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Small-Essentially Quasi-Dedekind $\mathcal{R}$-Modules

Haithab A. Shahad*, Nuhad S. Al-Muthafar
Department of Mathematics, Collage of Science, University of Baghdad, Baghdad, Iraq

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Abstract
In this research, we introduce a small essentially quasi−Dedekind R-module to generalize the term of an essentially quasi.−Dedekind R-module. We also give some of the basic properties and a number of examples that illustrate these properties.

Keywords: Quasi-Ded. modules , Essentially quasi-Ded. modules, K-nonsingular modules.

المقاسات شبه – الديديكاندية الصغيرة الجوهرية

هيذب عبود شهد، نهاد سالم المظفر
قسم الرياضيات، كلية العلوم ، جامعة بغداد، بغداد، العراق

الخلاصة
في هذا البحث تم تعميم مفهوم المقاسات شبه الديديكاندية بواسطة تقديم ما يسمى بالمقاسات شبه– الديديكاندية الصغيرة الجوهرية . كما اعطينا بعض من الخواص الاساسية وعدد من الامثلة التي تحقق تلك الخواص.

1- Introduction

Throughout this research , all rings R are commutative with unity, and all R-modules $\mathcal{A}$ are unitary. Recall that a non-zero sub-module $\mathcal{H}$ of R-module $\mathcal{A}$ is quasi−invertible if $Hom(\mathcal{A}/\mathcal{H}, \mathcal{A}) = 0$,[1]. $\mathcal{A}$ is said to be quasi.−Dedekind if all non-zero $\mathcal{H}$ $sub-$mod. of $\mathcal{A}$ is quasi−invertible that is $Hom(\mathcal{A}/\mathcal{H}, \mathcal{A}) = 0$ for each non-zero $\mathcal{H}$ sub-module of $\mathcal{A}$. Equivalently, $\mathcal{A}$ is said to be quasi−Dedekind if for each $f \in End(\mathcal{A}), f \neq 0$, $then\ ker(f) = 0$ [1]. As a generalization of quasi−Dedekind $\mathcal{R}-$ $m$odule, Inaam M. A. and Thaar Y. G. in [2] reviewed the notion essentially quasi.−Dedekind (briefly, ess.quasi.−Ded.) by restricting the definition of quasi.-Dedekind on essential $sub - mod$. The concept of essentially quasi.−Dedekind is equivalently to k−nonsingular which is introduced by Shyaa F. D. and Ali I. M.,[3]. Where $\mathcal{A}$ is ess.quasi.−Ded. R-module if for all $f \in End(\mathcal{A})$, $ker(f) \leq_e \mathcal{A}$ implies $f = 0$.

In [5], Zhou introduced the concept $s - ess$ential $sub -$mod., where a sub-module $\mathcal{H}$ of $\mathcal{A}$ is called s-essential (denoted by $\mathcal{H} \trianglelefteq_s \mathcal{A}$) if $\mathcal{H} \cap L \neq 0$ for each non-zero small sub-module L of $\mathcal{A}$.

*Email: hshahad@uowasit.edu.iq

The outlines of this paper is to introduce the small essentially quasi.−Dedekind $\mathcal{R}-\mathcal{m}$odule. We also discuss and give an equivalent notion to $s-ess.$ quasi.-Ded. R-module that is an $\mathcal{R}-\mathcal{m}$odule A is called sk-nonsingular if for all $f \in End(\mathcal{A})$, $ker(f) \trianglelefteq_s \mathcal{A}$ implies $f=0$. It is clear that every $s-ess.$ Quasi.−Ded. R-module is essentially quasi−Dedekind, however the converse is not true. In addition, we show that every quasi.−Dedekind R-module is $s-ess.$ Quasi.−Ded., but the converse does not hold. Several results are given in this work.

2- Small-Essentially Quasi−Dedekind Modules:

Definition(2-1): Let $\mathcal{A}$ be an R-module, then we have

1- $\mathcal{A}$ is said to be small essentially quasi−Dedekind(briefly $s-ess.$ Quasi.−Ded.) if for each non-zero s-essential sub-module $\mathcal{H}$ of $\mathcal{A}$ is quasi.− invertible, that is if $Hom(\mathcal{A}/\mathcal{H}, \mathcal{A})=0$ for all non-zero s-essential sub-module $\mathcal{H}$ of $\mathcal{A}$.

2- A ring $\mathcal{R}$ is $s-ess.$ quasi.−Ded., if it is $s-ess.$ quasi.−Ded. R-module.

Remarks and Examples(2-2):

1- Every nonsingular R-module is an $s-ess.$ quasi.−Ded.

Proof: By [6, p.19], we know that every essential sub-module is quasi.−invertible with the fact that every s −essential sub-module is quasi.−invertible[5], so that $\mathcal{A}$ is $s-ess.$ quasi.−Ded.

2- Every $s-ess.$ quasi.−Ded. R-module is ess quasi.−Ded. R-module. Because every essential sub-module is s-essential. Generally, the next example shows that the converse does not hold. Example: Consider $A=Z_6$ as Z-module is $ess.$ quasi.-Ded. R module[6].However, it is not $s-ess.$ quasi.−Ded, because of $Hom\left(\frac{Z_6}{<\bar{3}>}, Z_6\right) \simeq Z_3 \neq 0$ and $<\bar{3}> \trianglerighteq_s Z_6$.

3- It is clear that every quasi.−Dedekind R-module is an $s-ess.$ quasi.−Ded. R-module. But, the converse does not hold in general. One can see the following example. The Z-module $\mathcal{A}=Z\oplus Z$ is nonsingular, so it is $s-ess.$ quasi.−Ded., but, $\mathcal{A}$ is not quasi.−Dedekind, because of $Hom\left(\frac{\mathcal{A}}{Z\oplus<\bar{0}>}, \mathcal{A}\right) \simeq Hom(Z, Z\oplus Z) \neq 0$.

4- A homomorphic image of $s-ess.$ quasi.−Ded. needs not to be $s-ess.$ quasi-Ded. One can see the following example: Z as $Z-m$odule is s-ess. quasi-Ded., let $\pi: Z \to \frac{Z}{<\bar{8}>} \cong Z_8$ be the natural epimorphism, hence $\pi(Z)=Z_8$ is not $s-ess.$ quasi.-Ded. since $Hom\left(\frac{Z_8}{<\bar{2}>}, Z_8\right) \neq 0$ and $<\bar{2}> \trianglerighteq_s Z_8$.

5- Every integral domain $\mathcal{R}$ is an $s-ess.$ quasi.−Ded. $\mathcal{R}-\mathcal{m}$odule, by [6, p 24] and Remark (3).

6- If $\mathcal{A}$ is $s-ess.$ quasi.−Ded., then $ann\mathcal{A}=ann\mathcal{H}$ for each $\mathcal{H} \trianglelefteq_s \mathcal{A}$. Since $\mathcal{A}$ is $s-ess.$ quasi.−Ded., then every $0 \neq \mathcal{H} \trianglelefteq_s \mathcal{A}$ is quasi−invertible $sub-$mod. Hence $ann\mathcal{A}=ann\mathcal{H}$ for each $0 \neq \mathcal{H} \trianglelefteq_s \mathcal{A}$.

7- $Z\oplus Z_2$ is not $ess.$ quasi.−Ded. as Z-module,[2]. So that it is not $s-ess.$ quasi.−Ded. as Z-module.

8- Next example shows that the direct sum of $s-ess.$ quasi−Ded. R-module needs not to be an $s-ess.$ quasi−Ded. R-module.

Example:- $\mathcal{A}=Z\oplus Z_2$ is not $s-ess.$ quasi-Ded. [2], but Z, Z_2 are $s-ess.$ quasi.-Ded. R-module.

Proposition(2-3):

A direct summand of $s-ess.$ quasi.−Ded. R-module $\mathcal{A}$ is an $s-ess.$ quasi.−Ded.

Proof:-

Let $\mathcal{A}=\mathcal{A}_1\oplus\mathcal{A}_2$ and let $\mathcal{A}_1$be a direct summand of $\mathcal{A}$ .To prove that $\mathcal{A}_1$ is an $s-ess.$ quasi.−Ded. Let $0 \neq f \in End(\mathcal{A}_1)$. We have the following diagram:

$$\mathcal{A}_1 \oplus \mathcal{A}_2 \xrightarrow{\pi} \mathcal{A}_1 \xrightarrow{f} \mathcal{A}_1 \xrightarrow{i} \mathcal{A}_1 \oplus \mathcal{A}_2$$

$i \circ f \circ \pi \in End(\mathcal{A})$. If $i \circ f \circ \pi(\mathcal{A}) = i \circ f(\mathcal{A}_1) = i(f(\mathcal{A}_1)) = f(\mathcal{A}_1) \neq 0$, then $Ker(i \circ f \circ \pi) \not\trianglelefteq_s \mathcal{A}$. $Ker(i \circ f \circ \pi) = \{a_1 + a_2 : i \circ f \circ \pi(a_1, a_2) = 0\} = \{a_1 + a_2 : i \circ f(a_1) = 0\} = \{a_1 + a_2 : f(a_1) = 0\} = Kerf \oplus \mathcal{A}_2 \not\trianglelefteq_s \mathcal{A}_1$
$\oplus \mathcal{A}_2$. But $\mathcal{A}_2 \trianglelefteq_s \mathcal{A}_2$, so $Kerf \not\trianglelefteq_s \mathcal{A}_1$from [5 prop.(2-7)].

Recall that a non-zero R-module $\mathcal{A}$ is said to be $u$niform, whenever all non-zero sub-module of is an essential[8].

Next function gives a new notion for $u$niform.

Definition(2-4):- A non-zero $\mathcal{R}-m$odule $\mathcal{A}$ is said to be s- $u$niform, if all non-zero $sub-$mod. of is s-essential [10].

Remark: Every $u$niform R-module is s- $u$niform R-module. But the converse is not true in general. As shown in the following example: Z_6 as Z$-m$odule does not represent a $u$niform R-module, while it is s- $u$niform R-module, since $<\bar{2}>, <\bar{3}>$ are s-essential sub-modules of Z_6, [5].

Proposition(2-5):-

Let $\mathcal{A}$ be an s- $u$niform R-module. Then $\mathcal{A}$ is a quasi.-Dedekind R-module if and only if is an $s-ess$. quasi.−Ded.

Proof:- It is clear.

Corollary(2-6):-

If $\mathcal{A}$ be an s- $u$niform R-module, then the following statements are equivalent:

1- $\mathcal{A}$ is a quasi-Dedekind R-module.
2- $\mathcal{A}$ is an s-ess. quasi.−Ded. R-module.
3- $\mathcal{A}$ is an essentially quasi−Ded. R-module.

We point out that an R-module is k-nonsingular if for each $f \in End(\mathcal{A})$, $ker(f) \leq_e \mathcal{A}$ implies $f = 0$[3].

Below, we will introduce a new notion.

Definition(2-7):-

An $\mathcal{R}-m$odule $\mathcal{A}$ is said to be sk-nonsingular if for each $f \in End(\mathcal{A})$, $ker(f) \trianglelefteq_s \mathcal{A}$ implies $f = 0$.

Theorem(2-8):-

Let $\mathcal{A}$ be an R-module , then $\mathcal{A}$ is an $s-ess$. quasi.−Ded. if and only if $\mathcal{A}$ is a sk-nonsingular R-module.

Proof:- ⇒) Suppose that $\mathcal{A}$ is $s-ess$. quasi.−Ded. R-module and let $f \in End(\mathcal{A})$, $f \neq 0$. To prove that $Kerf \not\trianglelefteq_s \mathcal{A}$, we assume that $Kerf \trianglelefteq_s \mathcal{A}$ and define $h: \frac{\mathcal{A}}{Kerf} \to \mathcal{A}$ by $h(a + Kerf) = f(a)$ for all $a \in \mathcal{A}$. It is clear that h is well defined and $h \neq 0$, thus $Hom\left(\frac{\mathcal{A}}{Kerf}, \mathcal{A}\right) \neq 0$ that is a contradiction.

⇐)Assume that there exists $h: \frac{\mathcal{A}}{\mathcal{H}} \to \mathcal{A}$, $h \neq 0$, for some $\mathcal{H} \trianglelefteq_s \mathcal{A}$. Consider the following : $\mathcal{A} \xrightarrow{\pi} \frac{\mathcal{A}}{\mathcal{H}} \xrightarrow{h} \mathcal{A}$, where π is natural projective mapping. Let $\psi = f \circ \pi \in End(\mathcal{A})$. Since $\mathcal{H} \subseteq Ker\psi$ and $\mathcal{H} \trianglelefteq_s \mathcal{A}$, it follows that $Ker\psi \trianglelefteq_s \mathcal{A}$,[5] $\psi(\mathcal{A}) = f \circ \pi(\mathcal{A}) = f(\mathcal{A}/\mathcal{H}) \neq 0$ which is a contradiction.

Remark: It is clear that every sk-nonsingular is k-nonsingular. The opposite does not hold in general. This is shown in the following example: Z_{10} as Z$-m$odule is k-nonsingular, since Z_{10} is an ess. quasi Ded. where $<\bar{5}>, <\bar{2}>$ are not essential sub-modules in Z_{10}, but it is not sk-nonsingular because it is not $s-ess$. quasi.−Ded., since $Hom\left(\frac{Z_{10}}{<\bar{5}>}, Z_{10}\right) \cong Z_5 \neq 0$.

Recall that a ring $\mathcal{R}$ is called semi-prime if $< 0 >$ is a semi-prime ideal of $\mathcal{R}$; in other word $\mathcal{R}$ does not contain non-zero nilpotent ideal. And an ideal I of $\mathcal{R}$ is semi-prime if for all ideal J of R with $J^2 \subseteq I$, then $J \subseteq I$[7] .

Proposition(2-9):-

Let $\mathcal{R}$ be a ring. The following sentences are equipollent:-

1- $\mathcal{R}$ is a semi-prime ring.

2- $Z(\mathcal{R}) = 0$ ($\mathcal{R}$ is a non-singular ring).

3- $\mathcal{R}$ is an s-essential quasi-Dedekind ring.

Proof:-

1) $\Leftrightarrow$ 2) : It achieves from [7].

2) $\Longrightarrow$ 3) :It follows from Remarks and Examples (2-2)(1) .

1) $\Longrightarrow$ 3)

Let $h \in End(\mathcal{R})$ with $kerh \trianglerighteq_s \mathcal{R}$, to prove $h = 0$. We say that $h \neq 0$, then we have $0 \neq r \in \mathcal{R}$ and $h(a) = ra$ for every $a \in \mathcal{R}$. Since $kerh \trianglerighteq_s \mathcal{R}$ and $0 \neq r \in \mathcal{R}$, there exists $0 \neq s \in \mathcal{R}$ such that $0 \neq rs \in kerh$, it follows that $0 = f(rs) = rf(s) = r^2 s$, implies that $(rs)^2 = 0 \Rightarrow rs = 0$ (since $\mathcal{R}$ is semi-prime) that is a contradiction. Therefore, $h = 0$ and $\mathcal{R}$ is an $s - ess$. quasi-Ded.

3) $\Longrightarrow$ 1) It follows that from [7 prop.(1.2.6)].

Recall that an R-module $\mathcal{A}$ is called a non-singular if $Z(\mathcal{A}) = 0$, such that $Z(\mathcal{A}) = \{a \in \mathcal{A} : ann(a) \leq_e R\}$, [7]

Now, we have the following notion:

Definition (2-10): A sub-module C of a R-module $\mathcal{A}$ is called $s - c$losed if $\frac{\mathcal{A}}{c}$ is non-singular,[8].

Proposition(2-11):-

Let $\mathcal{A}$ be an R-module, then $\frac{\mathcal{A}}{C}$ is $s - ess$. quasi.$-$Ded. for each s-closed sub-module C of $\mathcal{A}$.

Proof:- Since C is an s-closed sub-module, then by definition(2-10), $\frac{\mathcal{A}}{C}$ is nonsingular. By Remark and Example(2-2) (1),we have $\frac{\mathcal{A}}{C}$ is an $s - ess$. quasi.$-$Ded. R-module.

Remark(2-12):-

Let $\mathcal{A}$ be an R-module and let $\mathcal{H} \leq \mathcal{A}$. If $\frac{\mathcal{A}}{\mathcal{H}}$ is an $s - ess$. quasi.$-$Ded. $\mathcal{R} - m$odule, then $\mathcal{A}$ is not necessary to be an $s - ess$. quasi.$-$Ded. R-module. As shown in the following example:- Let $A = Z_8$ as Z-module and $H =< \bar{4} > \leq Z_8$, then $\frac{Z_8}{<\bar{4}>} \simeq Z_4$ is an $s - ess$. quasi.-Ded., but $A = Z_8$ is not an $s - ess$. quasi.-Ded. Z-module.

Proposition(2-13):-

Let $\mathcal{A}$ be an R-module and let $\bar{\mathcal{R}} = \mathcal{R}/J$ where J is an ideal of $\mathcal{R}$ and $J \subseteq ann(\mathcal{A})$. $\mathcal{A}$ is an $s - ess$. quasi.$-$Ded. R-module if and only if $\mathcal{A}$ is an $s - ess$. quasi.$-$Ded. $\bar{\mathcal{R}}$-module.

Proof: We have $Hom_{\mathfrak{R}}\left(\frac{\mathcal{A}}{\mathcal{H}}, \mathcal{A}\right) = Hom_{\bar{\mathfrak{R}}}\left(\frac{\mathcal{A}}{\mathcal{H}}, \mathcal{A}\right), for\ all\ \mathcal{H} \leq \mathcal{A}$ [7]. Thus, if is an $s - ess$. quasi.$-$Ded. R-module, then $Hom_{\mathfrak{R}}\left(\frac{\mathcal{A}}{\mathcal{H}}, \mathcal{A}\right) = \{0\}$ for all $\mathcal{H} \trianglerighteq_s \mathcal{A}$, so $Hom_{\bar{\mathfrak{R}}}\left(\frac{\mathcal{A}}{\mathcal{H}}, \mathcal{A}\right) = \{0\}$ for all $\mathcal{H} \trianglerighteq_s \mathcal{A}$. It follows that $\mathcal{A}$ is an $s - ess$. quasi.$-$Ded. R-module. Similarly, we can prove the direction.

Proposition(2-14):-

Let $\mathcal{A}$ and $\bar{\mathcal{A}}$ are two isomorphic R-modules. Then $\mathcal{A}$ is an $s - ess$. quasi.$-$Ded. module if and only if $\bar{\mathcal{A}}$ is an $s - ess$. quasi.$-$Ded. module.

Proof: Let $\mathcal{A}$ is an $s - ess$. quasi.$-$Ded. R-module and let : $\mathcal{A} \to \bar{\mathcal{A}}$, ρ is an isomorphism. To achieve that $\bar{\mathcal{A}}$ is an $s - ess$. quasi.$-$Ded. R-module, we take $f \in End(\bar{\mathcal{A}})$ such that

$f \neq 0$. Hence, there exists $\xrightarrow{\rho} \bar{\mathcal{A}} \xrightarrow{f} \bar{\mathcal{A}} \xrightarrow{\rho^{-1}} \mathcal{A}$, let $h = \rho^{-1} o f o \rho \in End(\mathcal{A})$, thus $h \neq 0$, it implies $kerh \ntrianglelefteq_s \mathcal{A}$. To prove that $kerf \ntrianglelefteq_s \bar{\mathcal{A}}$, we claim that $kerf = \{y \in \bar{\mathcal{A}}: \rho^{-1}(y) \in kerh\}$. The assertion will be installed as follows: let $y \in kerf$, $f(y) = 0$, $h(\rho^{-1}(y)) = (\rho^{-1} o f o \rho)(\rho^{-1}(y)) = (\rho^{-1} o f)(y) = \rho^{-1}(f(y)) = \rho^{-1}(0) = 0$. Therefore, for each $y \in kerf$, $\rho^{-1}(y) \in kerh$, so $\rho^{-1}(kerf) \subseteq kerh \ntrianglelefteq_s \mathcal{A}$ which follows that $\rho^{-1}(kerf) \ntrianglelefteq_s \mathcal{A}$. Thus $kerf \ntrianglelefteq_s \bar{\mathcal{A}}$ and $\bar{\mathcal{A}}$ is an $\mathcal{s}-ess$. quasi.−Ded. R-module. Similarly, the other direction is obtained.

Remark(2-15):-

Let $\mathcal{A}$ be an $\mathcal{s}-ess$. quasi.−Ded. R-module and let $\mathcal{H} \leq \mathcal{A}$. Then it is not necessary that to be an $\mathcal{s}-ess$. quasi.−Ded. For this , one can see the following example: Consider $\mathcal{A} = Z \oplus Z$ as a Z-module is $\mathcal{s}-ess$. quasi.-Ded. , but $\mathcal{H} = Z \oplus 2Z$ is not $\mathcal{s}-ess$. quasi.-Ded. R-module. Since $Hom\left(\frac{Z\oplus Z}{Z\oplus 2Z}, Z\oplus Z\right) \simeq Hom(Z_2, Z\oplus Z) \neq 0$.

Now, we will give a proposition that explains a condition that makes a sub-module of an $\mathcal{s}-ess$. quasi−Ded. R-module is $\mathcal{s}-ess$. quasi.−Ded. sub-module. But before that, we need to know the concept of a quasi-injective as follows:

An R-module $\mathcal{A}$ is called a quasi-injective, if for all monomorphism $f: \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{H}} \leq \boldsymbol{\mathcal{A}}$ and any homomorphism $g: \mathcal{H} \to \mathcal{A}$**,** there exists a homomorphism $h: \mathcal{A} \to \mathcal{A}$ where $h \circ f = g$ [2].

Proposition(2-16):

Let $\mathcal{A}$ be an $\mathcal{s}-ess$. quasi−Ded. and quasi−injective R-module. If $\mathcal{H} \trianglelefteq_s \mathcal{A}$, then $\boldsymbol{\mathcal{H}}$ is an $\mathcal{s}-ess$. quasi.−Ded. R sub-module.

Proof :-

Let $f \in End(\mathcal{H})$ *such that* $f \neq 0$ to achieve that $kerf \ntrianglelefteq_s \mathcal{H}$. We say that $kerf \trianglelefteq_s \mathcal{H}$. Since $\mathcal{A}$ is quasi injective, then we have $g \in End(\mathcal{A})$ with $goi = iof$,where i is an inclusion mapping. It follows that $g \neq 0$. Thus, $kerg \ntrianglelefteq_s \mathcal{A}$, since $\mathcal{A}$ is an $\mathcal{s}-ess$. quasi−Ded. and $kerf \subseteq kerg$,we get $kerf \ntrianglelefteq_s \mathcal{A}$. Since by hypothesis $kerf \trianglelefteq_s \mathcal{H}$ and $\mathcal{H} \trianglelefteq_s \mathcal{A}$, then $kerf \trianglelefteq_s \mathcal{A}$. To explain that, since $\mathcal{H} \trianglelefteq_s \mathcal{A}$ for every $C \ll \mathcal{A}, C \neq 0$, then $\mathcal{H} \cap C \neq 0$ and $\mathcal{H} \cap C \leq \mathcal{H}$. But, $kerf \trianglelefteq_s \mathcal{H}$, so we have $kerf \cap (\mathcal{H} \cap C) \neq 0$ that means $(kerf \cap C) \cap \mathcal{H} \neq 0$, which implies $kerf \cap C \neq 0$ and this is a contradiction. Thus, $kerf \ntrianglelefteq_s \mathcal{H}$ and hence $\boldsymbol{\mathcal{H}}$ is an $\mathcal{s}-ess$. quasi−Ded. R-sub-module.

Recall that an injective R-module $E(\mathcal{A})$ is called an injective hull (injective envelope) of an R-module $\mathcal{A}$ if there exists a monomorphism $f: \mathcal{A} \to E(\mathcal{A})$, where $Imf \leq_e E(\mathcal{A})$[8].

Corollary (2-17):-

Let $\mathcal{A}$ be an R-module. If an R-module $E(\mathcal{A})$ is an $\mathcal{s}-ess$ quasi−Ded. , then $\mathcal{A}$ is an $\mathcal{s}-ess$. quasi−Ded. R-module.

Proof:- The proof is clear, so it is omitted.

In general, the converse of corollary(2-18) does not hold. For example:-

Consider $\mathcal{A} = Z_2$ as Z-module is an $\mathcal{s}-ess$. quasi.−Ded. Z-module. But $E(Z_2) = Z_2^{\infty}$ is not an $\mathcal{s}-ess$. quasi.−Ded. Z-module, because Z_2^{∞} is not ess. quasi.−Ded.,[2].

Proposition(2-18):-

If is an R-module with $f \neq 0$ for all $f \in Hom(\mathcal{A}, E(\mathcal{A}))$ such that $kerf \ntrianglelefteq_s \mathcal{A}$, then $\mathcal{A}$ is an $\mathcal{s}-ess$. quasi.−Ded. R-module.

Proof:-

Let $0 \neq h \in End(\mathcal{A})$, then $0 \neq i \circ h \in Hom(\mathcal{A}, E(\mathcal{A}))$, where i is an inclusion mapping. It implies that $ker(i \circ h) \ntrianglelefteq_s \mathcal{A}$. Since $kerh = ker(i \circ h)$.Therefore, $kerh \ntrianglelefteq_s \mathcal{A}$ and $\mathcal{A}$ is an $\mathcal{s}-ess$. quasi.−Ded. R-module.

Conclusions

This work aims to generalize the essentially quasi.−Dedekind R-module. The generalized module is called a small essentially quasi−Dedekind R-module. Some basic properties of the new generalized module are given and illustrated by many examples. Also, the relationship with other modules is investigated and discussed, namely with quasi.−Dedekind R-module. Several results are provided in this work

References

[1] E. A. Mohamed-Ali, " On Ikeda Nakayama Modules " Ph. D. Thesis, University of Baghdad, College of Education Ibn Al-Haitham, 2017.

[2] T. Y. Ghawi, " Some Generalization of Quasi-Dedekind Modules" M.Sc. Thesis , University of Baghdad, College of Education Ibn Al-Haitham, 2010.

[3] I. M. Ali and T. Y. Ghawi , " Some Results on Essentially Quasi-Dedekind Modules" Iraqi Journal of Science , Vol. 56, No. 2A, pp:1130-1139, Nov. 2015.

[4] F. D. Shyaa and I. M. Ali, " S-K-nonsingular Modules " Iraqi Journal of Science , Vol. 62, No. 4, pp:1314-1320. 2021.

[5] D.X. Zhou and X.R. Zhang, " Small-essential submodules and Morita duality" Southeast Asian Bull.Math., Vol. 35, pp:1051-1062. 2011.

[6] A. S. Mijbass , "Quasi-Dedekind Modules " Ph. D. Thesis, University of Baghdad, College of Science, 1997.

[7] G. Desale, and W. K. Nicholson, " Endoprimetive Rings " , J. Algebra, Vol.70, pp. 548-560, 1981.

[8] F. Kasch, "*Modules and Rings*", London, Academic press, 1982.

[9] Y. Durgun and S. Ozdemir , " On S-closed Submodules" J. Korean Math. Soc., No. 4, pp.1281-1299, 2017.

[10] F. H. Mohammed, " Generalizations of Extending Modules and Related Topics" M.Sc. Thesis, University of Al-Mustansiriya, College of Science, 2016.