{S}(T, c, \eta\ b)$ for every $\varphi \in K$, $c \in C$, $0 \leq \eta \leq 1$, $b \in G$ and $T \in \mathcal{H}$.

Theorem 1 (Botelho *et al.* [1]) If **R** and **S** satisfy conditions **1**, **2** and **3** and $0 < \mathrm{p} < \infty$, then $\mathrm{T} \in \mathcal{H}$ is **R**–**S**-abstract p-summing map if and only if there are constant $\delta > 0$ and Borel probability measure ν on K such that

$$\boldsymbol{S}(T, c, b) \leq \delta \cdot \left[\int_K \boldsymbol{R}(c, b, \varphi)^p d\nu(\varphi)\right]^{\frac{1}{p}},$$

whenever $c \in C$ and $b \in G$.

Building upon the observation was made by M. Mendel and G. Schechtman that appears in P. Farmer and Johnson [2]. D. Pellegrino and J. Santos [3]defined the equivalent to inequality (1) as follows. A mapping $T \in \mathcal{H}$ is said to be **R**–**S**-abstract p-summing if there is a constant $\delta > 0$ such that

$$\left[\sum_{j=1}^{m} \lambda_j\ \boldsymbol{S}(T, c_j, b_j)^p\right]^{\frac{1}{p}} \leq \delta \cdot \sup_{\varphi\in K}\left[\sum_{j=1}^{m} \lambda_j\ \boldsymbol{R}(\varphi, c_j, b_j)^p\right]^{\frac{1}{p}} \tag{2}$$

for all $c_1, \cdots, c_m \in C$, $b_1, \cdots, b_m \in G$, $\lambda_1, \cdots, \lambda_m \in \mathbb{R}^+$, and $m \in \mathbb{N}$. From inequality (2) and invoking Theorem 2.1 in [1] Boteho et al. proved a general Pietsch Domination-type Theorem with no assumption on **S** and just supposing that **R** satisfies condition **2** as follows.

Theorem 2 (Pellegrino and Santos [3].) If **R** satisfies condition **2**, and $0 < \mathrm{p} < \infty$, then $\mathrm{T} \in \mathcal{H}$ be **R**–**S**-abstract p-summing map if and only if there is constant $\delta > 0$ and Borel probability measure ν on K such that

$$\boldsymbol{S}(T, c, b) \leq \delta \cdot \left[\int_K \boldsymbol{R}(c, b, \varphi)^p d\nu(\varphi)\right]^{\frac{1}{p}},$$

whenever $c \in C$ and $b \in G$.

Pellegrino *et al.* [4] defined the concept of $\boldsymbol{R_1}, \ldots, \boldsymbol{R}_t$–**S**-abstract $(p_1, \ldots, p_t)$-summing map as follows. Let $0 < p < \infty$. A map T from $A_1 \times \cdots \times A_t$ into B is called $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_t$–**S**-abstract $(p_1, \ldots, p_t)$-summing if there is a constant $\delta \geq 0$ such that

$$\left[\sum_{j=1}^{m} \boldsymbol{S}(T, c_j^1, \ldots, c_j^r, b_j^1, \ldots, b_j^t)^p\right]^{\frac{1}{p}} \leq \delta \cdot \prod_{k=1}^{s} \sup_{\varphi\in K_k}\left[\sum_{j=1}^{m} \left|\boldsymbol{R}_k(c_j^1, \ldots, c_j^r, b_j^k, \varphi)\right|^{p_k}\right]^{\frac{1}{p_k}}$$

for all $c_1^1, \cdots, c_m^r \in C_s$, $b_1^1, \cdots, b_m^l \in G_l$, $m \in \mathbb{N}$ and $(s, l) \in \{1, \dots, r\} \times \{1, \dots, t\}$. They proved a quite general Pietsch Domination Theorem as follows.

Theorem 3 (Pellegrino *et al.* [4]) A map $T \in \mathcal{H}$ is **R**–**S**-abstract p-summing if and only if there are constant $\delta > 0$ and Borel probability measure ν on K such that

$$\boldsymbol{S}(T, c^1, \dots, c^r, b^1, \dots, b^t) \leq \delta \cdot \prod_{j=1}^{t} \left(\int_{K_j} \boldsymbol{R}_j(c^1, \dots, c^r, b^j, \varphi)^{p_j} d\nu_j(\varphi)\right)^{\frac{1}{p_j}}$$

for all $c^l \in C_l$, $l = 1, \dots, r$ and $b^j \in G_j$, with $j = 1, \dots, t$.

Several authors have investigated a special case version of the class of **R**–**S**-abstract p-summing maps starting with the seminal papers [4], [5], [6] and [7] (linear version) and (Lipschitz version) and further explored applications in the nonlinear case can be found in [8] and [9].

This paper consists of 7 sections. In Section 3, inequality (2) is modified to construct the concept of **H**–**Q**-abstract p-summing map which is quite useful to prove the main results under certain assumptions in the forthcoming sections. In Section 4, the nonlinear version concept of **M**–mixed $(s; q)$-summable family is defined in which the spaces are just arbitrary sets and establish an important characterization for this notion under certain hypotheses in abstract settings. In Section 5, the concept of **H**–**M**-$((s; q), p)$-mixing maps between arbitrary sets is constructed and several characterizations are proved. Afterwards, various compositions and inclusion results between different classes of mappings in abstract setting and a quite general of Pietsch [10] Domination-type Theorem are proved. Section 6 presented the proof of how Proposition 11 and Proposition 13 can be appealed in order to get some of the familiar characterizations that have appeared in the different generalizations of the concept of $(s; q)$-mixing operators. It is obvious to see that for suitable choices of A, B, C, G, $\mathcal{H}$, K, W, $\boldsymbol{H}$, and $\boldsymbol{M}$, for a mapping to belong to one of such classes of mixing maps is equivalent to be **H**–**M**-$((s; q), p)$-mixing map and the corresponding characterizations that hold for this class is nothing but Proposition 11 and Proposition 13. Fnally, in Section 7, a notion of mixing maps is generalized and characterization for this notion to a more general setting is showed.

3. Properties of H–Q-abstract p-summing maps

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be non-void families of mappings from B into D and A into D, respectively, and let

$$\boldsymbol{Q}: \mathcal{H} \times A \times C \times G \longrightarrow \mathbb{R},\ \boldsymbol{H}: A \times C \times G \times K \longrightarrow \mathbb{R},$$
$$\boldsymbol{Q}_1: \mathcal{H}_1 \times B \times C_1 \times G \longrightarrow \mathbb{R},\ \boldsymbol{H}_1: B \times C_1 \times G \times W \longrightarrow \mathbb{R},$$
$$\boldsymbol{Q}_2: \mathcal{H}_2 \times A \times C \times G \longrightarrow \mathbb{R}$$

be arbitrary maps satisfy the following conditions:

I- The mapping $\boldsymbol{H}_{a,c,g}: K \longrightarrow \mathbb{R}$ is defined by

$$\boldsymbol{H}_{a,c,g}(\varphi) = \boldsymbol{H}(a, c, g, \varphi)$$

Which is continuous for every $a \in A$, $c \in C$ and $g \in G$.

II- $\boldsymbol{Q}_2(S \circ T, a, c, g) \leq \boldsymbol{Q}_1(S, Ta, c, g)$ for every $T \in \mathcal{H}$, $S \in \mathcal{H}_1$, $a \in A$, $c \in C$ and $g \in G$.

Definition 4 Let $0 < p < \infty$, a map $T \in \mathcal{H}$ is said to be **H**–**Q**-abstract p-summing if there is a constant $\delta > 0$ such that

$$\left[\sum_{j=1}^{m} |\sigma_j|^p \left|\boldsymbol{Q}(T, a_j, c_j, g_j)\right|^p\right]^{\frac{1}{p}} \leq \delta \cdot \sup_{\varphi \in K} \left[\sum_{j=1}^{m} |\sigma_j|^p \left|\boldsymbol{H}(\varphi, a_j, c_j, g_j)\right|^p\right]^{\frac{1}{p}} \tag{3}$$

for all nonzero $\sigma_1, \dots, \sigma_m$ in $\mathbb{R}$, $a_1, \dots, a_m$ in A, $c_1, \dots, c_m$ in C, $g_1, \dots, g_m$ in G and $m \in \mathbb{N}$. The infimum of such constants δ is denoted by $\pi_{\boldsymbol{HQ},p}(T)$. Let $\Pi_p^{\boldsymbol{H}-\boldsymbol{Q}}(A, B)$ be the class of all **H**–**Q**-abstract p-summing maps from A into B. The next proposition has a similar implications as the nonlinear general Pietsch [10] Domination-type Theorem [Theorem 3.1], therefore it is omitted.

Proposition 5 Suppose that **Q** is an arbitrary map and **H** satisfies condition **I** and let $0 < \mathrm{p} < \infty$. A map $\mathrm{T} \in \mathcal{H}$ be **H**–**Q**-abstract p-summing if and only if there are constant $\delta > 0$ and Borel probability measure ν on K such that

$$|\boldsymbol{Q}(T,a,c,g)| \leq \delta \cdot \left[\int_K |\boldsymbol{H}(a,c,g,\varphi)|^p d\nu(\varphi)\right]^{\frac{1}{p}},$$

whenever $a \in A$, $c \in C$, and $g \in G$.

4. Properties of M–mixed $(s;q)$-summable families

Throughout this section we assume $0 < q \leq s \leq \infty$ and that r can be determined by the equation $\frac{1}{r}+\frac{1}{s}=\frac{1}{q}$. Let $\boldsymbol{M}: \mathcal{H} \times A \times C \times G \times W \longrightarrow \mathbb{R}$ be an arbitrary map satisfy the following conditions:

III- The mapping $\boldsymbol{M}_{T,a,c,g}: W \longrightarrow \mathbb{R}$ is defined by

$$\boldsymbol{M}_{T,a,c,g}(\psi) = \boldsymbol{M}(T,a,c,g,\psi)$$

Which is continuous for every $T \in \mathcal{H}$, $a \in A$, $c \in C$ and $g \in G$.

IV- The mapping $\boldsymbol{M}$ is a homogeneous of degree 1 in the variable W if

$$\boldsymbol{M}(T,a,c,g,\lambda\ \psi) = \lambda\ \boldsymbol{M}(T,a,c,g,\psi).$$

V- $\boldsymbol{M}_2(S \circ T,a,c,g,\psi) \leq \boldsymbol{M}_1(S,Ta,c,g,\psi)$ for every $T \in \mathcal{H}$, $S \in \mathcal{H}_1$, $a \in A$, $c \in C$, $g \in G$ and $\psi \in W$.

VI- $\boldsymbol{H}_1(Ta,c,g,\psi) \leq \boldsymbol{M}(T,a,c,g,\psi)$ for every $T \in \mathcal{H}$, $a \in A$, $c \in C$, $g \in G$ and $\psi \in W$.

VII- Let $1 \leq s < \infty$, and $\mu \in \boldsymbol{W}(W)$. Consider the map $\boldsymbol{J}_\mu \in \mathfrak{P}_s^{H_1-\boldsymbol{Q}_1}(B,D)$ with $\pi_{\boldsymbol{H_1}\boldsymbol{Q}_1,s}(\boldsymbol{J}_\mu) \leq 1$ such that

$$\left[\int_W |\boldsymbol{M}(T,a,c,g,\psi)|^s d\mu(\psi)\right]^{\frac{1}{s}} \leq \left|\boldsymbol{Q}_2(\boldsymbol{J}_\mu \circ T,a,c,g)\right| \tag{4}$$

for every $T \in \mathcal{H}$, $\boldsymbol{J}_\mu \in \mathcal{H}_1$, $a \in A$, $c \in C$ and $g \in G$.

Remark 6 Condition **VII** can be applied in the following special cases.

1. Set $A := E$, $B := F$, $C = C_1 := \{1\}$, $D := L_s(B_{F^*},\mu)$, $G := \mathbb{R}$, and $W := B_{F^*}$. Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be non-void families of mappings from E into F, F into $L_s(B_{F^*},\mu)$, and E into $L_s(B_{F^*},\mu)$, respectively. Let T be an operator from E into F and let $\mu \in \boldsymbol{W}(B_{F^*})$. Consider an operator $\boldsymbol{J}_\mu$ from F into $L_s(B_{F^*},\mu)$ assigning to $y \in F$ the function f_y with $f_y(y^*) := \langle y,y^*\rangle$, for more information see [4]. Now, define the following maps.

$$\boldsymbol{H}_1: F \times \{1\} \times \mathbb{R} \times B_{F^*} \longrightarrow \mathbb{R},\ \boldsymbol{H}_1(y,1,\sigma,y^*) = \langle y,y^*\rangle,$$
$$\boldsymbol{Q}_1: \mathcal{H}_1 \times F \times \{1\} \times \mathbb{R} \longrightarrow \mathbb{R},\ \boldsymbol{Q}_1(\boldsymbol{J}_\mu,y,1,\sigma) = \left\|f_y|L_s(B_{F^*},\mu)\right\|,$$
$$\boldsymbol{Q}_2: \mathcal{H}_2 \times E \times \{1\} \times \mathbb{R} \longrightarrow \mathbb{R},\ \boldsymbol{Q}_2(\boldsymbol{J}_\mu \circ T,x,1,\sigma) = \|f_{Tx}|L_s(B_{F^*},\mu)\|,$$
$$\boldsymbol{M}: \mathcal{H} \times E \times \{1\} \times \mathbb{R} \times B_{F^*} \longrightarrow \mathbb{R},\ \boldsymbol{M}(T,x,1,\sigma,y^*) = \langle Tx,y^*\rangle.$$

With these choices one can obtain $\boldsymbol{J}_\mu \in \mathfrak{P}_s^{H_1-Q_1}(F,L_s(B_{F^*},\mu))$ with $\pi_{\boldsymbol{H_1}\boldsymbol{Q}_1,s}(\boldsymbol{J}_\mu) = 1$, and he can satisfy inequality (4).

2. Set $A := X$, $B := Y$, $C := X$, $C_1 := Y$, $D := L_s(B_{Y^\#},\mu)$, $G := \mathbb{R}$, and $W := B_{Y^\#}$. Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be non-void families of mappings from X into Y, Y into $L_s(B_{Y^\#},\mu)$, and X into $L_s(B_{Y^\#},\mu)$, respectively. Let T be a Lipschitz map from X into Y and let $\mu \in \boldsymbol{W}(B_{Y^\#})$, consider Lipschitz map $\boldsymbol{J}_\mu$ from Y into $L_s(B_{Y^\#},\mu)$ assigning to points y_1 and y_2 in Y the function $f_{(y_1,y_2)}$ with $f_{(y_1,y_2)}(\tilde{g}) := \langle y_1,\tilde{g}\rangle - \langle y_2,\tilde{g}\rangle$, for more information see [4]. Define the following maps.

$$\boldsymbol{H}_1: Y \times Y \times \mathbb{R} \times B_{Y^\#} \longrightarrow \mathbb{R},\ \boldsymbol{H}_1(y_1,y_2,\sigma,\tilde{g}) = \langle y_1,\tilde{g}\rangle - \langle y_2,\tilde{g}\rangle,$$
$$\boldsymbol{Q}_1: \mathcal{H}_1 \times Y \times Y \times \mathbb{R} \longrightarrow \mathbb{R},\ \boldsymbol{Q}_1(J_\mu,y_1,y_2,\sigma) = \left\|f_{(y_1,y_2)}|L_s(B_{Y^\#},\mu)\right\|,$$
$$\boldsymbol{Q}_2: \mathcal{H}_2 \times X \times X \times \mathbb{R} \longrightarrow \mathbb{R},\ \boldsymbol{Q}_2(J_\mu \circ T,x_1,x_2,\sigma) = \left\|f_{(Tx_1,Tx_2)}|L_s(B_{Y^\#},\mu)\right\|,$$
$$\boldsymbol{M}: \mathcal{H} \times X \times X \times \mathbb{R} \times B_{Y^\#} \longrightarrow \mathbb{R},\ \boldsymbol{M}(T,x_1,x_2,\sigma,\tilde{g}) = \langle Tx_1,\tilde{g}\rangle - \langle Tx_2,\tilde{g}\rangle.$$

With these choices one can obtain $\boldsymbol{J}_\mu \in \mathfrak{P}_s^{\boldsymbol{H}_1-\boldsymbol{Q}_1}(Y, L_s(B_{Y^\#}, \mu))$ with $\pi_{\boldsymbol{H}_1\boldsymbol{Q}_1,s}(\boldsymbol{J}_\mu) = 1$ and satisfy Inequality (4). The concept of **M**–mixed $(s;q)$-summable family can be constructed as follows.

Definition 7 A family $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} \subset \mathbb{R} - \{0\} \times \mathcal{H} \times A \times C \times G \times W$ is called **M**–mixed $(s;q)$-summable if there exists a nonzero family $(\tau_j)_{j\in I} \in \ell_r(I)$ such that $\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\mathbf{M}(T, a_j, c_j, g_j, \psi)\right|^s < \infty$. The class of all **M**-mixed $(s;q)$-summable families is denoted by $\mathfrak{M}_{(s;q)}^{\mathbf{M}}(\mathbb{R} - \{0\} \times \mathcal{H} \times A \times C \times G \times W, I)$. Moreover, for a family $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} \in \mathfrak{M}_{(s;q)}^{\mathbf{M}}(\mathbb{R} - \{0\} \times \mathcal{H} \times A \times C \times G \times W, I)$. Define

$$\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} = \inf\left[\sum_I \left|\tau_j\right|^r\right]^{\frac{1}{r}} \sup_{\psi\in W}\left[\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}, \qquad (5)$$

where the infimum is taken over all nonzero families $(\tau_j)_{j\in I} \in \ell_r(I)$. The next result will be used in the forthcoming section.

Lemma 8 Let $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I}$ be an arbitrary family in $\mathfrak{M}_{(s;q)}^{\mathbf{M}}(\mathbb{R} - \{0\} \times \mathcal{H} \times A \times C \times G \times W, I)$. If $q = s$, then $\mathfrak{m}_{(s;q)}^{\mathbf{M}}\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} = \sup_{\psi\in W}\left[\sum_I \left|\sigma_j\right|^q \left|\mathbf{M}(T, a_j, c_j, g_j, \psi)\right|^q\right]^{\frac{1}{q}}$.

Proof. Suppose that $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} \in \mathfrak{M}_{(s;q)}^{\boldsymbol{M}}(\mathbb{R} - \{0\} \times \mathcal{H} \times A \times C \times G \times W, I)$, since $s = q$, then $r = \infty$. By Definition 7, there exists a family $(\tau_j)_{j\in I} \in \ell_\infty(I)$ such that $\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^q \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^q < \infty$. Therefore

$$\sup_{\psi\in W}\left[\sum_I \left|\sigma_j\right|^q \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^q\right]^{\frac{1}{q}}$$
$$\leq \left\|(\tau_j)_{j\in I}|\ell_\infty(I)\right\| \cdot \sup_{\psi\in W}\left[\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}.$$

Hence $\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} \geq \sup_{\psi\in W}\left[\sum_I \left|\sigma_j\right|^q \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^q\right]^{\frac{1}{q}}$. For the other direction, choose $(\tau_j)_{j\in I} = 1 \in \ell_\infty(I)$. Then $\left\|(\tau_j)_{j\in I}|\ell_\infty(I)\right\| = 1$ and

$$\sup_{\psi\in W}\left[\sum_I \left|\sigma_j\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}} = \sup_{\psi\in W}\left[\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}.$$

Then

$$\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} = \inf\left\|(\tau_j)_{j\in I}|\ell_\infty(I)\right\| \sup_{\psi\in W}\left[\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^q \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^q\right]^{\frac{1}{q}}$$
$$\leq \sup_{\psi\in W}\left[\sum_I \left|\sigma_j\right|^q \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^q\right]^{\frac{1}{q}}.$$

Inspired by analogous result in the linear theory of Pietsch [10] [Theorem 16.4.3], an important characterization of **M**–mixed $(s;q)$-summable family can be given.

Proposition 9 Let $0 < q < s < \infty$ and let **M** satisfies Condition (**III**). A family $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I} \in \mathfrak{M}_{(s;q)}^{\mathbf{M}}(\mathbb{R} - \{0\} \times \mathcal{H} \times A \times C \times G \times W, I)$ be **M**–mixed $(s;q)$-summable if and only if

$$\left[\sum_I \left|\sigma_j\right|^q \left(\int_W \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s d\mu(\psi)\right)^{\frac{q}{s}}\right]^{\frac{1}{q}} < \infty \tag{6}$$

for every $\mu \in \boldsymbol{W}(W)$. In this case

$$\sup_{\mu \in \boldsymbol{W}(W)} \left[\sum_I \left|\sigma_j\right|^q \left(\int_W \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s d\mu(\psi)\right)^{\frac{q}{s}}\right]^{\frac{1}{q}} = \mathfrak{m}^{\boldsymbol{M}}_{(s;q)}\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I}.$$

Proof. Suppose that the family $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I}$ satisfies (6). Define a number N as follows.

$$N = \sup_{\mu \in \boldsymbol{W}(W)} \left[\sum_I \left|\sigma_j\right|^q \left(\int_W \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s d\mu(\psi)\right)^{\frac{q}{s}}\right]^{\frac{1}{q}}.$$

Then N is finite. Put $u = \frac{r}{q}$ and $v = \frac{s}{q}$. Then $\frac{1}{u} + \frac{1}{v} = 1$. Now consider the compact, convex subset

$$Z = \left\{\xi = \left(\xi_j\right)_{j\in I} : \sum_I \xi_j^u \leq N^q \ \ and \ \ \xi_j \geq 0\right\}$$

of $\ell_u(I)$. Note that the equation

$$\phi(\xi) = \sum_I \left|\sigma_j\right|^s \left(\xi_j + \epsilon\right)^{-v} \cdot \int_W \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s d\mu(\psi),$$

where $\mu \in \boldsymbol{W}(W)$, $\epsilon > 0$, it defines a continuous convex function ϕ on Z. Take the special family $\left(\xi_j\right)_{j\in\mathbb{N}}$ with

$$\xi_j = \left(\int_W \left|\sigma_j\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s d\mu(\psi)\right)^{\frac{1}{u\cdot v}}.$$

Then $\xi \in Z$ and $\phi(\xi) \leq N^q$. Since the collection $\mathfrak{Q}$ of all functions ϕ obtained in this way is concave, by Saleh [7] [Lemma E.4.2], it can be found that $\xi^0 \in Z$ such that $\phi(\xi^0) \leq N^q$ for all $\phi \in \mathfrak{Q}$. In particular, considering the Dirac measure δ_ψ at $\psi \in W$, hence

$$\sum_I \left|\sigma_j\right|^s \left(\xi_j^0 + \epsilon\right)^{-v} \left|M(T, a_j, c_j, g_j, \psi)\right|^s \leq N^q.$$

Set $\tau_j(\epsilon) = \left(\xi_j^0 + \epsilon\right)^{\frac{1}{q}}$. Then

$$\left[\sum_I \left|\tau_j\right|^r\right]^{\frac{1}{r}} = \lim_{\epsilon\to 0^+} \left[\sum_I \left|\tau_j(\epsilon)\right|^r\right]^{\frac{1}{r}} = \left[\sum_I (\xi_j^0)^{\frac{r}{q}}\right]^{\frac{1}{r}} = \left[\sum_I (\xi_j^0)^u\right]^{\frac{1}{r}} \leq N^{\frac{q}{r}} = N^{\frac{1}{u}}, \tag{7}$$

and for $\psi \in W$

$$\left[\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}} = \lim_{\epsilon\to 0^+} \left[\sum_I \left|\frac{\sigma_j}{\tau_j(\epsilon)}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}$$

$$= \lim_{\epsilon\to 0^+} \left[\sum_I \left|\frac{\sigma_j^s}{(\xi_j^0+\epsilon)^v}\right| \ \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}} \leq N^{\frac{q}{s}} = N^{\frac{1}{v}}.$$

Hence

$$\left[\sum_I \left|\tau_j\right|^r\right]^{\frac{1}{r}} \left[\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}} \leq N.$$

This proves the necessity of the above condition. Conversely, suppose that a family $\left((\sigma_j, T, a_j, c_j, g_j, \psi)\right)_{j\in I}$ is $\boldsymbol{M}$-mixed (s; q)-summable. Take any family $(\tau_j)_{j\in I} \in \ell_r(I)$ such that $\sum_I \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s < \infty$. Applying Hölder inequality, hence

$$\left[\sum_{I} |\sigma_j|^q \left(\int_{W} |\boldsymbol{M}(T,a_j,c_j,g_j,\psi)|^s d\mu(\psi)\right)^{\frac{q}{s}}\right]^{\frac{1}{q}}$$
$$\leq \left[\sum_{I} |\tau_j|^r\right]^{\frac{1}{r}} \sup_{\psi\in W}\left[\sum_{I} \left|\frac{\sigma_j}{\tau_j}\right|^s |\boldsymbol{M}(T,a_j,c_j,g_j,\psi)|^s\right]^{\frac{1}{s}}$$

whenever $\mu \in \boldsymbol{W}(W)$. This proves the sufficiency of the above condition.

5. Properties of H–M-$((s,q),p)$-mixing maps

Throughout this section, assume that V be a vector space over the field $\mathbb{K}$ and let $\mathcal{P} = (\boldsymbol{P}_\iota)_{\iota=1}^r$ be a finite family of semi-norms on V. The topology induced by a finite family of semi-norms on V is denoted by $\mathcal{P}$-topology on V. If $\boldsymbol{P}_\iota$ $(1 \leq \iota \leq r)$ be a finite family of semi-norms on a vector space V, then the function $\boldsymbol{P} = \max \boldsymbol{P}_\iota$ defined by

$$\boldsymbol{P}(v) = \max_{1\leq\iota\leq r}\boldsymbol{P}_\iota(v)$$

be also a semi-norm on Pellegrino and Santos [3] [Proposition 2.16] it is known that the topology associated with the semi-norm P is identical with the $\mathcal{P}$-topology on V. Suppose that $\mathfrak{B}_{\boldsymbol{P}}$ be a compact unit P-ball defined as follows.

$$\mathfrak{B}_{\boldsymbol{P}} = \{v \in V: \boldsymbol{P}(v) \leq 1\}.$$

The concept of **H**–**M**-$((s;q),p)$-mixing map can be constructed as follows.

Definition 10 A map T from A into B is called **H**–**M**-$((\mathrm{s};\mathrm{q}),\mathrm{p})$-mixing, where $0 < \mathrm{q} \leq \mathrm{s} \leq \infty$ and $\mathrm{p} \leq \mathrm{q}$, if there is a constant $\delta \geq 0$ such that

$$\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\left((\sigma_j,T,a_j,c_j,g_j,\psi)\right)_{j=1}^m \leq \delta \cdot \sup_{\varphi\in K}\left[\sum_{j=1}^m |\sigma_j|^p |\boldsymbol{H}(a_j,c_j,g_j,\varphi)|^p\right]^{\frac{1}{p}}$$

for all nonzero $\sigma_1,\cdots,\sigma_m \in \mathbb{R}$, $a_1,\cdots,a_m \in A$, $c_1,\cdots,c_m \in C$, $g_1,\cdots,g_m \in G$ and $m \in \mathbb{N}$. The infimum of such constants δ is denoted by $\boldsymbol{HM}_{((s;q),p)}(T)$. The class of all **H**–**M**-$((s;q),p)$-mixing maps from A into $B$is denoted by $\mathfrak{M}_{((s;q),p)}^{\boldsymbol{H-M}}(A,B)$.

Inspired by analogous result in the linear theory of A. Pietsch [10] [Theorem 20.1.4] and the similar proof [4] of [Theorem 4.1], the following characterization of **H**–**M**-$((s;q),p)$-mixing map can be given.

Proposition 11 Let $0 < \mathrm{q} < \mathrm{s} < \infty$,and $\mathrm{p} \leq \mathrm{q}$ and let **H** and **M** satisfy conditions **I**, **III** and **IV**, respectively. A map T from A into B is **H**–**M**-$((\mathrm{s};\mathrm{q}),\mathrm{p})$-mixing if and only if there is a constant $\delta \geq 0$ such that

$$[\sum_{j=1}^m |\sigma_j|^q [\sum_{k=1}^n |\boldsymbol{M}(T,a_j,c_j,g_j,v_k)|^s]^{\frac{q}{s}}]^{\frac{1}{q}} \leq \delta \cdot \sup_{\varphi\in K}\left[\sum_{j=1}^m |\sigma_j|^p |\boldsymbol{H}(a_j,c_j,g_j,\varphi)|^p\right]^{\frac{1}{p}} \cdot [\sum_{k=1}^n \boldsymbol{P}(v_k)^s]^{\frac{1}{s}} \tag{8}$$

for every nonzero $\sigma_1,\cdots,\sigma_m \in \mathbb{R}$, $a_1,\cdots,a_m \in A$, $c_1,\cdots,c_m \in C$, $g_1,\cdots,g_m \in G$; $v_1,\cdots,v_n \in V$ and $m,n \in \mathbb{N}$. Moreover

$$\boldsymbol{HM}_{((s;q),p)}(T) = \inf\delta.$$

Proof. Assume that T is **H**–**M**-$((s,q),p)$-mixing map. Consider $v_1,\cdots,v_n \in V$ and define the discrete probability $\mu = \sum_{k=1}^n t_k\delta_k$, where $t_k = \boldsymbol{P}(v_k)^s \cdot [\sum_{h=1}^n \boldsymbol{P}(v_h)^s]^{-1}$ and δ_k denotes the Dirac measure at $y_k = \frac{v_k}{\boldsymbol{P}(v_k)} \in \mathfrak{B}_{\boldsymbol{P}}$; $k = 1,\cdots,n$. Then $\mu \in \boldsymbol{W}(\mathfrak{B}_{\boldsymbol{P}})$. For $\sigma_1,\cdots,\sigma_m \in \mathbb{R}$, $a_1,\cdots,a_m \in A$, $c_1,\cdots,c_m \in C$, and $g_1,\cdots,g_m \in G$, from Proposition 9, it can be obtained that:

$$[\sum_{j=1}^{m} |\sigma_j|^q [\sum_{k=1}^{n} |\boldsymbol{M}(T, a_j, c_j, g_j, v_k)|^s]^{\frac{q}{s}}]^{\frac{1}{q}}$$

$$= [\sum_{j=1}^{m} |\sigma_j|^q [\int_{\mathfrak{B}_P} |\boldsymbol{M}(T, a_j, c_j, g_j, v)|^s d\mu(v)]^{\frac{q}{s}}]^{\frac{1}{q}} \cdot \left[\sum_{k=1}^{n} \boldsymbol{P}(v_k)^s\right]^{\frac{1}{s}}$$

$$\leq \mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\big((\sigma_j, T, a_j, c_j, g_j, v)\big)_{j=1}^{m} \cdot \left[\sum_{k=1}^{n} \boldsymbol{P}(v_k)^s\right]^{\frac{1}{s}}$$

$$\leq \boldsymbol{HM}_{((s;q),p)}(T) \cdot \sup_{\varphi \in K} \left[\sum_{j=1}^{m} |\sigma_j|^p |\boldsymbol{H}(a_j, c_j, g_j, \varphi)|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{k=1}^{n} \boldsymbol{P}(v_k)^s\right]^{\frac{1}{s}}.$$

In order to show the converse, (8) can be explained as

$$[\sum_{j=1}^{m} |\sigma_j|^q [\int_{\mathfrak{B}_P} |\boldsymbol{M}(T, a_j, c_j, g_j, v)|^s d\mu(v)]^{\frac{q}{s}}]^{\frac{1}{q}} \leq \delta \cdot \sup_{\varphi \in K} [\sum_{j=1}^{m} |\sigma_j|^p |\boldsymbol{H}(a_j, c_j, g_j, \varphi)|^p]^{\frac{1}{p}} \quad (9)$$

for every discrete probability measure μ on $\mathfrak{B}_{\boldsymbol{P}}$ and $\sigma_1, \cdots, \sigma_m \in \mathbb{R}$, $a_1, \cdots, a_m \in A$, $c_1, \cdots, c_m \in C$, and $g_1, \cdots, g_m \in G$. Since $\sigma(C(\mathfrak{B}_{\boldsymbol{P}})^*, C(\mathfrak{B}_{\boldsymbol{P}}))$-dense the set of all finitely supported probability measures on $\mathfrak{B}_{\boldsymbol{P}}$, then (9) in the set of all probability measures on $\mathfrak{B}_{\boldsymbol{P}}$, it follows that (9) satisfies all probability measures μ on $\mathfrak{B}_{\boldsymbol{P}}$ and $\sigma_1, \cdots, \sigma_m \in \mathbb{R}$, $a_1, \cdots, a_m \in A$, $c_1, \cdots, c_m \in C$, $g_1, \cdots, g_m \in G$. Taking the supremum over $\mu \in \boldsymbol{W}(\mathfrak{B}_P)$ on the left side of (9) and using Proposition 9, it can be found that

$$\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\big((\sigma_j, T, a_j, c_j, g_j, v)\big)_{j=1}^{m} \leq \delta \cdot \sup_{\varphi \in K} [\sum_{j=1}^{m} |\sigma_j|^p |\boldsymbol{H}(a_j, c_j, g_j, \varphi)|^p]^{\frac{1}{p}}.$$

The following multiplication formula represents the main-point of the theory of **H**–**M**-$((s;q),q)$-mixing maps and it is somewhat inspired by analogous result in the linear theory.

Proposition 12 Let $0 < \mathrm{q} \leq \mathrm{s} \leq \infty$. If the maps $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{H}_1$ and $\mathbf{M}$ satisfy conditions **II** and **VI**, respectively, then

$$\left[\mathfrak{P}_{\boldsymbol{s}}^{\boldsymbol{H_1-Q_1}}(B, D), (\boldsymbol{H}_1\boldsymbol{Q}_1)_s\right] \circ \left[\mathfrak{M}_{((s;q),q)}^{\boldsymbol{H-M}}(A, B), \boldsymbol{HM}_{((s;q),q)}\right] \subseteq \left[\mathfrak{P}_{\boldsymbol{q}}^{\boldsymbol{H-Q_2}}(A, C), (\boldsymbol{HQ_2})_q\right].$$

Proof. Suppose that $S \in \mathfrak{P}_{\boldsymbol{s}}^{\boldsymbol{H_1-Q_1}}(B, D)$ and $T \in \mathfrak{M}_{((s,q),q)}^{\boldsymbol{H-M}}(A, B)$. Given $\sigma_1, \dots, \sigma_m$ in $\mathbb{R}$, $a_1, \dots, a_m$ in A, $c_1, \dots, c_m$, $g_1, \dots, g_m$ in G, and $\epsilon > 0$, then

$$\left[\sum_{j=1}^{m} |\tau_j|^r\right]^{\frac{1}{r}} \sup_{\psi \in W} \left[\sum_{j=1}^{m} \left|\frac{\sigma_j}{\tau_j}\right|^s |\boldsymbol{M}(T, a_j, c_j, g_j, \psi)|^s\right]^{\frac{1}{s}} \leq (1+\epsilon) \cdot \mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\big((\sigma_j, T, a_j, c_j, g_j, \psi)\big)_{j=1}^{m}$$

$$\leq (1+\epsilon) \cdot \boldsymbol{HM}_{((s;q),q)}(T) \cdot \sup_{\varphi \in K} [\sum_{j=1}^{m} |\sigma_j|^q |\boldsymbol{H}(a_j, c_j, g_j, \varphi)|^q]^{\frac{1}{q}}.$$

It can be noticed from

$$[\sum_{j=1}^{m} |\sigma_j|^s |\boldsymbol{Q}_1(S, b_j, c_j, g_j)|^s]^{\frac{1}{s}} \leq (\boldsymbol{H}_1\boldsymbol{Q}_1)_s(S) \sup_{\psi \in W} [\sum_{j=1}^{m} |\sigma_j|^s |\boldsymbol{H}_1(b_j, c_j, g_j, \psi)|^s]^{\frac{1}{s}}$$

that

$$[\sum_{j=1}^{m} |\sigma_j|^s |\boldsymbol{Q}_1(S, Ta_j, c_j, g_j)|^s]^{\frac{1}{s}} \leq (\boldsymbol{H}_1\boldsymbol{Q}_1)_s(S) \sup_{\psi \in W} [\sum_{j=1}^{m} |\sigma_j|^s |\boldsymbol{H}_1(Ta_j, c_j, g_j, \psi)|^s]^{\frac{1}{s}}.$$

By applying Hölder inequality and conditions (**II**) and (**VI**) one can obtain

$$[\sum_{j=1}^{m} |\sigma_j|^q |\boldsymbol{Q}_2(S \circ T, a_j, c_j, g_j)|^q]^{\frac{1}{q}} \leq [\sum_{j=1}^{m} |\sigma_j|^q |\boldsymbol{Q}_1(S, Ta_j, c_j, g_j)|^q]^{\frac{1}{q}}$$

$$\leq \left[\sum_{j=1}^{m} \left|\tau_j\right|^r\right]^{\frac{1}{r}} \left[\sum_{j=1}^{m} \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{Q}_1(S, Ta_j, c_j, g_j)\right|^s\right]^{\frac{1}{s}}$$

$$\leq (\boldsymbol{H}_1\boldsymbol{Q}_1)_s(S)\left[\sum_{j=1}^{m} \left|\tau_j\right|^r\right]^{\frac{1}{r}} \sup_{\psi\in W} \left[\sum_{j=1}^{m} \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{H}_1(Ta_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}$$

$$\leq (\boldsymbol{H}_1\boldsymbol{Q}_1)_s(S)\left[\sum_{j=1}^{m} \left|\tau_j\right|^r\right]^{\frac{1}{r}} \sup_{\psi\in W} \left[\sum_{j=1}^{m} \left|\frac{\sigma_j}{\tau_j}\right|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}$$

$$\leq (1+\epsilon)\cdot(\boldsymbol{H}_1\boldsymbol{Q}_1)_s(S)\ \ \boldsymbol{HM}_{((s;q),q)}(T) \sup_{\varphi\in K}\left[\sum_{j=1}^{m} \left|\sigma_j\right|^q \left|\boldsymbol{H}(a_j, c_j, g_j, \varphi)\right|^q\right]^{\frac{1}{q}}.$$

Hence $S \circ T \in \mathfrak{P}_{\boldsymbol{q}}^{\boldsymbol{H-Q_2}}(A, D)$ with $(\boldsymbol{HQ}_2)_q(S\circ T) \leq (\boldsymbol{H}_1 Q_1)_s(S)\cdot \boldsymbol{HM}_{((s;q),q)}(T)$.

The following characterization is a quite general of unified Pietsch [10] domination theorem [Theorem 3.1].

Proposition 13 Let $0 < q \leq s \leq \infty$,and let the maps $\mathbf{M}$, $\mathbf{H}$, $\mathbf{J}_\mu$ and $\mathbf{Q_2}$ satisfy conditions **I**, **III**, and **VII**, respectively. A map T is $\mathbf{H}$–$\mathbf{M}$-$((s;q),q)$-mixing if and only if there exists a constant $\delta \geq 0$ such that for any probability measure μ on W there exists a probability measure ν on K such that

$$\left[\int_W |\boldsymbol{M}(T,a,c,g,\psi)|^s d\mu(\psi)\right]^{\frac{1}{s}} \leq \delta\cdot\left[\int_K |\boldsymbol{H}(a,c,g,\varphi)|^q d\nu(\varphi)\right]^{\frac{1}{q}},$$

whenever $a \in A$, $c \in C$ and $g \in G$. Moreover $\boldsymbol{HM}_{((s;q),q)}(T) = \inf\delta$.

Proof. Let σ be an arbitrary nonzero sequence in $\mathbb{R}$. By the assumptions, one can have

$$\left[\sum_{j=1}^{m} \left|\sigma_j\right|^q \left[\int_W \left|\boldsymbol{M}(T,a_j,c_j,g_j,\psi)\right|^s d\mu(\psi)\right]^{\frac{q}{s}}\right]^{\frac{1}{q}} \leq$$

$$\delta\cdot\left[\sum_{j=1}^{m} \left|\sigma_j\right|^q \int_K \left|\boldsymbol{H}(a_j,c_j,g_j,\varphi)\right|^q d\nu(\varphi)\right]^{\frac{1}{q}} \leq \delta\cdot\sup_{\varphi\in K}\left[\sum_{j=1}^{m} \left|\sigma_j\right|^q \left|\boldsymbol{H}(a_j,c_j,g_j,\varphi)\right|^q\right]^{\frac{1}{q}}. \quad (10)$$

Taking the supremum over μ on W on the left side of (10) and from Proposition 9, then

$$\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}\left(\left(\sigma_j, T, a_j, c_j, g_j, \psi\right)\right)_{j=1}^{m} \leq \delta\cdot\sup_{\varphi\in K}\left[\sum_{j=1}^{m} \left|\sigma_j\right|^q \left|\boldsymbol{H}(a_j,c_j,g_j,\varphi)\right|^q\right]^{\frac{1}{q}}.$$

Conversely, suppose that T is $\boldsymbol{H}$–$\boldsymbol{M}$-$((s;q),q)$-mixing map. From Proposition 12 and using condition **VII**, $\boldsymbol{J}_\mu \circ T$ be $\boldsymbol{H}$–$\boldsymbol{Q}_2$-abstract q-summing map with $\pi_{\boldsymbol{HQ_2},q}(\boldsymbol{J}_\mu \circ T) \leq \boldsymbol{HM}_{((s;q),q)}(T)$. Hence, by using Proposition 5, there exists a probability measure ν on K such that

$$\left[\int_W |\boldsymbol{M}(T,a,c,g,\psi)|^s d\mu(\psi)\right]^{\frac{1}{s}} \leq \left|\boldsymbol{Q}_2\left(J_\mu\circ T, a, c, g\right)\right|$$

$$\leq \boldsymbol{HM}_{((s,q),q)}(T)\cdot\left[\int_K |\boldsymbol{H}(a,c,g,\varphi)|^q d\nu(\varphi)\right]^{\frac{1}{q}}$$

for all $a \in A$, $c \in C$ and $g \in G$. The next inclusion result follows immediately from Proposition 13.

Proposition 14 If $q_1 \leq q_2 \leq s_2 \leq s_1$, then

$$\mathfrak{M}_{((s_1;q_1),q_1)}^{\boldsymbol{H-M}}(A,B) \subseteq \mathfrak{M}_{((s_2;q_2),q_2)}^{\boldsymbol{H-M}}(A,B).$$

Proposition 15 Let the maps $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{H}_1$ and $\mathbf{M}$ satisfy conditions **V** and **VI**, respectively. If $0 < p \leq s \leq t \leq \infty$, then

$$\mathfrak{M}_{((t;s),s)}^{\boldsymbol{H_1-M_1}}(B,D)\circ\mathfrak{M}_{((s;q),q)}^{\boldsymbol{H-M}}(A,B) \subseteq \mathfrak{M}_{((t;q),q)}^{\boldsymbol{H-M_2}}(A,D).$$

Proof. From Definition 7, one can have

$$\mathfrak{m}^{\boldsymbol{M}_2}_{(t;q)}\big((\sigma_j, S\circ T, a_j, c_j, g_j, \psi)\big)_{j=1}^m = \inf_{\tau}\left[\sum_{j=1}^m |\tau_j|^r\right]^{\frac{1}{r}} \sup_{\psi\in W}\left[\sum_{j=1}^m \left|\frac{\sigma_j}{\tau_j}\right|^t \left|\boldsymbol{M}_2(S\circ T, a_j, c_j, g_j, \psi)\right|^t\right]^{\frac{1}{t}} = \inf_{\tau_1\cdot\tau_2}\left[\sum_{j=1}^m |\tau_j^1\cdot\tau_j^2|^r\right]^{\frac{1}{r}} \sup_{\psi\in W}\left[\sum_{j=1}^m \left|\frac{\sigma_j}{\tau_j^1\cdot\tau_j^2}\right|^t \left|\boldsymbol{M}_2(S\circ T, a_j, c_j, g_j, \psi)\right|^t\right]^{\frac{1}{t}}.$$

Put $\sigma' = \frac{\sigma}{\tau_1}$. Since $\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r}$ with Hölder inequality give us

$$\mathfrak{m}^{\boldsymbol{M}_2}_{(t;q)}\big((\sigma_j, S\circ T, a_j, c_j, g_j, \psi)\big)_{j=1}^m$$

$$\le \inf_{\tau_1\cdot\tau_2}\left[\sum_{j=1}^m |\tau_j^1|^{r_1}\right]^{\frac{1}{r_1}}\cdot\left[\sum_{j=1}^m |\tau_j^1|^{r_2}\right]^{\frac{1}{r_2}}\cdot \sup_{\psi\in W}\left[\sum_{j=1}^m \left|\frac{\sigma_j}{\tau_j^1\cdot\tau_j^2}\right|^t \left|\boldsymbol{M}_1(S, Ta_j, c_j, g_j, \psi)\right|^t\right]^{\frac{1}{t}}$$

$$= \inf_{\tau_1}\left[\sum_{j=1}^m |\tau_j^1|^{r_1}\right]^{\frac{1}{r_1}}\cdot \inf_{\tau_2}\left[\sum_{j=1}^m |\tau_j^1|^{r_2}\right]^{\frac{1}{r_2}}\cdot \sup_{\psi\in W}\left[\sum_{j=1}^m \left|\frac{\sigma_j}{\tau_j^1\cdot\tau_j^2}\right|^t \left|\boldsymbol{M}_1(S, Ta_j, c_j, g_j, \psi)\right|^t\right]^{\frac{1}{t}}$$

$$= \inf_{\tau_1}\left[\sum_{j=1}^m |\tau_j^1|^{r_1}\right]^{\frac{1}{r_1}}\cdot \mathfrak{m}^{\boldsymbol{M}_1}_{(t;s)}\big((\sigma'_j, S, Ta_j, c_j, g_j, \psi)\big)_{j=1}^m$$

$$\le \boldsymbol{HM}_{((t;s),s)}(S)\inf_{\tau_1}\left[\sum_{j=1}^m |\tau_j^1|^{r_1}\right]^{\frac{1}{r_1}} \sup_{\psi\in W}\left[\sum_{j=1}^m |\sigma'_j|^s \left|\boldsymbol{H}_1(Ta_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}$$

$$\le \boldsymbol{HM}_{((t;s),s)}(S)\inf_{\tau_1}\left[\sum_{j=1}^m |\tau_j^1|^{r_1}\right]^{\frac{1}{r_1}} \sup_{\psi\in W}\left[\sum_{j=1}^m |\sigma'_j|^s \left|\boldsymbol{M}(T, a_j, c_j, g_j, \psi)\right|^s\right]^{\frac{1}{s}}$$

$$= \boldsymbol{HM}_{((t;s),s)}(S)\cdot \mathfrak{m}^{\boldsymbol{M}}_{(s;q)}\big((\sigma_j, T, a_j, c_j, g_j, \psi)\big)_{j=1}^m$$

$$\le \boldsymbol{HM}_{((t;s),s)}(S)\cdot \boldsymbol{HM}_{((s;q),q)}(T)\sup_{\varphi\in K}\left[\sum_{j=1}^m |\sigma_j|^q \left|\boldsymbol{H}(a_j, c_j, g_j, \varphi)\right|^q\right]^{\frac{1}{q}}.$$

6. Recovering the known fundamental characterizations of mixing maps

6.1 The characterizations of $(s;q)$-mixing operators

1. If T is a bounded operator from E into F, then T is $(s;q)$-mixing if and only if there is a constant $\delta \ge 0$ such that

$$\left[\sum_{j=1}^m \left[\sum_{k=1}^n \left|\langle b_k^*, Tx_j\rangle\right|^s\right]^{\frac{q}{s}}\right]^{\frac{1}{q}} \le \delta\cdot \sup_{x^*\in B_{E^*}}\left[\sum_{j=1}^m \left|x^*(x_j)\right|^q\right]^{\frac{1}{q}}\cdot\left[\sum_{k=1}^n \|b_k^*\|^s\right]^{\frac{1}{s}}.$$

for every $x_1,\cdots,x_m\in E$, functional $b_1^*,\cdots,b_n^*\in F^*$ and $m,n\in\mathbb{N}$. Set $A:=E$, $B:=F$, $C:=\{1\}$, $G:=\mathbb{R}$, $V=F^*$, $W:=B_{F^*}$, $K:=B_{E^*}$, $\mathcal{H}$ be a family of bounded linear operators from E into F, and the family of semi-norms $\mathcal{P}$ can be taken to be the single norm $\boldsymbol{P}_{b^*}$ defined on F^* by $\boldsymbol{P}_{b^*} = \sup_{x\in B_E}|\langle x, b^*\rangle|$ (for more details find [10] [Theorem 20.1.4]). Define the maps as follows.

$$\boldsymbol{M}: \mathcal{H}\times E\times\{1\}\times\mathbb{R}\times B_{F^*}\longrightarrow\mathbb{R},\ \boldsymbol{M}(T,x,1,\sigma,b^*) = \frac{\langle Tx, b^*\rangle}{\sigma},$$

$$\boldsymbol{H}: E\times\{1\}\times\mathbb{R}\times B_{E^*}\longrightarrow\mathbb{R},\ \boldsymbol{H}(x,1,\sigma,x^*) = \frac{\langle x, x^*\rangle}{\sigma},$$

where $\sigma\ne 0$. With these choices and applying Proposition 11,we get T is $(s;q)$-mixing operator if and only if T be **H**–**M**-$((s;q),q)$-mixing operator. In this context Proposition 11 coincides with Theorem 20.1.4 Saleh [7] for $(s;q)$-mixing operator.

2. [Saleh [7], Theorem 20.1.7] says that a bounded operator T from E into F is $(s;q)$-mixing if and only if there is a constant $\delta\ge 0$ such that for any probability measure μ on B_{F^*} there exists a probability measure ν on B_{E^*} such that

$$\left[\int_{B_{F^*}} |\langle Tx, b^*\rangle|^s d\mu(b^*)\right]^{\frac{1}{s}} \le \delta\cdot\left[\int_{B_{E^*}} |\langle x, x^*\rangle|^q d\nu(x^*)\right]^{\frac{1}{q}},$$

whenever $x \in E$. Define the maps as follows.

$$\boldsymbol{M}: \mathcal{H} \times E \times \{1\} \times \mathbb{R} \times B_{F^*} \longrightarrow \mathbb{R},\ \boldsymbol{M}(T, x, 1, \sigma, b^*) = \langle Tx, b^* \rangle,$$

$$\boldsymbol{H}: E \times \{1\} \times \mathbb{R} \times B_{E^*} \longrightarrow \mathbb{R},\ \boldsymbol{H}(x, 1, \sigma, x^*) = \langle x, x^* \rangle.$$

With the previous choices, and applying Proposition 13, we get T is $(s;q)$-mixing operator if and only if T be **H**–**M**-$((s;q),q)$-mixing operator. In this context Proposition 13 coincides with Theorem 20.1.7 in [7] for $(s;q)$-mixing operator.

6.2 The characterizations of Lipschitz $(s;q)$-mixing maps

1. [Saleh [8], Theorem 4.4] says that a Lipschitz map T from X into Y is Lipschitz $(s;q)$-mixing if and only if there is a constant $\delta \geq 0$ such that

$$[\sum_{j=1}^{m} |\sigma_j|^q [\sum_{k=1}^{n} \left|\langle g_k, Tx'_j\rangle_{(Y^\#,Y)} - \langle g_k, Tx''_j\rangle_{(Y^\#,Y)}\right|^s]^{\frac{q}{s}}]^{\frac{1}{q}}$$

$$\leq C \cdot \sup_{f \in B_{X^\#}} [\sum_{j=1}^{m} |\sigma_j|^q |fx'_j - fx''_j|^q]^{\frac{1}{q}} [\sum_{k=1}^{n} \mathrm{L}ip(g_k)^s]^{\frac{1}{s}}$$

for every nonzero $\sigma_1,\cdots,\sigma_m \in \mathbb{R}$, $x'_1,\cdots,x'_m, x''_1,\cdots,x''_m \in X$, $g_1,\cdots,g_n \in Y^\#$ and $m,n \in \mathbb{N}$. Set $A := X$, $B := Y$, $C := X$, $G := \mathbb{R}$, $V := Y^\#$, $W := B_{Y^\#}$, $K := B_{X^\#}$, $\mathcal{H}$ be a family of Lipschitz maps from X into Y, and the family of semi-norms $\mathcal{P}$ can be taken to be the single norm $P_{\tilde{g}}$ defined on $Y^\#$ by $P_{\tilde{g}} = \sup_{x' \neq x''} \frac{|\tilde{g}x' - \tilde{g}x''|}{d_X(x', x'')}$. Define the maps as follows.

$$\boldsymbol{M}: \mathcal{H} \times X \times X \times \mathbb{R} \times B_{Y^\#} \longrightarrow \mathbb{R},\ \boldsymbol{M}(T, x', x'', \sigma, \tilde{g}) = \langle \tilde{g}, Tx' \rangle_{(Y^\#,Y)} - \langle \tilde{g}, Tx'' \rangle_{(Y^\#,Y)},$$

$$\boldsymbol{H}: X \times X \times \mathbb{R} \times B_{X^\#} \longrightarrow \mathbb{R},\ \boldsymbol{H}(x', x'', \sigma, f) = \langle f, x' \rangle_{(X^\#,X)} - \langle f, x'' \rangle_{(X^\#,X)}.$$

With these choices, and applying Proposition 11, we get T is Lipschitz $(s;q)$-mixing map if and only if T is **H**–**M**-$((s;q),q)$-mixing map. In this context Proposition 11 coincides with Theorem 4.4 in for Lipschitz $(s;q)$-mixing map.

2. [Chàvez-Domínguez [9], Theorem 4.1] says that a Lipschitz map T from X into Y is Lipschitz $(s;q)$-mixing if and only if there is a constant $\delta \geq 0$ such that for any probability measure μ on $B_{Y^\#}$ there exists a probability measure ν on $B_{X^\#}$ such that

$$\left[\int_{B_{Y^\#}} \left|\langle g, Tx' \rangle_{(Y^\#,Y)} - \langle g, Tx'' \rangle_{(Y^\#,Y)}\right|^s d\mu(g)\right]^{\frac{1}{s}}$$

$$\leq \delta \cdot \left[\int_{B_{X^\#}} \left|\langle f, x' \rangle_{(X^\#,X)} - \langle f, x'' \rangle_{(X^\#,X)}\right|^q d\nu(f)\right]^{\frac{1}{q}},$$

whenever $x'_1,\cdots,x'_m, x''_1,\cdots,x''_m \in X$, and $m \in \mathbb{N}$. With the above choices and applying Proposition 13, T is Lipschitz $(s;q)$-mixing map if and only if T is **H**–**M**-$((s;q),q)$-mixing map. In this context Proposition 13 coincides with Theorem 4.1 in [2] for Lipschitz $(s;q)$-mixing map.

6.3 Properties of R $_1$, ..., R $_t$-S-((s, q), p $_1$, ..., p $_t$)-mixing maps

Let $A_1,\cdots,A_t$, B and $C_1,\cdots,C_r$ be non-void sets, $\mathcal{H}$ be a non-void family of mappings from $A_1 \times \cdots \times A_t$ into B, and $G_1,\cdots,G_s$ be Banach spaces. Let W and $K_1,\cdots,K_s$ be compact Hausdorff topological spaces. Put $\tilde{A} := A_1 \times \cdots \times A_t$, $\tilde{C} := C_1 \times \cdots \times C_r$ and $\tilde{G} := G_1 \times \cdots \times G_s$.

Let $\boldsymbol{M}: \mathcal{H} \times \tilde{A} \times \tilde{C} \times \tilde{G} \times W \longrightarrow \mathbb{R}$ and $\boldsymbol{H}_k: \tilde{A} \times \tilde{C} \times G_k \times K_k \longrightarrow \mathbb{R}$, $k = 1, \dots, s$ be arbitrary maps satisfy the following conditions:

VIII- The mapping $\boldsymbol{M}_{T,a_1,\dots,a_t,c_1,\dots,c_r,g_1,\dots,g_s}: W \longrightarrow \mathbb{R}$ defined by

$$\boldsymbol{M}_{T,a_1,\dots,a_t,c_1,\dots,c_r,g_1,\dots,g_s}(\psi) = \boldsymbol{M}(T, a_1, \dots, a_t, c_1, \dots, c_r, g_1, \dots, g_s, \psi)$$

is continuous for every $T \in \mathcal{H}$, $a_1, \dots, a_t \in \tilde{A}$, $c_1, \dots, c_r \in \tilde{C}$ and $g_1, \dots, g_s \in \tilde{G}$.

VIIII- The mapping $\boldsymbol{M}$ be a homogeneous of degree 1 in the variable W if

$$\boldsymbol{M}(T,a_1,\dots,a_t,c_1,\dots,c_r,g_1,\dots,g_s,\lambda\ \psi)=\lambda\ \boldsymbol{M}(T,a_1,\dots,a_t,c_1,\dots,c_r,g_1,\dots,g_s,\psi).$$

X- The mapping $(\boldsymbol{H}_k)_{a_1,\dots,a_t,c_1,\dots,c_r,g}\colon K_k\longrightarrow\mathbb{R}$ defined by

$$(\boldsymbol{H}_k)_{a_1,\dots,a_t,c_1,\dots,c_r,g}(\varphi)=\boldsymbol{H}_k(a_1,\dots,a_t,c_1,\dots,c_r,g,\varphi)$$

is continuous for every $a_1,\dots,a_t\in\tilde{A}$, $c_1,\dots,c_r\in\tilde{C}$ and $g\in G_k$.

The concept of **H** $_1$,..., **H** $_t$-**M**-$((s;q),p_1,\dots,p_t)$-mixing map can be constructed as follows.

Definition 16 Let $0<q\le s\le\infty$, and $p\le q$. A map T from $\tilde{A}$ into B is called **H** $_1$,..., **H** $_t$-**M**-$((s;q),p_1,\dots,p_t)$-mixing if there is a constant $\delta\ge 0$ such that

$$\mathfrak{m}^{\boldsymbol{M}}_{(s;q)}\big((\sigma_j,T,a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^1,\dots,g_j^s,\psi)\big)_{j=1}^m$$
$$\le\delta\cdot\prod_{k=1}^s\sup_{\varphi\in K_k}\Big[\sum_{j=1}^m|\sigma_j|^{p_k}\big|\boldsymbol{H}_k(a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^k,\varphi)\big|^{p_k}\Big]^{\frac{1}{p_k}}\qquad(11)$$

for all nonzero $\sigma_1,\cdots,\sigma_m\in\mathbb{R}$, $a_1,\dots,a_t\in\tilde{A}$, $c_1^1,\cdots,c_m^r\in\tilde{C}$, $g_1^1,\cdots,g_m^s\in\tilde{G}$ and $m\in\mathbb{N}$. The infimum of such constants δ is denoted by **H** $_1$,..., **H** $_t$-**M** $_{((s;q),p_1,\dots,p_t)}(T)$. Let us denote by $\mathfrak{M}^{\boldsymbol{H}_1,\dots,\boldsymbol{H}_t-\boldsymbol{M}}_{((s;q),p_1,\dots,p_t)}(\tilde{A},B)$ the class of all **H** $_1$,..., **H** $_t$-**M**-$((s;q),p_1,\dots,p_t)$-mixing maps from $\tilde{A}$ into B.

Proposition 17 Let $0<q\le s\le\infty$,and $p\le q$ and let H_k and **M** satisfy conditions **X**, **VIII** and **VIIII**, respectively. A map T from $\tilde{A}$ into B is **H** $_1$,..., **H** $_t$-**M**-$((s;q),p_1,\dots,p_t)$-mixing if and only if there is a constant $\delta\ge 0$ such that

$$\Big[\sum_{j=1}^m|\sigma_j|^q\Big[\sum_{\zeta=1}^n\big|\boldsymbol{M}(T,a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^1,\dots,g_j^s,v_\zeta)\big|^s\Big]^{\frac{q}{s}}\Big]^{\frac{1}{q}}$$
$$\le\delta\cdot\prod_{k=1}^s\sup_{\varphi\in K_k}\Big[\sum_{j=1}^m|\sigma_j|^{p_k}\big|\boldsymbol{H}_k(a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^k,\varphi)\big|^{p_k}\Big]^{\frac{1}{p_k}}\cdot\Big[\sum_{\zeta=1}^n\boldsymbol{P}(v_\zeta)^s\Big]^{\frac{1}{s}}\qquad(12)$$

for every nonzero $\sigma_1,\cdots,\sigma_m\in\mathbb{R}$, $a_j^1,\dots,a_j^t\in\tilde{A}$, $c_1^1,\cdots,c_m^r\in\tilde{C}$, $g_1^1,\cdots,g_m^s\in\tilde{G}$, $v_1,\cdots,v_n\in V$ and $m,n\in\mathbb{N}$. Moreover

$$\boldsymbol{H}_1,\dots,\boldsymbol{H}_t-\boldsymbol{M}_{((s;q),p_1,\dots,p_t)}(T)=\inf\delta.$$

Proof. There are two cases.

Case 1: When $s=q$. Assume that inequality (12) holds and take $\zeta=1$, then

$$\Big[\sum_{j=1}^m|\sigma_j|^q\big|\boldsymbol{M}(T,a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^1,\dots,g_j^s,v)\big|^q\Big]^{\frac{1}{q}}$$
$$\le\delta\cdot\prod_{k=1}^s\sup_{\varphi\in K_k}\Big[\sum_{j=1}^m|\sigma_j|^{p_k}\big|\boldsymbol{H}_k(a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^k,\varphi)\big|^{p_k}\Big]^{\frac{1}{p_k}}$$

for all $v\in\mathfrak{B}_{\boldsymbol{P}}$. Hence

$$\sup_{v\in\mathfrak{B}_P}\Big[\sum_{j=1}^m|\sigma_j|^q\big|\boldsymbol{M}(T,a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^1,\dots,g_j^s,v)\big|^q\Big]^{\frac{1}{q}}$$
$$\le\delta\cdot\prod_{k=1}^s\sup_{\varphi\in K_k}\Big[\sum_{j=1}^m|\sigma_j|^{p_k}\big|\boldsymbol{H}_k(a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^k,\varphi)\big|^{p_k}\Big]^{\frac{1}{p_k}}.\qquad(13)$$

From Lemma 8 and using of inequality (13), one can obtain

$$\mathfrak{m}^{\boldsymbol{M}}_{(s;q)}\big((\sigma_j,T,a_j^1,\dots,a_j^1,\dots,a_j^t,c_j^r,g_j^1,\dots,g_j^s,v)\big)_{j=1}^m$$
$$=\sup_{v\in\mathfrak{B}_P}\left[\sum_{j=1}^m|\sigma_j|^q\big|\boldsymbol{M}(T,a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^1,\dots,g_j^s,v)\big|^q\right]^{\frac{1}{q}}$$
$$\le\delta\cdot\prod_{k=1}^s\sup_{\varphi\in K_k}\Big[\sum_{j=1}^m|\sigma_j|^{p_k}\big|\boldsymbol{H}_k(a_j^1,\dots,a_j^t,c_j^1,\dots,c_j^r,g_j^k,\varphi)\big|^{p_k}\Big]^{\frac{1}{p_k}}.$$

Hence T is **H** $_1$,..., **H** $_t$-**M**-$((s;q),p_1,\dots,p_t)$-mixing map and $\boldsymbol{H}_1,\dots,\boldsymbol{H}_t-\boldsymbol{M}_{((s;q),p_1,\dots,p_t)}(T)\le\delta$. Conversely, suppose that T is **H** $_1$,..., **H** $_t$-**M**-

$((s;q),p_1,\ldots,p_t)$-mixing map. Given $\sigma_1,\cdots,\sigma_m\in\mathbb{R}$, $a_1^1,\cdots,a_m^t\in\tilde{A}$, $c_1^1,\cdots,c_m^r\in\tilde{C}$, $g_1^1,\cdots,g_m^s\in\tilde{G}$, $v_1,\cdots,v_n\in V$ and $m,n\in\mathbb{N}$.

$$[\sum_{j=1}^m |\sigma_j|^q \sum_{\zeta=1}^n |\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v_\zeta)|^q]^{\frac{1}{q}}$$
$$\le$$
$$\left\|(\tau_j)_{j=1}^m|\ell_\infty\right\|\cdot[\sum_{j=1}^m \left|\frac{\sigma_j}{\tau_j}\right|^q \sum_{\zeta=1}^n \boldsymbol{P}(v_\zeta)^q \left|\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,\frac{v_\zeta}{P(v_\zeta)})\right|^q]^{\frac{1}{q}}$$
$$\le$$
$$\left\|(\tau_j)_{j=1}^m|\ell_\infty\right\| \sup_{v\in\mathfrak{B}_P}\left[\sum_{j=1}^m \left|\frac{\sigma_j}{\tau_j}\right|^q |\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v)|^q\right]^{\frac{1}{q}}[\sum_{\zeta=1}^n \boldsymbol{P}(v_\zeta)^q]^{\frac{1}{q}}. \quad (14)$$

Taking the infimum over all sequences $(\tau_j)_{j=1}^m\in\ell_\infty$ of inequality (14), then

$$[\sum_{j=1}^m |\sigma_j|^q \sum_{\zeta=1}^n |\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v_\zeta)|^q]^{\frac{1}{q}}$$
$$\le [\sum_{\zeta=1}^n \boldsymbol{P}(v_\zeta)^q]^{\frac{1}{q}}\mathfrak{m}_{(q;q)}^{\boldsymbol{M}}((\sigma_j,T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,\psi))_{j=1}^m$$
$$\le$$
$$[\sum_{\zeta=1}^n \boldsymbol{P}(v_\zeta)^q]^{\frac{1}{q}}\boldsymbol{H}_1,\ldots,\boldsymbol{H}_t-$$
$$\boldsymbol{M}_{((q;q),p_1,\ldots,p_t)}(T)\prod_{k=1}^s \sup_{\varphi\in K_k}[\sum_{j=1}^m |\sigma_j|^{p_k}|\boldsymbol{H}_k(a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^k,\varphi)|^{p_k}]^{\frac{1}{p_k}}.$$

Case 2: When $q<s$. Suppose that T is $\mathbf{H}_1,\ldots,\mathbf{H}_t$-$\mathbf{M}$-$((s;q),p_1,\ldots,p_t)$-mixing map. Given $\sigma_1,\cdots,\sigma_m\in\mathbb{R}$, $a_1^1,\cdots,a_m^t\in\tilde{A}$, $c_1^1,\cdots,c_m^r\in\tilde{C}$, $g_1^1,\cdots,g_m^s\in\tilde{G}$. From Proposition 9 and condition (**VIIII**), one can have

$$[\sum_{j=1}^m |\sigma_j|^q[\sum_{\zeta=1}^n |\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v_\zeta)|^s]^{\frac{q}{s}}]^{\frac{1}{q}}$$
$$=[\sum_{j=1}^m |\sigma_j|^q[\int_{\mathfrak{B}_P}|\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v)|^s d\mu(v)]^{\frac{q}{s}}]^{\frac{1}{q}}\cdot\left[\sum_{\zeta=1}^n \boldsymbol{P}(v_k)^s\right]^{\frac{1}{s}}$$
$$\le \mathfrak{m}_{(s;q)}^{\boldsymbol{M}}((\sigma_j,T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v))_{j=1}^m\cdot[\sum_{\zeta=1}^n \boldsymbol{P}(v_k)^s]^{\frac{1}{s}}$$
$$\le \boldsymbol{H}_1,\ldots,\boldsymbol{H}_t-\boldsymbol{M}_{((s,q),p_1,\ldots,p_t)}(T)\cdot[\sum_{\zeta=1}^n \boldsymbol{P}(v_k)^s]^{\frac{1}{s}}$$
$$\cdot\prod_{k=1}^s \sup_{\varphi\in K_k}[\sum_{j=1}^m |\sigma_j|^{p_k}|\boldsymbol{H}_k(a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^k,\varphi)|^{p_k}]^{\frac{1}{p_k}}.$$

To show the converse, notethat inequality (12) means

$$[\sum_{j=1}^m |\sigma_j|^q[\int_{\mathfrak{B}_P}|\boldsymbol{M}(T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,v)|^s d\mu(v)]^{\frac{q}{s}}]^{\frac{1}{q}}$$
$$\le \delta\cdot\prod_{k=1}^s \sup_{\varphi\in K_k}[\sum_{j=1}^m |\sigma_j|^{p_k}|\boldsymbol{H}_k(a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^k,\varphi)|^{p_k}]^{\frac{1}{p_k}} \quad (15)$$

for every discrete probability measure μ on $\mathfrak{B}_{\boldsymbol{P}}$ and $\sigma_1,\cdots,\sigma_m\in\mathbb{R}$, $a_1^1,\cdots,a_m^t\in\tilde{A}$, $c_1^1,\cdots,c_m^r\in\tilde{C}$, $g_1^1,\cdots,g_m^s\in\tilde{G}$. It follows that (15) holds for all probability measures μ on $\mathfrak{B}_{\boldsymbol{P}}$ and $\sigma_1,\cdots,\sigma_m\in\mathbb{R}$, $a_1^1,\cdots,a_m^t\in\tilde{A}$, $c_1^1,\cdots,c_m^r\in\tilde{C}$, $g_1^1,\cdots,g_m^s\in\tilde{G}$. Taking the supremum over $\mu\in\boldsymbol{W}(\mathfrak{B}_{\boldsymbol{P}})$ on the left side of (15) and using Proposition 9, therefore

$$\mathfrak{m}_{(s;q)}^{\boldsymbol{M}}((\sigma_j,T,a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^1,\ldots,g_j^s,\psi))_{j=1}^m$$
$$\le \delta\cdot\prod_{k=1}^s \sup_{\varphi\in K_k}[\sum_{j=1}^m |\sigma_j|^{p_k}|\boldsymbol{H}_k(a_j^1,\ldots,a_j^t,c_j^1,\ldots,c_j^r,g_j^k,\varphi)|^{p_k}]^{\frac{1}{p_k}}.$$

7. Conclusions and discussion

This work is concerned with the development of the unified version of mixing maps between arbitrary sets. The innovative general approach has been avoided the multiplication and the appearance of apparently different proofs of Pietsch Domination-type theorems. Based on the good results are achieved in the present proofs, it has encouraged the forthcoming work to focus on developing new nonlinear prototypes of mixing operators with new general settings.

8. Disclosure and conflict of interest

The authors declare that they have no conflicts of interest.

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