DOI: 10.24996/ijs.2022.63.10.28

ISSN: 0067-2904

Some Applications of Quasi-Subordination for Bi-Univalent Functions Using Jackson's Convolution Operator

Waggas Galib Atshan , Reaam Abd Al-sajjad *
Department of Mathematics, College of Science, University of Al-Qadisiyah, Diwaniyah-Iraq

Received: 29/9/2021 Accepted: 4/4/2022 Published: 30/10/2022

Abstract
In this paper, subclasses of the function class $\sum$ of analytic and bi-univalent functions associated with operator $L_q^{k,\lambda}$ are introduced and defined in the open unit disk $\triangle$ by applying quasi-subordination. We obtain some results about the corresponding bound estimations of the coefficients a_2 and a_3 .

Keywords: bi−univalent function, Quasi–subordination, coefficient estimates, subordination.

بعض تطبيقات شبه التابعية للدوال ثنائية التكافؤ باستخدام مؤثر الالتواء لجاكسون

وقاص غالب عطشان , رئام عبد السجاد جبار*
قسم الرياضيات , كلية العلوم , جامعة القادسية , الديوانية , العراق

الخلاصة
في هذا البحث تم تقديم أصناف جزئية لصنف الدالة $\sum$ من الدوال التحليلية الثنائية التكافؤ المرتبطة بالمؤثر $L_q^{k,\lambda}$ المعرف في قرص الوحدة المفتوح $\triangle$ بواسطة تطبيق شبه التابعية . حصلنا على بعض النتائج حول التخمينات المقيدة المقابلة للمعاملات a_2 و a_3 .

1.Introduction

Let A be the class of normalized analytic functions in the open unit disk $\Delta = \{z \in C : |z| < 1\}$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let S be the class of all univalent functions from A in Δ. According to the Koebe One Quarter Theorem [1,2], the inverse f^{-1} of every $f \in S$ satisfies :

$$f^{-1}(f(z)) = z \quad (z \in \Delta) \quad \text{and} \quad f(f^{-1}(w)) = w \quad (w \in \Delta_p),$$

where $p \geq \frac{1}{4}$ denotes the radius of the image $f(\Delta)$ and $\Delta_p = \{ z \in C : |z| < p \}$. It is recalled that

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \quad (1.2)$$

*Email: sci2021@uoanbar.edu.iq

If both functions $f \in A$ and its inverse f^{-1} are univalent in Δ, then it is bi-univalent. Denote to the class of all bi-univalent functions $f \in A$ in Δ by $\sum$.

In 1967, Le [3] introduced the analytic and bi-univalent function and proved that $|a_2| \leq 1.51$. Moreover, Br and Cl [4] conjectured that $|a_2| \leq \sqrt{2}$, Ne [5] obtained that $|a_2| = \frac{4}{3}$. Later, Styer and Wright [6] showed that there exists the function $f(z)$ so that $|a_2| > \frac{4}{3}$. However, the upper bound estimate $|a_2| < 1.485$ of coefficient for any function in $\sum$ by Tan [7] is the best. Based on the works of Br and Ta [8] and Sr et al. [9], many subclasses of analytic and bi-univalent functions class $\sum$ were introduced and investigated and the non-sharp estimates of first two Taylor- Maclaurin coefficients $|a_2|$ and $|a_3|$. Recently, Srivastava et al.[10, 11] gave some new subclasses of the function class $\sum$ of analytic and bi-univalent functions to unify the works of Deniz [12].

Now we mention the concept of subordination between analytic functions. Let f and g are analytic functions in Δ. Then we state that the function f is subordinate to g, if there exists a Schwarz function w, such that $f(z) = g(w(z)), (z \in \triangle)$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$, $(z \in \triangle)$. Specifically, if the function g is univalent in $\triangle$, the above subordination is equivalent to the conditions $f(0) = g(0)$, $f(\triangle)$ $g(\triangle)$. In year 1970, the concept of the subordination was extended to quasi-subordination by Ro in [13]. We refer a function f quasi-subordinate to a function g in $\triangle$ if there exists the Schwarz function w and an analytic function φ satisfying $|\varphi(z)| < 1$ such that $f(z) = \varphi(z)\, g(w(z))$ in $\triangle$. We then write $f \prec_q g$. If $\varphi(z) = 1$ then the quasi-subordination reduces to the subordination. If we set $w(z) = z$, then $f(z) = \varphi(z)\, g(z)$ and we say that f is majorized by g. It is denoted as $f(z) \ll g(z)$ in $\triangle$. Therefore quasi-subordination is an extension of the definition of the subordination. In addition, the majorization emphasizes its significance. The related works of quasi-subordination can be found in [14,13]. See [15] for the subclasses of analytic and bi-univalent associated with quasi-subordination. Ma– Minda [16] introduced the following classes using subordination:

$$S^*(\phi) = \left\{ f \in A : \frac{z f'(z)}{f(z)} \prec \phi(z), z \in \Delta \right\},$$

where ϕ is an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ on a starlike area with respect to 1 and which is symmetric consider to the real axis. A function $f \in S^*(\phi)$ is called Ma–Minda starlike. The class $S^*(\phi)$ contains various well - known subcategories of starlike function as private case.

Let $f \in A$ be given by (1.1) and g be given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.3}$$

Hadamard product of $f(z)$ and $g(z)$ is denoted by $(f * g)(z)$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.4}$$

For $\lambda > -1$ and $0 < q < 1$, El-Deeb et al. [17] defined the linear operator $H_{\Upsilon}^{\lambda,q} : A \to A$ by

$H_{\Upsilon}^{\lambda,q} f(z) * M_{q,\lambda+1}(z) = z\, D_q\, (f * \Upsilon)(z) \qquad (z \in \triangle),$

where the function $M_{q,\lambda}(z)$ is given by

$$M_{q,\lambda}(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda]_{q,n-1}}{[n-1]_q!} z^n, \qquad (z \in \triangle).$$

A simple calculation indicates that

$$H_{\Upsilon}^{\lambda,q} f(z) = z + \sum_{n=2}^{\infty} \frac{[n]_q!}{[\lambda+1]_{q,n-1}} a_n \psi_n z^n \quad (\lambda > -1;\ 0 < q < 1;\ z \in \triangle). \tag{1.5}$$

For the function $f \in A$, Jackson's q – derivative [18] ($0 < q < 1$) is expressed by:

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ f'(z), & z = 0 \end{cases} \tag{1.6}$$

and $D_q^2 f(z) = D_q\left(D_q f(z)\right)$. Thus, from Eq (1.6), we deduce that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_2 \, z^{n-1},$$

$$where \ [n]_q = \frac{1-q^n}{1-q} \text{. If } q \to 1^-, \quad we\ get\ [n]_q \to n.$$

Lately, in [19] the Sălăgean type q-differential operator has been introduced and is given as follows :

$$D_q^0 f(z) = f(z)$$
$$D_q^1 f(z) = z \, D_q f(z)$$
$$D_q^k f(z) = z \, D_q\left(D_q^{k-1} f(z)\right)$$

$$D_q^k f(z) = z + \sum_{n=2}^{\infty} [n]_q^k \, a_n z^n, \qquad (k \in N_0, z \in \Delta). \tag{1.7}$$

Hadamard product of the operators $H_{\Upsilon}^{\lambda,q} f(z)$ and $D_q^k f(z)$ defined as

$$L_q^{k,\lambda} f(z) = H_{\Upsilon}^{\lambda,q} f(z) * D_q^k f(z) = z + \sum_{n=2}^{\infty} \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \, [n]_q^k \, a_n \, \psi_n \, z^n,$$

$$L_q^{k,\lambda} f(z) = z + \sum_{n=2}^{\infty} \Omega_{n,q} \, a_n \, z^n,$$

where $\Omega_{n,q} = \left(\left(\frac{[n]_q!}{[\lambda+1]_{q,n-1}}\right) [n]_q^k \, \psi_n\right)$.

Several authors studied quasi-subordination of bi-univalent for another conditions, like, [20-38]. Throughout this idea, it is assumed $\phi(z)$ is analytic and univalent with positive real part in Δ and let

$$\phi(z) = 1 + G_1 z + G_2 z^2 + \cdots, \qquad (G \in R^+) \tag{1.8}$$

Also, let $\Gamma(z)$ be an analytic function in Δ and

$$\Gamma(z) = C_0 + C_1 z + C_2 z^2 + \cdots, \qquad (z \in \triangle). \tag{1.9}$$

Lemma 1.1. (See [39,40,41]). Let P be class of all analytic functions p in U such that $Re(p(z)) > 0$ and have the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in \Delta$, then $|p_i| \leq 2$ for each $i \in N$.

2. Coefficient Estimates For The Class $B_{\Sigma,q}^{k,\lambda}(\rho,\sigma,\phi)$

Definition 2.1. For $0 \leq \rho \leq 1$ and $0 \leq \sigma \leq 1$, a function $f \in \Sigma$ defined in (1.1) is said to be in the class $B_{\Sigma,q}^{k,\lambda}(\rho,\sigma,\phi)$ if the following quasi-subordination holds:

$$\left[(1+\rho)\left(\frac{z\left(L_q^{k,\lambda} f(z)\right)'}{\left(L_q^{k,\lambda} f(z)\right)}\right) - \rho\left(\frac{\sigma z^2\left(L_q^{k,\lambda} f(z)\right)'' + z\left(L_q^{k,\lambda} f(z)\right)'}{\sigma z\left(L_q^{k,\lambda} f(z)\right)' + (1-\sigma)\left(L_q^{k,\lambda} f(z)\right)}\right) - 1\right] \prec_q (\phi(z) - 1),$$

$$\left[(1+\rho)\left(\frac{w\left(L_q^{k,\lambda}\, g(w)\right)'}{\left(L_q^{k,\lambda}\, g(w)\right)}\right)-\rho\left(\frac{\sigma\, w^2\left(L_q^{k,\lambda}\, g(w)\right)''+w\left(L_q^{k,\lambda}\, g(w)\right)'}{\sigma\, w\left(L_q^{k,\lambda}\, g(w)\right)'+(1-\sigma)\left(L_q^{k,\lambda}\, g(w)\right)}\right)-1\right] \prec_q (\phi(w)-1),$$

where the function g is the extension of $f^{-1}\ in\ \Delta$.

Remark 2.1. For $\rho = 0$, a function $f \in \sum$ defined in (1.1) is said to be in the class $B_{\Sigma,q}^{k,\lambda}(0,\sigma,\phi)$ if the following conditions are satisfied:

$$\left[\left(\frac{z\left(L_q^{k,\lambda}\, f(z)\right)'}{\left(L_q^{k,\lambda}\, f(z)\right)}\right)-1\right] \prec_q (\phi(z)-1),$$

$$\left[\left(\frac{w\left(L_q^{k,\lambda} g(w)\right)'}{\left(L_q^{k,\lambda} g(w)\right)}\right)-1\right] \prec_q (\phi(w)-1).$$

Theorem 2.1. Let $f(z)$ given by (1.1) be in the class $B_{\Sigma,q}^{k,\lambda}(\rho,\sigma,\phi)$. For $0 \le \rho \le 1$, $0 \le \sigma \le 1$, then

$$|a_2| \le min\left\{\frac{|C_0|\, G_1}{\left|(1-\rho\sigma)\Omega_{2,q}\right|}, \sqrt{\frac{|C_0|\,(G_1+|G_2-G_1|)}{\left|2(1-2\rho\sigma)\Omega_{3,q}-\{(1+\rho)-\rho(\sigma+1)^2\}\Omega_{2,q}^2\right|}}\right\}, \tag{2.1}$$

$$|a_3| \le min\left\{\frac{(|C_1|+2|C_\circ|)G_1}{\left|4(1-2\rho\sigma)\Omega_{3,q}\right|}+\frac{C_\circ^2 G_1^2}{(1-\rho\sigma)^2\,\Omega_{2,q}^2}, \frac{(|C_1|+2|C_\circ|)G_1}{\left|4(1-2\rho\sigma)\Omega_{3,q}\right|} + \frac{|C_\circ|\,(G_1+|G_2-G_1|)}{\left|2(1-2\rho\sigma)\Omega_{3,q}-\{(1+\rho)-\rho(\sigma+1)^2\}\Omega_{2,q}^2\right|}\right\}. \tag{2.2}$$

Proof. Since $f \in B_{\Sigma,q}^{k,\lambda}(\rho,\sigma,\phi)\ and\ g=f^{-1}$. Then, there are analytic functions $x,y:\Delta \to \Delta\ with\ x(z)=s_1z+\sum_{j=2}^{\infty} s_j\, z^j,\ \ y(w)=t_1w+\sum_{j=2}^{\infty} t_j\, w^j, x(0)=y(0)=0$, such that

$$\left[(1+\rho)\left(\frac{z\left(L_q^{k,\lambda}\, f(z)\right)'}{\left(L_q^{k,\lambda}\, f(z)\right)}\right)-\rho\left(\frac{\sigma\, z^2\left(L_q^{k,\lambda}\, f(z)\right)''+z\left(L_q^{k,\lambda}\, f(z)\right)'}{\sigma\, z\left(L_q^{k,\lambda}\, f(z)\right)'+(1-\sigma)\left(L_q^{k,\lambda}\, f(z)\right)}\right)-1\right] = \Gamma(z)\,(\phi(x(z)-1)), \tag{2.3}$$

$$\left[(1+\rho)\left(\frac{w\left(L_q^{k,\lambda}\, g(w)\right)'}{\left(L_q^{k,\lambda}\, g(w)\right)}\right)-\rho\left(\frac{\sigma\, w^2\left(L_q^{k,\lambda}\, g(w)\right)''+w\left(L_q^{k,\lambda}\, g(w)\right)'}{\sigma\, w\left(L_q^{k,\lambda}\, g(w)\right)'+(1-\sigma)\left(L_q^{k,\lambda}\, g(w)\right)}\right)-1\right] = \Gamma(w)\,(\phi(y(w)-1)). \tag{2.4}$$

Define the functions $b(z)$ and $e(w)$ by

$$b(z)=\frac{1+x(z)}{1-x(z)}=1+s_1z+s_2z^2+\cdots, \tag{2.5}$$

$$e(w)=\frac{1+y(w)}{1-y(w)}=1+t_1w+t_2w^2+\cdots. \tag{2.6}$$

Or equivalently,

$$x(z) = \frac{b(z)-1}{b(z)+1} = \frac{1}{2}\,[s_1 z + \left(s_2 - \frac{s_1^2}{2}\right) z^2 + \cdots, \tag{2.7}$$

$$y(w) = \frac{e(w)-1}{e(w)+1} = \frac{1}{2}\,[t_1 w + \left(t_2 - \frac{t_1^2}{2}\right) w^2 + \cdots. \tag{2.8}$$

It is clear that $b(z)$ and $e(w)$ are analytic in Δ with $b(0) = e(0) = 1$. Since $x, y : \Delta \to \Delta$, the functions $b(z)$ and $e(w)$ have a positive real part in Δ, and $|s_i| \le 2$ *and* $|t_i| \le 2$ $(i = 1,2)$.
In the view of (2.3), (2.4), (2.7) and (2.8) clearly we have

$$\left[(1+\rho)\left(\frac{z\left(L_q^{k,\lambda}\ f(z)\right)'}{\left(L_q^{k,\lambda}\ f(z)\right)}\right) - \rho\left(\frac{\sigma\ z^2\left(L_q^{k,\lambda}\ f(z)\right)'' + z\left(L_q^{k,\lambda}\ f(z)\right)'}{\sigma\, z\left(L_q^{k,\lambda}\ f(z)\right)' + (1-\sigma)\left(L_q^{k,\lambda}\ f(z)\right)}\right) - 1\right] =$$

$$\Gamma(z)\left[\phi\left(\left(\frac{b(z)-1}{b(z)+1}\right) - 1\right)\right], \tag{2.9}$$

$$\left[(1+\rho)\left(\frac{w\left(L_q^{k,\lambda}\ g(w)\right)'}{\left(L_q^{k,\lambda}\ g(w)\right)}\right) - \rho\left(\frac{\sigma\, w^2\left(L_q^{k,\lambda}\ g(w)\right)'' + w\left(L_q^{k,\lambda}\ g(w)\right)'}{\sigma\, w\left(L_q^{k,\lambda}\ g(w)\right)' + (1-\sigma)\left(L_q^{k,\lambda}\ g(w)\right)}\right) - 1\right] =$$

$$\Gamma(w)\left[\phi\left(\left(\frac{e(w)-1}{e(w)+1}\right) - 1\right)\right]. \tag{2.10}$$

Since $f \in \sum$ has the Maclaurin series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion given by (1.2), hence, we get

$$\left[(1+\rho)\left(\frac{z\left(L_q^{k,\lambda}\ f(z)\right)'}{\left(L_q^{k,\lambda}\ f(z)\right)}\right) - \rho\left(\frac{\sigma\ z^2\left(L_q^{k,\lambda}\ f(z)\right)'' + z\left(L_q^{k,\lambda}\ f(z)\right)'}{\sigma\, z\left(L_q^{k,\lambda}\ f(z)\right)' + (1-\sigma)\left(L_q^{k,\lambda}\ f(z)\right)}\right) - 1\right] =$$

$$(1+\rho)\left[1 + \Omega_{2,q} a_2\, z + \left(2\,\Omega_{3,q} a_3 - \Omega_{2,q}^2 a_2^2\right) z^2\right]$$

$$-\rho\left[\,1 + (\sigma+1)\,\Omega_{2,q}\,a_2\,z + \left\{2\,(2\,\sigma+1)\,\Omega_{3,q}\,a_3 - (\sigma+1)^2\Omega_{2,q}^2\,a_2^2\right\} z^2\,\right] - 1 =$$

$$(1-\rho\sigma)\,\Omega_{2,q} a_2 z + \left[2(1-2\rho\sigma)\Omega_{3,q} a_3 - \{(1+\rho) - \rho(\sigma+1)^2\}\Omega_{2,q}^2\, a_2^2\right] z^2, \tag{2.11}$$

$$\left[(1+\rho)\left(\frac{w\left(L_q^{k,\lambda}\ g(w)\right)'}{\left(L_q^{k,\lambda}\ g(w)\right)}\right) - \rho\left(\frac{\sigma\, w^2\left(L_q^{k,\lambda}\ g(w)\right)'' + w\left(L_q^{k,\lambda}\ g(w)\right)'}{\sigma\, w\left(L_q^{k,\lambda}\ g(w)\right)' + (1-\sigma)\left(L_q^{k,\lambda}\ g(w)\right)}\right) - 1\right] =$$

$$(1+\rho)[1 - \Omega_{2,q} a_2 w + \left\{4\,\Omega_{3,q} a_2^2 - \Omega_{2,q}^2 a_2^2 - 2\,\Omega_{3,q} a_3\right\} w^2] - \rho[1 - (\sigma+1)\Omega_{2,q} a_2 w$$

$$+\left\{4\,(2\,\sigma+1)\,\Omega_{3,q}\,a_2^2 - (\sigma+1)^2\,\Omega_{2,q}^2\,a_2^2 - 2\,(2\,\sigma+1)\,\Omega_{3,q}\,a_3\right\} w^2\,] - 1 =$$

$$-(1-\rho\,\sigma)\,\Omega_{2,q}\,a_2\,w + [4\,(1-2\,\rho\,\sigma)\,\Omega_{3,q}\,a_2^2 - \{(1+\rho) - \rho\,(\sigma+1)^2\}\,\Omega_{2,q}^2\,a_2^2$$

$$-2\,(1-2\,\rho\,\sigma)\,\Omega_{3,q}\,a_3\,]\,w^2. \tag{2.12}$$

Using (2.7) and (2.8) together with (1.8), (1.9) it is evident that

$$\Gamma(z)\left[\phi\left(\left(\frac{b(z)-1}{b(z)+1}\right) - 1\right)\right]$$

$$= \frac{1}{2} C_0\, G_1 s_1 z + \left[\frac{1}{2} C_1 G_1 s_1 + \frac{1}{2} C_0 G_1\left(s_2 - \frac{s_1^2}{2}\right) + \frac{C_0 G_2}{4} s_1^2\right] z^2 + \cdots, \tag{2.13}$$

$$\Gamma(w)\left[\phi\left(\left(\frac{e(w)-1}{e(w)+1}\right)-1\right)\right]$$
$$=\frac{1}{2}C_0\,G_1t_1w+\left[\frac{1}{2}C_1G_1t_1+\frac{1}{2}C_0\,G_1\left(t_2-\frac{t_1^2}{2}\right)+\frac{C_0\,G_2}{4}t_1^2\right]w^2+\cdots. \quad (2.14)$$

Now, using (2.11) and (2.13) in view of (2.9) and comparing the coefficients of z and z^2, we obtain

$$(1-\rho\sigma)\,\Omega_{2,q}a_2=\frac{1}{2}C_0G_1s_1\ , \quad (2.15)$$

$$2(1-2\rho\sigma)\,\Omega_{3,q}\,a_3-\{(1+\rho)-\rho(\sigma+1)^2\}\,\Omega_{2,q}^2\,a_2^2$$
$$=\frac{1}{2}C_1\,G_1s_1+\frac{1}{2}C_0\,G_1\left(s_2-\frac{s_1^2}{2}\right)+\frac{C_0\,G_2}{4}s_1^2\,. \quad (2.16)$$

Similarly it follows from (2.12) and (2.14) in (2.10) that

$$-(1-\rho\sigma)\,\Omega_{2,q}a_2=\frac{1}{2}C_0\,G_1t_1\,, \quad (2.17)$$

$$4\,(1-2\,\rho\,\sigma)\,\Omega_{3,q}\,a_2^2\ -\{\,(1+\rho\,)-\rho\,(\,\sigma+1\,)^2\,\}\Omega_{2,q}^2a_2^2\ -\ 2\,(\,1-2\,\rho\,\sigma\,)\,\Omega_{3,q}\,a_3$$
$$=\frac{1}{2}C_1G_1t_1+\frac{1}{2}C_0\,G_1\left(t_2-\frac{t_1^2}{2}\right)+\frac{C_0\,G_2}{4}t_1^2\,. \quad (2.18)$$

From the two equations are equal (2.15) and (2.17), we find that

$$a_2=\frac{C_0\,G_1s_1}{2(1-\rho\sigma)\,\Omega_{2,q}}\ =\ -\ \frac{C_0\,G_1t_1}{2(1-\rho\sigma)\,\Omega_{2,q}}\ , \quad (2.19)$$

$$s_1=-\,t_1\,. \quad (2.20)$$

It follows that we multiply by 2 and square both sides, then we add the two equations (2.15) and (2.17)

$$8(1-\rho\sigma)^2\Omega_{2,q}^2\,a_2^2\ =C_0^2G_1^2(s_1^2+t_1^2)\,. \quad (2.21)$$

Adding (2.16) and (2.18) in light of (2.19), we get

$$4\big[4(1-2\rho\sigma)\,\Omega_{3,q}\ -2\{(1+\rho)-\rho(\sigma+1)^2\}\Omega_{2,q}^2\big]a_2^2$$
$$=2C_0\,G_1(s_2+t_2)+C_0(G_2-G_1)(t_1^2+s_1^2). \quad (2.22)$$

Applying Lemma (1.1) for the coefficients s_1, s_2, t_1 *and* t_2, it follows from (2.21) and (2.22) that

$$|a_2|\le\ \frac{|C_0|\,G_1}{\big|(1-\rho\,\sigma)\,\Omega_{2,q}\big|}\,,$$

$$|a_2|\le\ \sqrt{\frac{|C_0|\,(G_1+\ |G_2-\ G_1|\,)}{|2\,(1\ -\ 2\rho\sigma\,)\,\Omega_{3,q}\ -\{\,(1\ +\ \rho\ )\ -\ \rho(\sigma\ +\ 1\,)^2\,\}\,\Omega_{2,q}^2\ |}}\ .$$

This yields the desired estimate on $|a_2|$ as asserted in (2.1).

Now, to find them bound on the coefficient $|a_3|$ by subtracting relations (2.18) from (2.16), we get

$$8(1-2\rho\sigma)\Omega_{3,q}\,(a_3-\ a_2^2\,)=\ C_1G_1s_1+C_0G_1(s_2-t_2). \quad (2.23)$$

In light of (2.21), (2.22) and putting (2.23), we have

$$a_3\ \le\frac{C_1G_1s_1+\ C_0\,G_1(s_2-t_2)}{8(1-2\rho\sigma)\Omega_{3,q}}+\ \frac{C_0^2G_1^2(s_1^2+t_1^2)}{8(1-\rho\sigma)^2\Omega_{2,q}^2}\ , \quad (2.24)$$

$$a_3\le\frac{C_1G_1s_1+C_0G_1(s_2-t_2)}{8(1-2\rho\sigma)\Omega_{3,q}}+\ \frac{2C_0\,G_1(\,s_2+t_2\,)+\ C_0\ (\,G_2-G_1)(t_1^2\ +\ s_1^2\,)}{8\big[2(1-2\rho\sigma)\Omega_{3,q}-\{(1+\rho)-\rho(\sigma+1)^2\}\,\Omega_{2,q}^2\big]}. \quad (2.25)$$

Applying Lemma (1.1) once again for the coefficients s_1, s_2, t_1 *and* t_2, we find that

$$|a_3| \leq \frac{(\,|C_1| + 2|C_0\,|\,)G_1}{\left|4(1-2\rho\sigma)\Omega_{3,q}\right|} + \frac{C_0^2 G_1^2}{(1-\rho\sigma)^2 \Omega_{2,q}^2}\ ,$$

$$|a_3| \leq \frac{(\,|C_1| + 2|C_0|\,)G_1}{\left|4(1-2\rho\sigma)\Omega_{3,q}\right|} + \frac{|C_0\,|\,(\,G_1 + |G_2 - G_1|\,)}{\left|2(1-2\rho\sigma)\,\Omega_{3,q} - \{(1+\rho) - \rho(\sigma+1)^2\}\,\Omega_{2,q}^2\right|}.$$

The proof of Theorem (2.1) is now complete.
For $\sigma = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. Let $f(z)$ given by (1.1) belong to the class $B_{\Sigma,q}^{k,\lambda}(\rho, 1, \phi)$. Then

$$|a_2| \leq min\left\{\frac{|C_0|\,G_1}{\left|(1-\rho\,)\Omega_{2,q}\right|}\,,\sqrt{\frac{|C_0|\,(G_1 + \,|G_2 - \,G_1|\,)}{\left|2\,(1\,-\,2\rho\,)\,\Omega_{3,q}\,-\,\{\,(1\,+\,\rho\,)\,-\,4\rho\,\}\,\Omega_{2,q}^2\,\right|}}\right\},$$

$$|a_3| \leq min\left\{\frac{(\,|C_1| + 2|C_0|\,)G_1}{\left|4(1-2\rho)\Omega_{3,q}\right|} + \frac{C_0^2 G_1^2}{(1-\rho)^2\Omega_{2,q}^2}\,,\frac{(\,|C_1| + 2|C_0|\,)G_1}{\left|4(1-2\rho)\Omega_{3,q}\right|} + \frac{|C_0|\,(\,G_1 + |G_2 - G_1|\,)}{\left|2(1-2\rho)\,\Omega_{3,q} - \{(1+\rho) - 4\rho\}\,\Omega_{2,q}^2\right|}\right\}.$$

Putting $\rho = 1$ and $\sigma = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. Let $f(z)$ given by (1.1) belong to the class $B_{\Sigma,q}^{k,\lambda}(1,0,\phi)$. Then

$$|a_2| \leq min\left\{\frac{|C_0|\,G_1}{\left|\Omega_{2,q}\right|}\,,\sqrt{\frac{|C_0|\,(G_1 + \,|G_2 - \,G_1|\,)}{\left|2\,\Omega_{3,q}\,-\,\Omega_{2,q}^2\,\right|}}\right\},$$

$$|a_3| \leq min\left\{\frac{(\,|C_1| + 2|C_0|\,)G_1}{\left|4\,\Omega_{3,q}\right|} + \frac{C_0^2\,G_1^2}{\Omega_{2,q}^2}\,,\frac{(\,|C_1| + 2|C_0|\,)G_1}{\left|4\,\Omega_{3,q}\right|} + \frac{|C_0|\,(\,G_1 + |G_2 - G_1|\,)}{\left|2\,\Omega_{3,q} - \,\Omega_{2,q}^2\right|}\right\}.$$

3. Coefficient Estimates For the Class $S_{\Sigma,q}^{k,\lambda}(\varrho, \zeta, \pi, \phi)$

Definition 3.1. A function $f \in \Sigma$ defined in (1.1) is said to be in the class $S_{\Sigma,q}^{k,\lambda}(\varrho\,, \zeta, \pi, \phi)$ if the following quasi-subordination holds:

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}f(z)}{z}\right) + \varrho z\left(L_q^{k,\lambda}f(z)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda}f(z)\right)' + \zeta z^2\left(L_q^{k,\lambda}f(z)\right)''' + 2\varrho - 1\right] \prec_q (\phi(z) - 1)\,,$$

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}g(w)}{w}\right) + \varrho w\left(L_q^{k,\lambda}g(w)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda}g(w)\right)' + \zeta w^2\left(L_q^{k,\lambda}g(w)\right)''' + 2\varrho - 1\right] \prec_q (\phi(w) - 1),$$

where the function g is the extension of f^{-1} *in* Δ, for $\pi \in C\,/\,\{0\}$, $0 \leq \varrho \leq 1$, $0 \leq \zeta \leq 1$.

Remark 3.1. For $\varrho = 0$, a function $f \in \sum$ defined in (1.1) is said to be in the class $S_{\sum,q}^{k,\lambda}\ (0,\zeta,\pi,\phi)$ if the following conditions are satisfied:

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}\, f(z)}{z}\right) - \zeta\left(L_q^{k,\lambda}\, f(z)\right)' + \zeta z^2\left(L_q^{k,\lambda}\, f(z)\right)''' - 1\right] \prec_q (\phi(z)-1),$$

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}\, g(w)}{w}\right) - \zeta\left(L_q^{k,\lambda}\, g(w)\right)' + \zeta w^2\left(L_q^{k,\lambda}\, g(w)\right)''' - 1\right] \prec_q (\phi(w)-1).$$

Theoremv3.1. Let $f(z)$ given by (1.1) be in the class $S_{\sum,q}^{k,\lambda}\ (\varrho,\zeta,\pi,\phi)$. Then

$$|a_2| \le \left\{\frac{\pi\, |C_0|\, G_1\sqrt{G_1}}{\sqrt{\left|\,\pi\, C_0\, G_1^2\, (1+4\zeta)\, \Omega_{3,q} + (G_1-G_2)\,(1-\zeta-2\varrho)^2\Omega_{2,q}^2\,\right|}}\right\}, \tag{3.1}$$

$$|a_3| \le \left\{\frac{\pi^2 |C_0|^2\, G_1^2}{(1-\zeta-2\varrho)^2\Omega_{2,q}^2} + \frac{\pi\, G_1\,(\,|C_1| + 2\,|C_0|\,)}{2(1+4\zeta)\,\Omega_{3,q}}\right\}. \tag{3.2}$$

Proof. If $f \in S_{\sum,q}^{k,\lambda}\ (\varrho,\zeta,\pi,\phi)$ *and* $g = f^{-1}$. Then, there are analytic functions $x, y: \Delta \to \Delta$ *with* $x(z) = s_1 z + \sum_{j=2}^{\infty} s_j\, z^j,\ \ y(w) = t_1 w + \sum_{j=2}^{\infty} t_j\, w^j, x(0) = y(0) = 0$, such that

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}\, f(z)}{z}\right) + \varrho z\left(L_q^{k,\lambda}\, f(z)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda} f(z)\right)' + \zeta z^2\left(L_q^{k,\lambda}\, f(z)\right)''' + 2\varrho - 1\right] = \Gamma(z)(\phi(x(z)-1)), \tag{3.3}$$

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda} g(w)}{w}\right) + \varrho w\left(L_q^{k,\lambda} g(w)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda} g(w)\right)' + \zeta w^2\left(L_q^{k,\lambda}\, g(w)\right)''' + 2\varrho - 1\right] = \Gamma(w)\big(\phi(y(w)-1)\big), \tag{3.4}$$

where $x(z)$ and $y(w)$ are defined by (2.7) and (2.8) respectively.
Proceeding similarly as in Theorem (2.1), we obtain

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda} f(z)}{z}\right) + \varrho z\left(L_q^{k,\lambda} f(z)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda} f(z)\right)' + \zeta z^2\left(L_q^{k,\lambda} f(z)\right)''' + 2\varrho - 1\right] = \Gamma(z)\left[\phi\left(\left(\frac{b(z)-1}{b(z)+1}\right) - 1\right)\right], \tag{3.5}$$

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda} g(w)}{w}\right) + \varrho w\left(L_q^{k,\lambda} g(w)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda} g(w)\right)' + \zeta w^2\left(L_q^{k,\lambda}\, g(w)\right)''' + 2\varrho - 1\right] = \Gamma(w)\left[\phi\left(\left(\frac{e(w)-1}{e(w)+1}\right) - 1\right)\right], \tag{3.6}$$

where the right-hand sides of (3.5) and (3.6) given by (2.13) and (2.14), respectively.
Since

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}\, f(z)}{z}\right) + \varrho z\left(L_q^{k,\lambda} f(z)\right)'' - (2\varrho+\zeta)\left(L_q^{k,\lambda} f(z)\right)' + \zeta z^2\left(L_q^{k,\lambda} f(z)\right)''' + 2\varrho - 1\right] = \frac{1}{\pi}\,(1-\zeta-2\varrho)\,\Omega_{2,q}\, a_2\, z + \frac{1}{\pi}\,(1+4\zeta\,)\,\Omega_{3,q}\, a_3\, z^2 + \cdots, \tag{3.7}$$

$$\frac{1}{\pi}\left[(1+\zeta)\left(\frac{L_q^{k,\lambda}g(w)}{w}\right)+\varrho w\left(L_q^{k,\lambda}g(w)\right)''-(2\varrho+\zeta)\left(L_q^{k,\lambda}g(w)\right)'+\zeta w^2\left(L_q^{k,\lambda}g(w)\right)''' + 2\varrho-1\right]$$

$$=-\frac{1}{\pi}(1-\zeta-2\varrho)\Omega_{2,q}a_2w+\frac{1}{\pi}\left[2(1+4\zeta)\Omega_{3,q}a_2^2-(1+4\zeta)\Omega_{3,q}a_3\right]w^2+\cdots. \quad (3.8)$$

Comparing the coefficient of (3.7) with (2.13) and (3.8) with (2.14), then, we have

$$2(1-\zeta-2\varrho)\,\Omega_{2,q}\,a_2=\pi\,C_0G_1s_1\,, \quad (3.9)$$

$$4(1+4\zeta)\,\Omega_{3,q}\,a_3=2\pi\,C_1G_1s_1+2\pi\,C_0G_1\left(s_2-\frac{s_1^2}{2}\right)+\pi\,C_0G_2\,s_1^2\,, \quad (3.10)$$

$$-2(1-\zeta-2\varrho)\,\Omega_{2,q}\,a_2=\pi\,C_0G_1t_1\,, \quad (3.11)$$

and

$$\left[4(1+4\zeta)\,\Omega_{3,q}\right](2\,a_2^2-a_3)=2\pi\,C_1G_1t_1+2\pi\,C_0G_1\left(t_2-\frac{t_1^2}{2}\right)+\pi\,C_0G_2\,t_1^2\,. \quad (3.12)$$

From (3.9) and (3.11), we find that

$$s_1=-t_1\,, \quad (3.13)$$

$$8(1-\zeta-2\varrho)^2\Omega_{2,q}^2\,a_2^2=\pi^2C_0^2G_1^2(s_1^2+t_1^2)\,. \quad (3.14)$$

Adding (3.10) and (3.12), by using (3.13) and (3.14), we get

$$\left[8(1+4\zeta)C_0G_1^2\Omega_{3,q}\right]a_2^2=2\,\pi\,C_0^2G_1^3(s_2+t_2)+(G_2-G_1)\,\pi\,C_0^2G_1^2(s_1^2+t_1^2)\,,$$

$$\left[8(1+4\zeta)\,C_0G_1^2\Omega_{3,q}\right]a_2^2=2\,\pi\,C_0^2G_1^3(s_2+t_2)+(G_2-G_1)\,\frac{8(1-\zeta-2\varrho)^2\Omega_{2,q}^2\,a_2^2}{\pi}\,, \quad (3.15)$$

which implies

$$a_2^2=\frac{2\,\pi^2\,C_0^2\,G_1^3\,(s_2+t_2)}{8\left[\pi\,C_0\,G_1^2\,(1+4\zeta)\,\Omega_{3,q}+(G_1-G_2)\,(1-\zeta-2\varrho)^2\Omega_{2,q}^2\right]}.$$

Applying Lemma (1.1) for the coefficients s_2 and t_2 , we can easily obtain

$$|a_2|\leq\left\{\frac{\pi\,|C_0|\,G_1\sqrt{G_1}}{\sqrt{\left|\pi\,C_0\,G_1^2\,(1+4\zeta)\,\Omega_{3,q}+(G_1-G_2)\,(1-\zeta-2\varrho)^2\Omega_{2,q}^2\right|}}\right\},$$

which is the bound on $|a_2|$ as asserted in (3.1).

Now, in order to find the bound on the coefficient $|a_3|$, by subtracting (3.12) from (3.10), in light of (3.13), we have

$$\left[8(1+4\zeta)\,\Omega_{3,q}\right](a_3-a_2^2)=2\,\pi\,C_1G_1s_1+2\,\pi\,C_0G_1(s_2-t_2)\,,$$

$$a_3=a_2^2+\frac{2\,\pi\,C_1G_1s_1+2\,\pi\,C_0\,G_1(s_2-t_2)}{8\,(1+4\zeta)\,\Omega_{3,q}}\,. \quad (3.16)$$

Upon substituting the value of a_2^2 from (3.14), we obtain

$$a_3=\frac{\pi^2C_0^2G_1^2(s_1^2+t_1^2)}{8(1-\zeta-2\varrho)^2\Omega_{2,q}^2}+\frac{\pi\,G_1\,(\,C_1\,s_1+\,C_0\,(s_2-t_2)\,)}{4\,(1+4\zeta)\,\Omega_{3,q}}\,.$$

Applying Lemma (1.1) once again for the coefficients s_1, s_2, t_1 *and* t_2, we find that

$$|a_3|\leq\left\{\frac{\pi^2|C_0|^2\,G_1^2}{(1-\zeta-2\varrho)^2\Omega_{2,q}^2}+\frac{\pi\,G_1\,(\,|C_1|+2\,|C_0|\,)}{2(1+4\zeta)\,\Omega_{3,q}}\right\}.$$

This completes the proof of Theorem (3.1).

Taking $\varrho=1$ and $\pi=1$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. Let $f(z)$ given by (1.1) belong to the class $S_{\Sigma,q}^{k,\lambda}\,(1,\zeta,1,\phi)$. Then

$$|a_2| \leq \left\{ \frac{|C_0|\, G_1 \sqrt{G_1}}{\sqrt{\left|\; C_0\, G_1^2\, (1+4\zeta)\, \Omega_{3,q} + (G_1 - G_2)\, [-(1+\zeta)]^2\, \Omega_{2,q}^2 \right|}} \right\},$$

$$|a_3| \leq \left\{ \frac{|C_0|^2\, G_1^2}{[-(1+\zeta)]^2 \Omega_{2,q}^2} + \frac{G_1\, (\,|C_1| + 2\,|C_0|\,)}{2(1+4\zeta)\, \Omega_{3,q}} \right\}.$$

By putting $\varrho = 1$ and $\zeta = 0$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. Let $f(z)$ given by (1.1) belongs to the class $S_{\Sigma,q}^{k,\lambda}\ (1,0,\pi,\phi)$. Then

$$|a_2| \leq \left\{ \frac{\pi\, |C_0|\, G_1 \sqrt{G_1}}{\sqrt{\left|\, \pi\, C_0\, G_1^2\, \Omega_{3,q} + (G_1 - G_2)\, \Omega_{2,q}^2 \right|}} \right\},$$

$$|a_3| \leq \left\{ \frac{\pi^2 |C_0|^2\, G_1^2}{\Omega_{2,q}^2} + \frac{\pi\, G_1\, (\,|C_1| + 2\,|C_0|\,)}{2\, \Omega_{3,q}} \right\}.$$

If we set $\Gamma(z) = 1$ and $\varrho = 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. Let $f(z)$ given by (1.1) belong to the class $S_{\Sigma,q}^{k,\lambda}\ (0,\zeta,\pi,\phi)$. Then

$$|a_2| \leq \left\{ \frac{\pi\;\, G_1 \sqrt{G_1}}{\sqrt{\left|\, \pi\, G_1^2\, (1+4\zeta)\, \Omega_{3,q} + (G_1 - G_2)\, (1-\zeta)^2 \Omega_{2,q}^2 \right|}} \right\},$$

$$|a_3| \leq \left\{ \frac{\pi^2\, G_1^2}{(1-\zeta)^2 \Omega_{2,q}^2} + \frac{\pi\, G_1}{2(1+4\zeta)\, \Omega_{3,q}} \right\}.$$

Conclusion

In this paper, we introduced subclasses of the function class $\sum$ of analytic and bi-univalent functions associated with operator $L_q^{k,\lambda}$ defined in the open unit disk $\triangle$ by applying quasi-subordination have been introduced and studied. Some results and properties about the corresponding bound estimations of the coefficients a_2 and a_3 are given and investigated. Here ,we opened some new windows to find the coefficients using quasi-subordination.

References

[1] W. K. Hayman. Univalent functions. Grundlehren der Mathematischen Wissenschaften **259**, New York, Springer, 1983.

[2] P. L. Duren, Univalent Functions, Springer - Verlag, New York, 1983

[3] M. Lewin, On a coefficient problem for bi- univalent functions, Proc. Amer. Math.Soc., **18**: 63–681967.

[4] D. A. Brannan and J. G. Clunie. **1980**. Aspect of contemporary complex analysis, Academi Press, New York, London.

[5] E. Netanyahau, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in | z |< 1, Archive for Rotional Mechanic and Analysis, **32(2)**: 100–112, 1969.

[6] H. M. Srivastava, A. K. Mishra and P. Gochhayat**.** Certain subclasses of analytic and bi- univalent functions, Appl. Math. Lett., **23(10)**: 1188–1192, 2010.
[7] D. Styer and J. Wright, Result on bi-univalent functions, Proc. Amer. Math. Soc., **(82)**: 243–248, 1981.
[8] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ .Babeş-Bolyai Math., **31(2)**: 70–77, 1986.
[9] H. M. Srivastava, S. Gaboury and F. Ghanim. Coefficient estimates for some general subclasses of analytic and bi- univalent functions, Afr. Math., **(28)**: 693–706, 2017.
[10] M. S. Robertson,. Quasi-subordination and coefficient conjectures, Bull. Amer. Math. Soc**., 76**: 1–9, 1970.
[11] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for a general subclass of analytic and bi- univalent functions of the Ma–Minda type, Rev. Ser. Math., **(112)**: 1157–1168, 2018.
[12] E. Deniz, Certain subclasses of bi- univalent functions satisfying subordinate conditions, J. Class. Anal., **2(1)**: 49–60, 2013.
[13] F. Y. Ren, S. Owa and S. Fukui. Some inequalities on quasi- subodrdinate functions, Bull. Aust. Math. Soc., **43(2)**: 317–324, 1991.
[14] M. Hajimohd and M. Darus, Fekete-Szegő problems for quasi-subordination classes, Abstr. Appl. Anal. Art., **ID 192956**, 1-14, 2012.
[15] S. P. Goyal, O. Singh and R. Mukherjee**,**. Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, Palest. J.Math., **5(1)**: 79–85, 2016.
[16] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In: Proc. Conf. on complex analysis, (Tianjin,1992), Conference Proceedings Lecturer Notes Analysis, International Press, Cambridge, Mass., USA, 157–169, 1994.
[17] S. M. El-Deeb, T. Bulboaca and B. M. El-Matary, Maclaurin coefficient estimates of bi-univalent functions connected with the q-derivative, Mathematics, **(8)**: 1–14, 2020.
[18] F. H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, **(46)**: 253 – 281, 1980.
[19] M. Govindaraj and S. Sivasubramanian, On a class of analytic functions related to conic domains involving q-Calculus, Analysis Math., **43(3)**: 475-487, 2017.
[20] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, Coefficients estimates of bi-univalent functions defined by new subclass function, Journal of Physics: Conference Series, **1530**: 012105, 1-8, 2020.
[21] S. A. Al-Ameedee, W. G. Atshan and F. A. Al-Maamori, Second Hankel determinant for certain subclasses of bi-univalent functions, Journal of Physics: Conference Series, **1664**:012044, 1-8, 2020.
[22] W. G. Atshan and E. I. Badawi, Results on coefficient estimates for subclasses of analytic and bi-univalent functions, Journal of Physics: Conference Series, **1294**: 032025, 1-10, 2019.
[23] W. G. Atshan, S. Yalcin and R. A. Hadi, Coefficient estimates for special subclasses of k-fold symmetric bi-univalent functions, Mathematics for Applications**, 9**: 83-90, 2020.
[24] S. Yalcin, W. G. Atshan and H. Z. Hassan.,Coefficients assessment for certain subclasses of bi-univalent functions related with quasi-subordination, Publications De L'Institut Mathematique, Nouvelle seŕie, tome **108(122)**: 155-162, 2020.
[25] W. G. Atshan, I. A. R. Rahman and A. A. Lupas, Some results of new subclasses for bi-univalent functions using quasi-subordination, Symmetry, **13(9)**: 1653, 1-12, 2021.
[26] W. G. Atshan and A. A. R. Ali**,** On sandwich theorems results for certain univalent functions defined by generalized operator, Iraqi Journal of Science, **62(7)**: pp: 2376 – 2383, 2021.
[27] A. A. Hussein and K. A. Jassim, Some geometric properties of generalized Class of meromorphic functions associated with higher Ruscheweyh derivatives, Iraqi Journal of Science, **60(9)**: pp: 2036-2042, 2019.

[28] S. K. Hussein, K. A. Jassim**,** On a class of meromorphic multivalent functiions convoluted with higher derivatives of fractional calculus operator, Iraqi Journal of Science, **60(10):** pp: 2237-2245, 2019.

[29] W. G. Atshan and A. A. R. Ali, On some sandwich theorems of analytic functions inolving Noor-Sălăgean operator. Advances in Mathematics: Scientific Journal. **9(10)**: 8455-8467, 2020.

[30] M. A. Sabri , W. G. Atshan and E. El-Seidy, On sandwich-type results for a subclass of certain univalent functions using a new Hadamard product operator, Symmetry, **14:** 931, 1-11, 2022.

[31] I. A. Rahman, W. G. Atshan and G. I. Oros, New concept on fourth Hankel determinant of a certain subclass of analytic functions, Afrika Mat., **33(1)**: 1-15, 2022.

[32] W. G. Atshan and S. R. Kulkarni, On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator ,Journal of Inequalities in Pure and Applied Mathematics, **10(2)**: 53, 11 pp, 2009.

[33] W. G. Atshan and R. A. Hadi, Some differential subordination and superordination results of p-valent functions defined by differential operator, Journal of Physics: Conference Series**, 1664**: 012043, 1-15, 2020.

[34] W. G. Atshan, A. H. Battor and A. F. Abaas, Some sandwich theorems for meromorphic univalent functions defined by new integral operator, Journal of Interdisciplinary Mathematics, **24(3)**: 579-591, 2021.

[35] S. D. Theyab, W. G. Atshan, A. A. Lupaș and H. K. Abdullah, New results on higher-order differential subordination and superordination for univalent analytic functions using a new operator, Symmetry, **14(8)**: 1576, 1-12, 2022

[36] B. K. Mihsin, W. G. Atshan, S. S. Alhily and A. A. Lupas, New results on fourth- order differential subordination and superordination for univalent analytic functions involving a linear operator, Symmetry, **14(2)**: 324, 1-12, 2022.

[37] W. G. Atshan , R. A. Hiress and S. Altinkaya, On third-order differential subordination and superordination properties of analytic functions defined by a generalized operator, Symmetry **14(2)**: 418, 1-17, 2022.

[38] A. M. Darweesh, W. G. Atshan, A. H. Battor and A. A. Lupaş**,** Third-order differential subordination results for analytic functions associated with a certain differential operator, Symmetry, **14(1**): 99, 1-15, 2022.

[39] Ch. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Gottingen, 1975.

[40] D. L. Tan, Coefficient estimates for bi-univalent functions, Chin. Ann. Math. Ser. A., **(5)**: 559–568, 1984.

[41] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generealized hypergometric function. Appl. Math. Comput., **(103)**: 1–13, 1990.