DOI: 10.24996/ijs.2022.63.12.32

ISSN: 0067-2904

On New Sandwich Results of Univalent Functions Defined by a Linear Operator

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Received: 29/9/2021 Accepted: 16/5/2022 Published: 30/12/2022

Abstract

In this research paper, we explain the use of the convexity and the starlikness properties of a given function to generate special properties of differential subordination and superordination functions in the classes of analytic functions that have the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk. We also show the significant of these properties to derive sandwich results when the Srivastava-Attiya operator $F_{\sigma,b} f(z)$ is used.

Keywords: Analytic function, univalent function, differential subordination, superordination, sandwich theorem, Srivastava-Attiya operator.

حول نتائج الساندوج جديدة للدوال احادية التكافؤ المعرفة بواسطة المؤثر الخطي

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الخلاصه

الهدف من هذا البحث هو توضيح كيفية استخدام الخصائص المحدبة والنجمية للدالة المعطاه لتوليد خصائص جديدة للتبعية التفاضلية وما فوق ذلك للدوال احادية التكافؤ $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ المعرفه في قرص الوحدة المفتوح , وكيف تلك الخصائص استخدمت في الحصول على نتائج الساندويتج بوجود مؤثر سرفيستافا – عطية $F_{\sigma,b} f(z)$.

1. Introduction.

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ be the class of all functions that are analytic in $\mathbb{U}$, where $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk. Let $\mathcal{H}[a, n]$ where $n \in N$, $a \in \mathbb{C}$, and $\mathcal{H}$ be a subclass of the functions $f \in \mathcal{H}$,which is given by

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots , \quad n \in N, a \in \mathbb{C} \quad . \tag{1.1}$$

We also assume $\mathbb{A} \subset \mathcal{H}$, where $\mathbb{A}$ is said to be the subclass of analytic and univalent functions in $\mathbb{U}$, that contains Maclaurin series and Taylor series that combined with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n . \tag{1.2}$$

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Now, we assume that $f, g \in \mathcal{H}$, so that the function f is a subordinate to function g, or the function g is superordinate to the function f. If there exists the Schwartz function w such that $f(z) = g(w(z))$, where $w(z)$ is analytic function in $\mho$ $with$ $|w(z)| < 1$ and $w(0) = 0, z \in \mho$, then one can say that $f \prec g$ or $f(z) \prec g(z)$ for $z \in \mho$ [1].
In addition, if g is univalent in $\mho$, then $f \prec g$ if and only if $f(\mho) = g(\mho)$ and $f(0) = g(0)$, ([1,2,3]).

Definition (1.1) [4, 5]: Let $\psi : \mathbb{C}^3 \times \mho \to \mathbb{C}$ and let $\hbar(z)$ is univalent in $\mho$. If $p(z)$ is analytic function in $\mho$ and satisfies the second-order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec \hbar(z), \tag{1.3}$$

then $p(z)$ is said to be a solution of the differential subordination (1.3). The solutions of equation (1.3) of differential subordination have dominant univalent function $\mathrm{Ч}(z)$ or more simply a dominant, if $p(z) \prec \mathrm{Ч}(z)$ to all $p(z)$ satisfying (1.3). A dominant function $\widetilde{\mathrm{Ч}}(z)$ that satisfies $\widetilde{\mathrm{Ч}}(z) \prec \mathrm{Ч}(z)$ for all dominant $\mathrm{Ч}(z)$ of (1.3) is called the best dominant of (1.3).

Definition (1.2) [4, 5, 6]: Let $\psi: \mathbb{C}^3 \times \mho \to \mathbb{C}$ and let $\hbar(z)$ be analytic function in $\mho$. If $\psi(p(z), zp'(z), z^2p''(z); z)$ and $p(z)$ are univalent functions in $\mho$ and $p(z)$ satisfiess the second-order differential superordination :

$$\hbar(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.4}$$

then $p(z)$ is said to be a solution of the differential superordination (1.4). The analytic function $\mathrm{Ч}(z)$ is said to be a subordinant of the solutions of equation (1.4) of the differential superordination, or more simply a subordinate, if $\mathrm{Ч}(z) \prec p(z)$ for all $p(z)$ satisfying eq. (1.4). If $\mathrm{Ч}(z) \prec \widetilde{\mathrm{Ч}}(z)$ for all subordinates $\mathrm{Ч}(z)$ of eq.(1.4) which is satisfied by univalent subordinate $\widetilde{\mathrm{Ч}}(z)$, then $\mathrm{Ч}(z)$ is said to be the best subordinate . Ali et al. [7,8] get sufficient consideration for normalize analytic functions to hold.

$$\mathrm{Ч}_1(z) \prec \frac{zf'(z)}{f(z)} \prec \mathrm{Ч}_2(z),$$

such that $\mathrm{Ч}_1(z)$ and $\mathrm{Ч}_2(z)$ in $\mho$ that take the form of univalent function with
$1 = \mathrm{Ч}_1(0) = \mathrm{Ч}_2(0)$.
The well-known monograph of Mocanu and Miller [5] and the supplemental recent book of Bulboacă [1,9] gave more details to the theory of differential subordination, subordination and superordination, with showing a definition of the Srivastara-Attiya transformation. The recent results are also given by several on Differential subordination, such as ([10-20]). So that we have to generalize the function of Hurwitz-Lerch that is defined in [21] with following sequence:

$$\Upsilon(z, \sigma, \mathfrak{b}) = \sum_{n=0}^{\infty} \frac{z^n}{(\mathfrak{b}+n)^\sigma}, \tag{1.5}$$

where $\mathfrak{b} \in \mathbb{C} \setminus z_0^-, \sigma \in \mathbb{C}$, $z \in \mho, Re(\mu) > 1$, and $z \in \partial\mho$.
Some cases of the function $\Upsilon(z, \sigma, \mathfrak{b})$ are involved. For further details see [22]. Srivastara-Attiya [22] considered the following normalized function :

$$R_{\sigma,\mathfrak{b}(z)} = (\mathfrak{b}+1)^\sigma[\gamma(z, \sigma, \mathfrak{b}) - \mathfrak{b}^{-\sigma}] = z + \sum_{n=2}^{\infty} \left(\frac{\mathfrak{b}+1}{\mathfrak{b}+n}\right)^\sigma z^n, \ \ z \in \mho \tag{1.6}$$

By using $R_{\sigma,\mathfrak{b}(z)}$ we obtain the linear operator $F_{\sigma,\mathfrak{b}}: \mathbb{A} \to \mathbb{A}$ that defines the convolution as follows:

$$F_{\sigma,\mathfrak{b}}f(z) = R_{\sigma,\mathfrak{b}(z)} * f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\mathfrak{b}+1}{\mathfrak{b}+n}\right)^\sigma a_n z^n, \ \ z \in \mho. \tag{1.7}$$

There are many various applications for $F_{\sigma,\mathfrak{b}}f(z)$ operator $F_{\sigma,b}f(z)$ see [2,23]. From (1.7), it is clear that

$$zF'_{\sigma,\mathfrak{b}}f(z) = (1+\mathfrak{b})F_{\sigma,\mathfrak{b}}f(z) - \mathfrak{b}F_{\sigma+1,\mathfrak{b}}f(z) \tag{1.8}$$

The major objective of present implementing is to discover enough conditions to a certain normalize analytic functions f to give:

$Ч_1(z) \prec \frac{F_{\sigma+1,b}f(z)}{z} \prec Ч_2(z)$,

such that $Ч_1(z)$ and $Ч_2(z)$ in U are called univalent functions with $Ч_1(0) = Ч_2(0) = 1$. Recently, several authors [2, 23-40] gave sandwich-type sequences for types of analytic functions using another sufficient conditions .

2. Preliminaries.

Definition $(\mathbf{2.1})[\mathbf{41}]$: Let Q be the set to all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, such that $\overline{U} = U \sqcup \{z \in \partial U\}$, and

$$E(f) = \{ \zeta \in \partial \mathfrak{U}:\ lim_{z \to \zeta} f(z) = \infty \}, \tag{2.1}$$

and $f'^{(\zeta)} \neq 0$ for $\zeta \in \partial U \setminus E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$, where $Q(1) = Q_1$ and $Q(0) = Q_0$.

Lemma (2.2) [1]: Assume θ and φ are analytic function in a domain D involving $Ч(U)$ with $\varphi(w) \neq 0$ such that $w \in Ч(U)$.

Take $ħ(z) = \theta(Ч(z)) + \psi(z)\ and\ \psi(z) = zЧ'(z)\,\varphi\,(Ч(z))$.

Furthermore, we assume that

(1) $\psi(z)$ be univalent starlike function in U,

(2) $\mathrm{Re}\left\{\frac{zħ'(z)}{\psi(z)}\right\} > 0\ for\ z \in$ U.

If $p(z)$ is analytic in U with $p(U) \subseteq D, p(0) = Ч(0)$ and

$$\theta(p(z)) + zp'(z)\varphi\,(p(z)) \prec \theta\,(Ч(z)) + zp'(z)\varphi\,(Ч(z))\ . \quad , (2.2)$$

Then Ч to be the best dominant and $p \prec Ч$.

Lemma (2.3) [41]: Assume that Re $\{ 1 + \frac{zЧ''(z)}{Ч'(z)} \} >$ max {0, - Re ($\frac{\alpha}{\beta}$) } and let $Ч(z)$ be univalent convex function in U and suppose $\alpha \in \mathbb{C}, \beta \in \mathbb{C} / \{0\}$ such that

$$\alpha p\,(z) + \beta zp'(z) \prec \alpha Ч(z) + \beta zЧ'(z), \tag{2.3}$$

then $p(z) \prec Ч(z)$ and $Ч(z)$ will be the best dominant.

Lemma (2.4) [5]: Assume $Ч(z)$ is univalent convex function in U and $Ч(0) = 1$. Suppose that $\beta \in \mathbb{C}$ such that $Re\,(\beta) > 0$. If $p(z) + \beta zp'(z)$ in U is univalent and $\rho(z) \in$ Ή$[Ч(0), 1] \sqcap$ Q, satisfies $Ч(z) + \beta zЧ'(z) \prec p(z) + \beta zp'(z)$, then

$Ч(z) \prec p(z) and\ Ч(z)$ is the best subordinant**.**

3. Subordination Results.

Theorem (3.1). Assume $Ч(z)$ is univalent convex function in U $with\ Ч(z) \neq 0\ , and\ Ч(0) = 1$. If $Ч(z)$ satisfies:

$$Re\{1 + \frac{zЧ''(z)}{Ч'(z)}\} > \max\{0\,, Re(\tfrac{1}{\vartheta})\ ,\ \text{for all } z \in U, \tag{3.1}$$

where $\vartheta \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We also assume

$$\psi(z) = \left(1 - \frac{\vartheta}{\mathcal{X}}\right)\left(\frac{F_{\sigma+1,b}f(z)}{z}\right) + \frac{\vartheta}{\mathcal{X}}\left(\frac{F_{\sigma,b}f(z)}{z}\right)\ , \mathcal{X} > 0\ . \tag{3.2}$$

If $Ч(z)$ satisfies the subordination $\psi(z) \prec Ч(z) + \vartheta\, z\, Ч'(z)$, (3.3)

then $\left(\frac{F_{\sigma+1,b}f(z)}{z}\right) \prec Ч(z)$, (3.4)

and $Ч(z)$ will be the best dominant of equation (3.3).

Proof. Consider the function $p(z) = \left(\frac{F_{\sigma+1,b}\,f(z)}{z}\right)\ , z \in U.$ (3.5)

By taking the differentiation of eq. (3.5) logarithmically with respect to z, then we obtain

$$\frac{z\,p'(z)}{p(z)}=\frac{z\left(F_{\sigma+1,\mathrm{b}}f(z)\right)'}{F_{\sigma+1,\mathrm{b}}f(z)}-1. \tag{3.6}$$

Now, we use the identity (1.8) in (3.6) ,then we get

$\frac{z\,p'(z)}{p(z)}=\frac{1}{\sigma}\left(\frac{F_{\sigma,\mathrm{b}}f(z)}{F_{\sigma+1,\mathrm{b}}f(z)}-1\right).$

Therefore, we apply Lemma (2.3) with $\alpha=1$ and $\beta=\vartheta$, so that the subordination (3.3) implies that $Ч(z)$ is best dominant and $p(z)\prec Ч(z)$.

The proof is complete.

In this step, we set $-1\le B\le A<1$,and $Ч(z)=\frac{1+Az}{1+Bz}$, in previous Theorem.

The condition (3.1) becomes

$$Re\left\{\frac{1-Bz}{1+Bz}\right\}>max\left\{0,-\mathrm{Re}\left(\frac{1}{\vartheta}\right)\right\}, z\in \mathrm{U}. \tag{3.7}$$

The function $\varphi(\partial)=\frac{1-\partial}{1+\partial}$, $|\partial|<|B|$ is a convex function in U and because of $\varphi(\overline{\partial})=\overline{\varphi(\partial)}$ for all $|\partial|<|\mathrm{B}|$, therefore the image $\varphi(\mathrm{U})$ will be a convex domain symmetrically according to real axis, as a result,

$\inf\left\{Re\left(\frac{1-Bz}{1+Bz}\right): z\in \mathrm{U}\right\}=\frac{1-|B|}{1+|B|}>0.$

The inequality (3.7) is equivalent to

$Re\left\{\frac{1}{\vartheta}\right\}>\frac{|B|-1}{1+|B|}.$

Therefore, we get the result corollary.

Corollary (3.1). Assume that $max\left\{0,-Re\left(\frac{1}{\vartheta}\right)\right\}\le\frac{1-|B|}{1+|B|}$ and $-1\le B\le A<1.$

If $\psi(z)\prec\frac{1-Az}{1+Bz}+\vartheta\frac{A-B}{(1+Bz)^2}$, then $\left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right)\prec\frac{1+\mathrm{A}z}{1+\mathrm{B}z}$, and the term of $\frac{1+Az}{1+Bz}$ will be the best dominant.

Corollary (3.2): Suppose that $Re\left(\frac{1}{\alpha}\right)\ge 0$. If $\psi(z)\prec\frac{1+z}{1-z}+\vartheta\frac{2z}{(1+z)^2}$.

Then $\mathrm{Re}\left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right)>0$, and the term of $\frac{1+z}{1-z}$ will be the best dominant.

Now, For $Ч(z)=e^{Az}$, $|A|<\pi$.

The next corollary is obtained by Theorem (3.1).

Corollary (3.3). Assume $Re(1+\mathrm{A}z)>\max\left\{0,-\mathrm{Re}\left(\frac{1}{\vartheta}\right)\right\}$, $|A|<\pi$, such that $\vartheta\in\mathbb{C}^*$, $\psi(z)\prec(1+\vartheta zA)e^{Az}$, then $\mathrm{Re}\left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right)<e^{Az}$, and e^{Az} will be the best dominant.

Theorem (3.2). Let $Ч(z)$ be univalent function in U, such that $Ч(z)\neq 0$ and $Ч(0)=1$ for $z\in U$ and suppose that $Ч(z)$ satisfies the term of $Re\left(\frac{zЧ'(z)}{Ч(z)}\right)>0$, which is univalent and starlike function in U. In addition, we suppose that $\gamma_1,\gamma_2,\propto,\partial\in\mathbb{C}^*$, with $\gamma_1+\gamma_2\neq 0$,

$$\frac{\gamma_1\, F_{\sigma+1,\mathrm{b}}f(z)+\gamma_2 F_{\sigma,\mathrm{b}}f(z)}{(\gamma_1+\gamma_2)z}\neq 0\ , z\in \mathrm{U}. \tag{3.8}$$

and

$$[1+\propto\partial\left(\frac{\gamma_1 z\left(F_{\sigma+1,\mathrm{b}}f(z)\right)'+\gamma_2 z\left(F_{\sigma,\mathrm{b}}f(z)\right)'}{\gamma_1\, F_{\sigma+1,\mathrm{b}}f(z)+\gamma_2\, F_{\sigma,\mathrm{b}}f(z)}-1\right)\prec 1+\partial\,\frac{zЧ'(z)}{Ч(z)}], \tag{3.9}$$

then $\daleth(z)$ will be the best dominant of equation (3.9) ,and $\left(\frac{\gamma_1 \; F_{\sigma+1,b}f(z)+\gamma_2 \; F_{\sigma,b}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \prec \daleth(z)$.

Proof. Suppose that $p(z)=\left(\frac{\gamma_1 \; F_{\sigma+1,b}f(z)+\gamma_2 \; F_{\sigma,b}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}, z\in \mho$. (3.10)
According to (3.8),we have $p(z)$ is analytic function in $\mho$.
By taking the differentiation of eq. (3.10) logarithmically according to z , we obtain
$\frac{zp\prime(z)}{p(z)}=\propto\left(\frac{\gamma_1 z\left(F_{\sigma+1,b}f(z)\right)'+\gamma_2 z\left(F_{\sigma,b}f(z)\right)'}{\gamma_1 \; F_{\sigma+1,b}f(z)+\;\gamma_2 \; F_{\sigma,b}f(z)}-1\right)$.
To prove our result, we use Lemma (2.2). Now, consider $(w)=\frac{\partial}{w}$ *and* $\theta(w)=1$,
we note that $w\in\mathbb{C}^*$, and θ is analytic function in $\mathbb{C}$.
We also suppose $\hbar(z)=\theta(\daleth(z))+\psi(z)=1+\partial\frac{z\daleth'(z)}{\daleth(z)}$
and $\psi(z)=z\daleth'(z)\varphi(\daleth(z))=\partial\frac{z\daleth'(z)}{\daleth(z)}$, so that the function $\psi(z)$ is starlike in $\mho$,and

$$Re\left(\frac{z\hbar'(z)}{\psi(z)}\right)=Re\left(1+\frac{z\daleth''(z)}{\daleth'(z)}-\frac{z\daleth'(z)}{\daleth(z)}\right)>0.$$

Therefore, the subordination (3.9) implies that $\daleth(z)$ is the best dominant and$p(z)\prec\daleth(z)$.
The next result can be obtained by setting $-1\le B\le A<1$, $\daleth(z)=\frac{1+Az}{1+Bz}$, $\gamma_2=0$ *and* $\partial=1$ in Theorem (3.3).

Corollary (3.4). Suppose that $-1\le B\le A<1$ ℵ $\left(\frac{F_{\sigma+1,b}f(z)}{z}\right)\neq 0, z\in\mho, \propto\in\mathbb{C}^*$.
If $[1+\propto\left(\frac{z\left(F_{\sigma+1,b}f(z)\right)'}{F_{\sigma+1,b}f(z)}-1\right)\prec 1+\frac{(A-B)z}{(1+Az)(1-Bz)}]$, then $\left(\frac{Az+1}{Bz+1}\right)$ will be best dominant and $\left(\frac{F_{\sigma+1,b}f(z)}{z}\right)^{\propto}\prec\frac{Az+1}{Bz+1}$.

Theorem (3.3). Suppose that the univalent function $\daleth(z)$ in $\mho$ *with* $\daleth(z)\neq 0, \daleth(0)=1$, $\forall z\in\mho, \propto, \partial\in\mathbb{C}^*, \gamma_1, \gamma_2, \varphi, \tau\in\mathbb{C}$, with $\gamma_1+\gamma_2\neq 0$ and the eq. (3.8) is satisfied.
Let $Re\left(1+\frac{z\daleth''(z)}{\daleth'(z)}\right)>\max\{0,-\operatorname{Re}\left(\frac{\varphi}{\partial}\right)\}$, and

$$\Theta(z)=\left(\frac{\gamma_1 \; F_{\sigma+1,b}f(z)+\gamma_2 \; F_{\sigma,b}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}\times\left[\varphi+\propto\partial\left(\frac{\gamma_1 z\left(F_{\sigma+1,b}f(z)\right)'+\gamma_2 z\left(F_{\sigma,b}f(z)\right)'}{\gamma_1 \; F_{\sigma+1,b}f(z)+\;\gamma_2 \; F_{\sigma,b}f(z)}-1\right)\right]+\tau . \quad (3.11)$$

If $q(z)$ holds the subordination $\Theta(z)\prec\varphi\daleth(z)+\partial z\daleth'(z)+\tau$, (3.12)
then $\left(\frac{\gamma_1 \; F_{\sigma+1,b}f(z)+\;\gamma_2 \; F_{\sigma,b}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}\prec\daleth(z)$, (3.13)
and $\daleth(z)$ will be best dominant of equation (3.12) .
Proof. Assume $\xi(z)=\left(\frac{\gamma_1 \; F_{\sigma+1,b}f(z)+\;\gamma_2 \; F_{\sigma,b}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}$, (3.14)
then the function $\xi(z)$ be analytic in $\mho$ and $\daleth(0)=1$, hence by taking the differentiation of eq. (3.14) logarithmically with respect to z , and by taking the eq. (1.8) in a recent equation ,

$$\Theta(z)=\left(\frac{\gamma_1 \; F_{\sigma+1,b}f(z)+\gamma_2 \; F_{\sigma,b}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}\times\left[\varphi+\propto\partial\left(\frac{\gamma_1 z\left(F_{\sigma+1,b}f(z)\right)'+\gamma_2 z\left(F_{\sigma,b}f(z)\right)'}{\gamma_1 \; F_{\sigma+1,b}f(z)+\;\gamma_2 \; F_{\sigma,b}f(z)}-1\right)\right]+\tau.$$

Thus the subordination (3.13) is against to
$\varphi\xi(z) + \partial z\xi'(z) + \tau \prec \varphi Ч(z) + \partial z Ч'(z) + \tau$.
Now, if we apply of Lemma (2.3) with $\beta = \frac{\partial}{\varphi}$ $and\ \alpha = 1$, then we have eq.(3.13) .
We also get the next corollary, if we substitute $-1 \leq B \leq A < 1$, $Ч(z) = \frac{Az+1}{Bz+1}$, in Theorem (3.3) and Theorem(3.2).

Corollary (3.5) Let $-1 \leq B \leq A < 1$, $\partial, \varphi \in \mathbb{C} \setminus \{0\}, Re\ \frac{1-Az}{1+Bz} > \max\{0, -Re\left(\frac{\varphi}{\partial}\right)\}$,
if $f \in \mathbb{A}$ will satisfy the condition of subordination :
$\Theta(z) \prec \frac{1+Az}{1+Bz} + \frac{\partial}{\varphi}\frac{(A-B)z}{(1+Bz)^2}$, where $\Theta(z)$ defined by eq.(3.11), then
$\frac{1+Az}{1+Bz}$ will be the best dominant $\left(\frac{\gamma_1\ F_{\sigma+1,\mathrm{b}}f(z) + \gamma_2 F_{\sigma,\mathrm{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \prec \frac{1+Az}{1+Bz}$.
The following corollary can be get by setting $B = -1$, $and\ A = 1$ in corollary (3.5).

Corollary (3.6). Suppose that $Re\ \frac{1-z}{1+z} > \max\{0, -Re\left(\frac{\varphi}{\partial}\right)\}$, ∂ , $\varphi \in \mathbb{C} \setminus \{0\}$,if $f \in \mathbb{A}$ satisfies
the following subordination condition $[\Theta(z) \prec \frac{1+z}{1+z} + \frac{\partial}{\varphi}\frac{2z}{(1+z)^2}]$,
such that $\Theta(z)$ is obtain by eq. (3.10), then $\frac{1+z}{1-z}$ will be best dominant
and $\left(\frac{\gamma_1\ F_{\sigma+1,\mathrm{b}}f(z) + \gamma_2\ F_{\sigma,\mathrm{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \prec \frac{1+z}{1+z}$.

4. Superordination Results.

Theorem (4.1). Let $q(z)$ be a convex and univalent function in the unit disk U, with $0 \neq \left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right) \in \mathcal{H}[Ч(0), 1] \sqcap \mathrm{Q}$ and $Ч(z) \neq 0$, $Ч(0) = 1$ for all $z \in U$ with $Re(\partial) > 0$.
Then the univalent function $\psi(z) = \left(1 - \frac{\vartheta}{\chi}\right)\left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right) + \frac{\vartheta}{\chi}\left(\frac{F_{\sigma,\mathrm{b}}f(z)}{z}\right)$ in U will be the best subordination of the following equation $Ч(z) + \vartheta\ z\ Ч'(z) \prec \psi(z)$ (4.1)
, and $Ч(z) \prec \left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right)$.

Proof: According to our assumptions, we let $p(z) = \left(\frac{F_{\sigma+1,\mathrm{b}}f(z)}{z}\right)$, $z \in$ U be analytic in U .
By taking the differentiation logarithmically with respect to z, so that one obtains
$\frac{z\mathrm{p}'(z)}{\mathrm{p}(z)} = \frac{z\left(\mathrm{F}_{\sigma+1,\mathrm{b}}f\ (z)\right)'}{\mathrm{F}_{\sigma+1,\mathrm{b}}f(z)} - 1.$
By some calculations, we obtain $\psi(z) = p(z) + \vartheta\ z\ p'(z)$, where $\psi(z)$ is known in eq. (3.2)
and from Lemma (2.4) ,we get the required result .

Theorem (4.2). Assume $Ч(z)$ be a convex function in U, with $Ч(z) \neq 0, Ч(0) = 1$
for all $z \in$ U, $\propto, \partial \in \mathbb{C}^*$, $\gamma_1, \gamma_2, \varphi, \tau \in \mathbb{C}$, with $Re\left(\frac{\varphi}{\partial}\right) > 0$, and $\gamma_1 + \gamma_2 \neq 0$.
Let $0 \neq \left(\frac{\gamma_1\ F_{\sigma+1,\mathrm{b}}f(z) + \gamma_2\ F_{\sigma,\mathrm{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \in \mathcal{H}[Ч(0), 1] \sqcap \mathrm{Q}$.
If the univalent $Ч(z)$ in U and the known function $\Theta(z)$ in (3.11) satisfy
$\varphi Ч(z) + \partial z\ Ч'(z) + \tau \prec \Theta(z)$, (4.2)
then $Ч(z)$ will be the best subordinant and $Ч(z) \prec \left(\frac{\gamma_1\ F_{\sigma+1,\mathrm{b}}f(z) + \gamma_2\ F_{\sigma,\mathrm{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}$.

Proof. Suppose that $\xi(z) = \left(\frac{\gamma_1\ F_{\sigma+1,\mathrm{b}}f(z) + \gamma_2\ F_{\sigma,\mathrm{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}$. (4.3)

By taking the differentiation of the eq. (4.3) with respect to logarithmically of z, we obtain

$$\frac{z\xi'(z)}{\xi(z)} = \propto \left(\frac{\gamma_1 z\left(F_{\sigma+1,\mathfrak{b}}f(z)\right)' + \gamma_2 z\left(F_{\sigma,\mathfrak{b}}f(z)\right)'}{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)}\right). \tag{4.4}$$

A simple computation and using eq.(1.8) from eq.(4.4), we have

$$\left(\frac{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}$$

$$\times \left[\varphi + \propto \partial\left(\frac{\gamma_1 z\left(F_{\sigma+1,\mathfrak{b}}f(z)\right)' + \gamma_2 z\left(F_{\sigma,\mathfrak{b}}f(z)\right)'}{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)} - 1\right)\right] + \tau = \varphi\xi(z) + \partial z\xi'(z) + \tau.$$

From Lemma (2.4), we have the required result.

The next corollary can be get by setting $-1 \le B \le A < 1$, $q(z) = \frac{1+Az}{1+Bz}$ in Theorem (4.2).

Corollary (4.1). Assume that $\left(\frac{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \in \acute{H}[Ч(0),1]\sqcap Q, -1 \le B \le A < 1 \; and \; Re\left(\frac{\varphi}{\partial}\right) > 0.$

If $f(z) \in \mathbb{A}$ holds and under superordination condition with $\Theta(z)$ is univalent function defined by (3.12)

$$\frac{1+Az}{1+Bz} + \frac{\partial}{\varphi}\frac{(A-B)z}{(1+Bz)^2} \prec \Theta(z),$$

then, $\frac{1+Az}{1+Bz} \prec \left(\frac{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto}$, and $\frac{1+Az}{1+Bz}$ will be the best subordinant.

5. Sandwich Results

It is important to point out that we can obtain the following final two results, sandwich theorems, by applying the starlike properties in theorem (3.2) and combining theorem (3.3) and theorem (4.2) to consequences of superordination and differential subordination.

Theorem (5.1). Assume $Ч_1(z) \; and \; Ч_2(z)$ are univalent and convex in $\mho$ with $1 = Ч_1(0) \ne Ч_2(0)$ where $Ч_1(z) \; and \; Ч_2(z)$ are not equal to zero, $\partial \in \mathbb{C}^*$, for all $z \in \mho$ and $0 \ne \left(\frac{F_{\sigma+1,\mathfrak{b}}f(z)}{z}\right) \in \acute{H}[1,1]\sqcap Q.$

Suppose that $\psi(z)$ be univalent function in $\mho$, where $\psi(z)$ is given by (3.2) satisfies

$$Ч_1 z + \partial z Ч_1'(z) \prec \psi(z) \prec Ч_2(z) + \partial z Ч_2'(z), \tag{5.1}$$

then $Ч_1(z) \prec \left(\frac{F_{\sigma+1,\mathfrak{b}}f(z)}{z}\right) \prec Ч_2(z)$, and $Ч_2(z), Ч_1(z)$ are to be the best dominant and best subordinant, respectively .

In order to get the next theorem we have to join the results in Theorem (3.3) and Theorem (4.2).

Theorem (5.2). Assume $Ч_1(z) \; and \; Ч_2(z)$ are univalent and convex in $\mho$ with $1 = Ч_1(0) = Ч_2(0)$ where $0 \ne \left(\frac{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \in \acute{H}[1,1]\sqcap Q \; and Ч_1(z), Ч_2(z)$ are not equal to zero, $\partial \in \mathbb{C}^*$ for all $z \in \mho$, and let $\Theta(z)$ be univalent function in $\mho$ that satisfies

$$\varphi Ч_1(z) + \partial z Ч_1'(z) + \tau \prec \Theta(z) \prec \varphi Ч_2(z) + \partial z Ч_2'(z), \tag{5.2}$$

then $Ч_2(z)$ and $Ч_1(z)$ are to be the best dominant and best subordinant respectively and

$$Ч_1(z) \prec \left(\frac{\gamma_1 \; F_{\sigma+1,\mathfrak{b}}f(z) + \gamma_2 \; F_{\sigma,\mathfrak{b}}f(z)}{(\gamma_1+\gamma_2)z}\right)^{\propto} \prec Ч_2(z).$$

Acknowledgments:

The authors express their sincere gratitude to Mustansiriyah University, College of Science, Department of Mathematics, and the University of Al-Qadisiyah for their assistance in carrying out this research.

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