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On Some Sandwich Results of Univalent Functions Related by Differential Operator

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Abstract:
The goal of the present paper is to obtain some differential subordination and superordination theorems for univalent functions related by differential operator $S^k_{\alpha,\beta,\lambda,\delta}$. Also, we discussed some sandwich-type results.

Keywords: Univalent functions, Subordination, Superordination, Hadamard product, Sandwich theorems

حول بعض نتائج الساندوج للدوال احادية التكافؤ المرتبطة بمؤثر تفاضلي

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الخلاصة :
الهدف من هذا البحث هو الحصول على بعض مبرهنات التابعية التفاضلية والتابعية التفاضلية العليا للدوال احادية التكافؤ المرتبطة بالمؤثر التفاضلي $S^k_{\alpha,\beta,\lambda,\delta}$ وايضا ناقشنا بعض النتائج من النوع الساندوج.

1-Introduction

Let M= M [$\mathbb{U}$] be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. For j a positive integer and $a \in \mathbb{C}$, let $\mathcal{M}[a, j]$ be the subclass of the functions $f \in \mathcal{M}$ of the form:

$$f(z) = a + a_j z^j + a_{j+1} z^{j+1} + \cdots \qquad (a \in \mathbb{C}, j \in \mathbb{N} = \{1,2,3, \dots\}) . \tag{1.1}$$

Also, let B be the subclass of $\mathcal{M}$ consisting of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \qquad (a_j \geq 0). \tag{1.2}$$

For the function f which is given by (1.2) and $g \in B$ is given by

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$$g(z) = z + \sum_{j=2}^{\infty} b_j z^j,$$

the Hadamard product (or Convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z) . \quad (1.3)$$

Let f and g are analytic functions in $\mathbb{U}$, we say that the function f is subordinate to g or g is said to be superordinate to f if there exists a Schwaz function w in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{C})$, where $f(z) = g(w(z))$. In such case, we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{C})$. In particular, if the function g is univalent in $\mathbb{U}$ then $f \prec g$ if and only if $f(0) = g(0)$, and $f(\mathbb{U}) = g(\mathbb{U})$ ([1,2]).

Definition1:([1,3]) Let $\Psi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If p and $\Psi(p(z), zp'(z), z^2p''(z); z)$ are univalent in $\mathbb{U}$ and if p satisfies the second-order differential superordinations $h(z) \prec \Psi(p(z), zp'(z), z^2p''(z); z)$, (1.4)
then p is called a solution of the differential superordination (1.4). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.4) or more simply a subordinant, if $q \prec p$ for all p satisfying (1.4). A univalent subordinant $\breve{q}(z)$ that satisfies $q \prec \breve{q}$ for all subordinants q of (1.4) is said to be the best Subordinant.

Definition 2: [1] Let $\Psi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent functions in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination,

$$\Psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) , \quad (1.5)$$

then p is called a solution of a differential subordination (1.5). The univalent function is called a dominant of the solution of the differential subordination (1.5), or more simply dominant if $p \prec q$ for all p satisfying (1.5). A dominant $\breve{q}(z)$ that satisfies $\breve{q} \prec q$ for all dominant q of (1.5) is said to be the best dominant.
Miller and Mocanu [1,2,4] studied the dual problem and determined conditions on Ψ such that (1.4) is satisfied, this implies $q(z) \prec p(z)$ for all function $q \in \mathbb{Q}$, that satisfy the superordination (1.4). They also found conditions so that the function q is the largest function with this property, which is called the best subordinant of the superordination (1.4). They also considered the problem of determining conditions and admissible functions Ψ such that (1.5) is satisfied which implies $p(z) \prec q(z)$, for all functions $p(z) \in \mathcal{M}$. Moreover, they found conditions so that q is the smallest function with this property which is called the best dominant of the subordination (1.5).See also [1,3,5-17].

Using the results (see [11,18-31]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$.

Also, Al-Ameedee et al. [18,19] and El-Ashwah and Aouf [3] derived some differential subordination and superordination results for analytic functions in $\mathbb{U}$. Recently, several researchers obtained sandwich theorems for subclasses of analytic functions (see [3,5-8,10,12-14,18,20,26-31]) . In [32], Catas extended the multiplier transformation and defined the operator $S^k_{\alpha,\beta,\lambda,\delta}$ on B, which is defined as follows:

$$S^k_{\alpha,\beta,\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} \left((\lambda - \delta)(\beta - \alpha)(n-1) + 1\right)^k a_n z^n, \quad (1.6)$$

where $\alpha, \beta, \delta, \lambda \geq 0, \lambda > \delta, \beta > \alpha$.

We note that from (1.6), we have

$$(\lambda-\delta)(\beta-\alpha)\left(S^{k}_{\alpha,\beta,\lambda,\delta}f(z)\right)' = S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)-\big(1-(\lambda-\delta)(\beta-\alpha)\big)S^{k}_{\alpha,\beta,\lambda,\delta}f(z). \quad (1.7)$$

See also [9].

The main object of this paper is to find sufficient conditions for certain normalized analytic functions h to satisfy:

$$q_1(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)+(\lambda-\delta)(\beta-\alpha)S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z)}{2(\lambda-\delta)(\beta-\alpha)z} \prec q_2(z),$$

and

$$q_1(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \prec q_2(z),$$

where $q_1(z)$ *and* $q_2(z)$ are given univalent functions in $\mathbb{U}$ with $q_1(0)=q_2(0)=1$.

2-Preliminaries

To prove our sudordination and superordination results, we need the following definitions and lemmas.

Definition2.1: [4] Let Q be the set of all functions t that are analytic and injective on $\overline{\mathbb{U}}\setminus E(t)$, *where* $\overline{\mathbb{U}}=\mathbb{U}\cup\{z\in\partial\mathbb{U}\}$, and

$$E(t)=\left\{\boldsymbol{\zeta}\in\partial\mathrm{U}: \lim_{z\to\boldsymbol{\zeta}} t(z)=\infty\right\}, \quad (2.1)$$

such that $t'(\boldsymbol{\zeta})\neq 0$ for $\boldsymbol{\zeta}\in\partial\mathrm{U}\setminus E(t)$. Further, let $Q(a)$ be the subclass of Q for which $t(0)=a$, such that $Q(0)\equiv Q_0$ *and* $Q(1)\equiv Q_1$.

Lemma2.1: [2] Let $t(z)$ be a convex univalent function in $\mathbb{U}$. Let $\sigma\in\mathbb{C}, \rho\in\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ and suppose that

$$Re\left(1+\frac{zt''(z)}{t'(z)}\right) > max\left\{0,-Re\left(\frac{\sigma}{\rho}\right)\right\}.$$

If $r(z)$ is analytic in $\mathbb{U}$ and

$\sigma r(z)+\rho z r'(z) \prec \sigma t(z)+\rho z t'(z)$, then $r(z)\prec t(z)$ and t is the best dominant.

Lemma 2.2: [1] Let t be univalent in $\mathbb{U}$ and let ϕ *and* θ be analytic in the domain D containing $t(\mathbb{U})$ with $\phi(w)\neq 0$, when $w\in t(\mathrm{U})$. Set $Q(z)=zt'(z)\phi\big(t(z)\big)$ *and* $h(z)=\theta\big(t(z)\big)+Q(z)$,

suppose that

1- Q is starlike univalent in $\mathbb{U}$.

2- $Re\left(\frac{zh\prime(z)}{Q(z)}\right) > 0,\ z\in\mathbb{U}$.

If r is analytic in $\mathbb{U}$ with $r(0)=t(0), r(\mathbb{U})\subseteq D$ and

$\phi\big(r(z)\big)+zr'(z)\,\phi\big(r(z)\big) \prec \phi\big(t(z)\big)+zt'(z)\phi\big(t(z)\big)$, then $r\prec t$, and t is the best dominant.

Lemma2.3: [11] Let $t(z)$ *be* a convex univalent in the unit disk$\mathbb{U}$ and let θ *and* ϕ be analytic in a domain D containing $t(\mathbb{U})$. Suppose that

1- $Re\left\{\frac{\theta\prime(t(z))}{\phi(t(z))}\right\} > 0$ *for* $z\in\mathrm{U}$,

2- zt'(z)ϕ(t(z)) is starlike univalent in $z\in\mathbb{U}$.

If $r\in\mathcal{H}[t(0),1]\cap Q$, with $r(\mathrm{U})\subseteq D$, and $\theta(r(z)+zr'(z)\phi(r(z))$ is univalent in , and

$$\theta(t(z)) + zt'(z)\phi(t(z)) \prec \theta(r(z)) + zr'(z)\phi(r(z)), \tag{2.2}$$
then $t \prec r$, and t is the best subordinant.

Lemma2.4: [11] Let $t(z)$ be a convex univalent in$\mathbb{U}$ and $t(0) = 1$. Let $\alpha \in \mathbb{C}, \gamma \in \mathbb{C}^*$ that $Re\left\{\frac{\alpha}{\gamma}\right\} > 0$. If $r(z) \in \mathcal{M}[t(0), 1] \cap Q$ and $r(z) + \gamma zr'(z)$ is univalent in U, then $\alpha t(z) + \gamma zt'(z) \prec \alpha r(z) + \gamma zr'(z)$, which implies that $t(z) \prec r(z)$ and $t(z)$ is the best subordinant.

3-Subordination Results

Theorem3.1: Let $t(z)$ be a convex univalent in O with $t(0) = 1$, and Suppose that
$$Re\left\{1 + \frac{zt''(z)}{t'(z)}\right\} \geq max\{0, -Re(1)\}. \tag{3.1}$$
If $h \in \mathcal{B}$ satisfies the subordination
$$\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z} + \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z) - (1-(\lambda-\delta)(\beta-\alpha))S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right]$$
$$\prec q(z) + zq'(z), \tag{3.2}$$
then
$$\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \prec t(z), \tag{3.3}$$
and $t(z)$ is the best dominant.

Proof : Define a function $k(z)$ by
$$k(z) = \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}, \tag{3.4}$$
then the function $k(z)$ is analytic in $\mathbb{U}$ and $k(0) = 1$. Therefore, differentiating (3.4) with respect to z and using the identity (1.7) in the resulting equation,we obtain
$$\frac{zk'(z)}{k(z)} = \frac{S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z) - (1-(\lambda-\delta)(\beta-\alpha))S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)} - 1. \tag{3.5}$$
Therefore,
$$zk'(z) = \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z) - (1-(\lambda-\delta)(\beta-\alpha))S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right].$$
Thus the subordination (3.2) is equivalent to
$$k(z) + zk'(z) \prec t(z) + zt'(z).$$
Putting $t(z) = \frac{1+z}{1-z}$ in Theorem (3.1), we obtain the following:

Corollary 3.1: Suppose that
$$Re\left(1 + \frac{2z}{1-z}\right) > max\{0, -Re(1)\}.$$
If $h \in \mathcal{B}$ is satisfy the following subordination condition:
$$\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z) - (1-(\lambda-\delta)(\beta-\alpha))S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right]$$
$$\prec \left(\frac{1-z^2+2z}{(1-z)^2}\right),$$

then

$$\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \prec \left(\frac{1+z}{1-z}\right),$$

and $t(z)=\frac{1+z}{1-z}$ is the best dominant.

Theorem 3.3: Let $q(z)$ be univalent in $\mathbb{U}$, with $q(0) = 1$,. Let $f \in \mathcal{B}$ and suppose that f and q satisfy the next conditions:

$$\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)}{2(\lambda-\delta)(\beta-\alpha)z} \neq 0 \quad (3.6)$$

and

$$Re\left\{1+\frac{zt''(z)}{t'(z)}\right\} \geq max\{0,-Re(1)\}. \quad (3.7)$$

If

$$\chi(z)=\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)}{2(\lambda-\delta)(\beta-\alpha)z}+\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)'+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)'}{2(\lambda-\delta)(\beta-\alpha)}-1 \quad (3.8)$$

and

$$\chi(z) \prec q(z)+\frac{zq'(z)}{q(z)}, \quad (3.9)$$

then

$$\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)}{2(\lambda-\delta)(\beta-\alpha)z} \prec q(z) \quad (3.10)$$

and $q(z)$ is the best dominant of (3.6).

Proof: Let

$$k(z)=\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)}{2(\lambda-\delta)(\beta-\alpha)z}. \quad (3.11)$$

According to (3.2) the function $p(z)$ is analytic in $\mathbb{U}$, and differentiating (3.11) with respect to z, we obtain

$$\frac{zk'(z)}{k(z)}=\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)'+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)'}{2(\lambda-\delta)(\beta-\alpha)}-1 \quad (3.12)$$

and hence

$$zk'(z)=\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)}{2(\lambda-\delta)(\beta-\alpha)z}+\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)'+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)'}{2(\lambda-\delta)(\beta-\alpha)}-1.$$

In order to prove our result, we have to use Lemma (2.2). In this Lemma consider

$$\theta(w)=w \text{ and } \varphi(w)=\frac{1}{w},$$

then θ is analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$ is analytic in $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)'+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)'}{2(\lambda-\delta)(\beta-\alpha)}-1$$

and

$$h(z) = \theta(q(z)) + Q(z)$$

$$=\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)}{2(\lambda-\delta)(\beta-\alpha)z}+\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)'+(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)'}{2(\lambda-\delta)(\beta-\alpha)}-1,$$

from (3.6), we see that $\mathbb{Q}(z)$ is a starlike function in $\mathbb{U}$. We also have

$$Re\left\{\frac{zh'(z)}{\mathbb{Q}'(z)}\right\} = Re\left\{2 + \frac{zt''(z)}{t'(z)}\right\} > 0, (z \in U)$$

and then, by using Lemma (2.2), we deduce that the subordination (3.9) implies $p(z) \prec q(z)$.

4-Superordination Results:

Theorem 4.1: Let $q(z)$ be a convex in $\mathbb{U}$ with $q(0)$=1, $\alpha, \beta, \lambda, \delta \geq 0, \lambda \geq 0, \lambda > \delta, \beta > \alpha$, $Re(1) > 0$, If

$f(z) \in \mathcal{B}$ such that $\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \in w[q(0),1] \cap Q$ and

$$\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z} + \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)-(1-(\lambda-\delta)(\beta-\alpha))S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)} - 1\right]$$

is univalent in $\mathbb{U}$, and satisfies the superordination

$$q(z) + zq'(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z} + \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)-(1-(\lambda-\delta)(\beta-\alpha))S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)} - 1\right], \quad (4.1)$$

then

$$q(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z}$$

and $q(z)$ is the best subordinant.

Proof: If we consider the analytic function

$$\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z}, z \in U \quad (4.2)$$

Differentiating (4.2) with respect to z and using the identity (1.7) in the resulting equation, we have

$$\frac{zk'(z)}{k(z)} = \frac{S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)-(1-((\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)} - 1,$$

that is

$$zk'(z) = \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S_{\alpha,\beta,\lambda,\delta}^{k+2}f(z)-(1-(\lambda-\delta)(\beta-\alpha))S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)}{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{k+1}f(z)} - 1\right]. \quad (4.3)$$

Thus, the subordination (4.1) is equivalent to

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

Applying Lemma (2.3), with the proof of Theorem (4.1) is complete.
Taking $k = 0$ in Theorem (4.1), we obtain the following result:

Corollary 4.1: Let $q(z)$ be convex in $\mathbb{U}$, with $q(0) = 1$, $\beta \in \mathbb{C}$, $Re(\beta) > 0$, and suppose that (3.1) holds. If f(z) $\in \mathcal{B}$,such that $\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{1}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \in w[q(0),1] \cap Q$, and

$$\frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{1}f(z)}{(\lambda-\delta)(\beta-\alpha)z} + \frac{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{1}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S_{\alpha,\beta,\lambda,\delta}^{2}f(z)-(1-(\lambda-\delta)(\beta-\alpha))S_{\alpha,\beta,\lambda,\delta,}^{1}f(z)}{(\lambda-\delta)(\beta-\alpha)S_{\alpha,\beta,\lambda,\delta}^{1}f(z)} - 1\right],$$

is univalent in $\mathbb{U}$ and satisfies the superordination

$$q(z)+zq'(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}+\frac{(\lambda-\delta)(\beta-\alpha)S^{1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S^{2}_{\alpha,\beta,\lambda,\delta}f(z)-(1-(\lambda-\delta)(\beta-\alpha))S^{1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{1}_{\alpha,\beta,\lambda,\delta}f(z)}-1\right],$$

then $q(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}$ and q(z) is the best subordinant.

5- Sandwich Results:

Combination Theorem (3.1) with Theorem (4.1) , we obtain the following sandwich Theorem:

Theorem (4.3): Let q_1, q_1 are two convex functions in $\mathbb{U}$ with $q_1(0)=q_2(0) = 1$ and q_2 satisfies (3.1), $Re(1) > 0$. If $f(z) \in \mathcal{B}$ such that

$$\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \in \mathcal{M}[q(0),1] \cap Q,$$

and $\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}+\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z)-(1-(\lambda-\delta)(\beta-\alpha))S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}-1\right]$ is univalent in $\mathbb{U}$, and satisfies

$$q(z)+zq'(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}+\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z}\left[\frac{S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z)-\left(1-(\lambda-\delta)(\beta-\alpha)\right)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}-1\right] \prec q(z)+zq'(z), \quad (5.1)$$

then

$$q_1(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)}{(\lambda-\delta)(\beta-\alpha)z} \prec q_2(z)$$

and q_1, q_2 are the best subordinant and the best dominant of (5.1), respectively.

Theorem (4.4) : Let q_1, q_2 be two convex functions in U, with $q_1(0)=q_2(0) = 1, Re(1) > 0$. Let $f \in \mathcal{B}$ and suppose that f satisfies the following conditions:

$$\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)+(\lambda-\delta)(\beta-\alpha)S^{k+2}_{\alpha,\beta,\lambda,\delta,}f(z)}{2(\lambda-\delta)(\beta-\alpha)z} \neq 0$$

and

$$\frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta}f(z)+(\lambda-\delta)(\beta-\alpha)S^{k+2}_{\alpha,\beta,\lambda,\delta}f(z)}{2(\lambda-\delta)(\beta-\alpha)z} \in \mathcal{M}[p(0),1].$$

If the function $\chi(z)$ given by (3.8) is univalent in $\mathbb{U}$ and,

$$q_1(z)+\frac{zq_1{}'(z)}{q_1(z)} \prec \chi(z) \prec q_2(z)+\frac{zq_2'(z)}{q_2(z)}\ , \quad (5.2)$$

then

$$q_1(z) \prec \frac{(\lambda-\delta)(\beta-\alpha)S^{k+1}_{\alpha,\beta,\lambda,\delta,p}f(z)+(\lambda-\delta)(\beta-\alpha)S^{k+2}_{\alpha,\beta,\lambda,\delta,p}f(z)}{2(\lambda-\delta)(\beta-\alpha)z} \prec q_2(z)$$

and q_1, q_2 are the best subordinate and the best dominant of (5.2), respectively.

Conclusion

In this work, some differential subordination and superordination theorems for univalent functions related by differential operator $S^{k}_{\alpha,\beta,\lambda,\delta}$ are obtained and investigated. Futher, some results of sandwich-type are studied and discussed. Finally, many properties and important outcomes are given.

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