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Oscillation and Asymptotic Behavior of Second Order Half Linear Neutral Dynamic Equations

Hussain Ali Mohamad[1]*, Faten Abbas Ahmed[2]
[1]Department of Mathematics, College of Science for Women, University of Baghdad, Iraq
[2]Department of Mathematics, College of Science, University of Baghdad, Iraq

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Abstract
The oscillation property of the second order half linear dynamic equation was studied, some sufficient conditions were obtained to ensure the oscillation of all solutions of the equation. The results are supported by illustrative examples.

Keywords: Oscillation, Nonoscillation, neutral; Delay; Dynamic equations; Differential equations, Time scales.

التذبذب والسلوك المقارب للمعادلات الديناميكية المحايدة الخطية من الدرجة الثانية

حسين علي محمد[1]*، فاتن عباس أحمد[2]
[1]قسم الرياضيات ، كلية العلوم للبنات ، جامعة بغداد ، العراق
[2]قسم الرياضيات ، كلية العلوم ، جامعة بغداد ، العراق

الخلاصه
تمت دراسة خاصية التذبذب للمعادلة الديناميكية المحايدة نصف الخطية من الرتبة الثانية ، وتم الحصول على بعض الشروط الكافية لضمان تذبذب جميع حلول هذه المعادلة. النتائج مدعومة بأمثلة توضيحية.

1. Introduction

In recent years, the study of dynamic equations on time scales has become an area of mathematics and has received a lot of attention. It was created to unify the study of differential and difference equations, and it also extends these classical cases to cases in between to the so-called q-difference equations. Many results concerning differential equations carry over quite easily to corresponding results for deference equations while other results seem to be completely different from their continuous counterparts. Bohner et al.[1], Peterson [2] and Zhang et al.[3] summarize and organize much of time scale calculus. Dynamic equations on a time scale. Ahmed et al.[4] investigated third order neutral dynamic equation and obtain some oscillation conditions for each solution. In [5-9] the author have

*Email: faten.ahmed1103@sc.uobaghdad.edu.iq

established some new oscillation criteria for second order non-linear neutral delay dynamic equations of the form

$$\left(\xi(t)\left(x(t)+p(t)y(t-\tau)\right)^{\Delta}\right)^{\gamma})^{\Delta}+g(t,y(t-\tau))=0.$$

Dosly et al.[10] obtained necessary and sufficient condition for oscillation of the Sturm Liouville dynamic equation on time scales $(r(t)x^{\Delta})^{\Delta}+\ c(t)x^{\sigma}=0$
on a time scale $\mathbb{T}$, where the function $c(t)$ is an rd-continuous function such that
$c(t)>0$, for $t\in\mathbb{T}$. In [11] Tripathy considered the non-linear dynamic equation of the form

$$\left(\xi(t)\left(\left(y(t)+p(t)y(t-\tau)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+q(t)|y(t-\alpha)|^{\gamma}\mathrm{sgn}(y(t-\alpha))=0.$$

On timescale $\mathbb{T}$ where $\xi(t)>0,\ 0<p(t)\le 1, g(t,w)\ge q(t)|w^{\gamma}|$, **and** $\gamma>0$ is a quotient of odd positive integer, such that $\xi(t)$, $p(t)$ and $q(t)$ are positive real valued functions defined on $\mathbb{T}$. The aim of this paper is to obtain sufficient conditions to ensure the oscillation of all solutions of second order half linear dynamic equations.

2. Preliminaries

In this section we will present some definitions and concepts of Δ derivatives that we will need in this research.

Definition 2. 1[1]

A time scale is an arbitrary nonempty closed subset of the real numbers. We denote a time scale by the symbol $\mathbb{T}$ [3]. A solution of delay dynamic equation is said to be oscillatory if it has arbitrarily large zeros, for $t\in\mathbb{T}$, hence a nonoscillatory solution is either eventually positive or eventually negative.

Definition 2. 2[1]

1- For $t\in\mathbb{T}$,we define $\sigma(t)=\inf\{r\in\mathbb{T}: r>t\}$ in the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ maybe note that $\sigma(t)\ge t$ for any $t\in\mathbb{T}$.
2- If $\sigma(t)>t$ then t is called right-scattered, if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then t is called right-dense.
3- The graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$ for all $t\in\mathbb{T}$
4- Assume that $\psi:\mathbb{T}\to R$ is said to be right-dense continuous if it is right continuous at each right-dense point andthere exists a finite left limit at all left-dense points. ψ is said to be differentiable if its derivative exists. If ψ is continuous at t, and t is right-scatter then ψ is differentiable and $\psi^{\Delta}(t)=\frac{\psi(\sigma(t))-\psi(t)}{\mu(t)}$,we call $\psi^{\Delta}(t)$ the delta or Hilger derivative of ψ at t.We will make use of the following product and quotient rules for the derivatives of the product $\psi\phi$ and the quotient $\frac{\psi}{\phi}$ (where $\phi\phi^{\sigma}\ne 0,\ \phi^{\sigma}=\phi\circ\sigma$) of two differentiable function ψ and ϕ:

$$(\psi\phi)^{\Delta}(t)=\psi^{\Delta}(t)\phi(t)+\psi(\sigma(t))\phi^{\Delta}(t)=\psi(t)\phi^{\Delta}(t)+\psi^{\Delta}(t)\,\phi(\sigma(t)).$$

And

$$\left(\frac{\psi}{g}\right)^{\Delta}(t)=\frac{\psi^{\Delta}(t)\,\phi(t)-\psi(t)\phi^{\Delta}(t)}{\phi(t)\phi(\sigma(t))}$$

5- Let $\varphi:\mathbb{T}\to\mathbf{R}$ be an antiderivative of $\psi:\mathbb{T}\to R$. The Cauchy integral of ψ is then defined by: $\int_a^b\psi(t)\Delta t=\varphi(b)-\varphi(a)$ where $a,b\in\mathbb{T}$.

And infinite integrals are defined as

$$\int_{a}^{\infty} \psi(t)\Delta t = \lim_{b\to\infty} \int_{a}^{b} \psi(t)\Delta t.$$

3. Oscillation results

In this section, some oscillation results for the second order non-linear neutral delay dynamic equation

$$\left(\xi(t)\left(\left[y(t)+p(t)y\left(\tau(t)\right)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + \sum_{i=1}^{n} q_i(t)y\left(\delta_i(t)\right) = 0. \qquad (3.1)$$

Where $\gamma > 0$ is a quotient of odd positive integer on $\mathbb{T}$, $\tau(t), \delta_i(t),\ p(t), q_i(t),\ i = 1, 2, \ldots, n$ are positive real valued functions defined on $\mathbb{T}$, $\delta_i(t) < t, \tau(t), \delta_i(t)$are increasing functions obtained on a time scale$\mathbb{T}$, in addition to the following assumption

$(\mathbf{H_1})\ \int_{t_0}^{\infty} \left(\frac{1}{\xi(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty.$

$(\mathbf{H_2})\ \xi(t) > 0,\ \ 0 < p(t) \le a \ \mathbf{and} \sum_{i=1}^{n} q_i(t) \ge 0.$

$(\mathbf{H_3}) Q(t) = \min_{t\ge t_0}\left\{q_i(t), q_i\left(\tau(t)\right)\right\}.$

$(\mathbf{H_4})\ \int_{T}^{\infty} Q(t)\Delta t = \infty,\ \ T \ge t_0.$

$$\omega(t) = y(t) + p(t)y\left(\tau(t)\right). \qquad (3.2)$$

It is noticed that the work in [2] initiates the further study of equation (3.1) in other ranges of $p(t)$ on a time scale $\mathbb{T}$.

Lemma **3.1.** If $y(t)$ is an eventually positive solution of equation (3.1), then there exists $t_1 \ge a \ge 0$, such that $\omega^{\Delta}(t) > 0$, for $t \ge t_1$.

Proof. Let $y(t)$ be an eventually positive solution of equation (3.1). Hence there exists $t \in \mathbb{T}$ with $t \ge a$ such that $y(t) > 0$ and $y\left(\tau(t)\right) > 0,\ t \ge t_0$from equation (3.1) we get:

$$[\xi(t)\ (\omega^{\Delta}(t))^{\gamma}]^{\Delta} = -\sum_{i=1}^{n} q_i(t)\ y^{\gamma}(\delta_i(t)) \le 0,\ \ t \ge t_0. \qquad (3.3)$$

It follows that $\xi(t)(\omega^{\Delta}(t))^{\gamma}$ is nonincreasing function, we claim that $\xi(t)(\omega^{\Delta}(t))^{\gamma} > 0,\ t \ge t_1 \ge t_0$. Otherwise, $\xi(t)(\omega^{\Delta}(t))^{\gamma} < 0$, for $t \ge t_1 \ge t_0$, hence there exists $L < 0$ such that $\xi(t)\ (\omega^{\Delta}(t))^{\gamma} \le L < 0$ for $t \ge t_2 \ge t_1$, whichimplies that

$$\omega^{\Delta}(t) \le L^{\frac{1}{\gamma}} \left(\frac{1}{\xi(t)}\right)^{\frac{1}{\gamma}}, \qquad t \ge t_2. \qquad (3.4)$$

Integrating (3.4) from t_2 to t we get

$$\omega(t) - \omega(t_2) \le L^{\frac{1}{\gamma}} \int_{t_2}^{t} \left(\frac{1}{\xi(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

as $t \to \infty$, the last inequality leads to $\lim_{t\to\infty} \omega(t) = -\infty$ contradicts the fact that

$\omega(t) > 0$. Hence $\xi(t)(\omega^{\Delta}(t))^{\gamma} > 0,\ t \geq t_1$ and so $\omega^{\Delta}(t) > 0$.

Theorem 3.2. Let $t \in [t_0, \infty)_{\mathbb{T}}$, $(\mathbf{H_1}) - (\mathbf{H_4})$ hold and there exists $\lambda > 0$, such that

$$v^{\gamma} + u^{\gamma} \geq \lambda(u+v)^{\gamma}, u, v \in R.\ \ uv > 0 \tag{3.5}$$

Then every solution of equation (3.1) oscillates.

Proof. Assume that equation (3.1) has a nonoscillatory solution $y(t)$. We may assume without loss of generality that $y(t) > 0,\ y(\tau(t)) > 0,\ y(\delta_i(t)) > 0,\ i = 1, 2, \dots, n,\ t \geq t_0$.
By Lemma 3.1 it follows that $\omega^{\Delta}(t) > 0,\ t \geq t_1$. And $\xi(t)\ (\omega^{\Delta}(t))^{\gamma}$ is eventually non-negative for $t \geq t_1$. To find λ in (3.5), let $v = y(t),\ u = y(\tau(t))$,
$\eta(t) = \max\{y(t), y(\tau(t))\},\ t \geq t_1$ then there exist $\lambda_a > 0$ such that

$$y^{\gamma}(t) + p^{\gamma}(t) y^{\gamma}(\tau(t)) \geq \lambda_a \eta^{\gamma}(t) \geq \frac{\lambda_a}{(1+a)^{\gamma}}(y(t) + a y(\tau(t)))^{\gamma}$$

$$\geq \frac{\lambda_a}{(1+a)^{\gamma}}\left(y(t) + p(t) y(\tau(t))\right)^{\gamma} = \lambda\left(y(t) + p(t) y(\tau(t))\right)^{\gamma},$$

Where $\lambda = \frac{\lambda_a}{(1+a)^{\gamma}}$, hence

$$y^{\gamma}(t) + p^{\gamma}(t) y^{\gamma}(\tau(t)) \geq \lambda\left(y(t) + p(t) y(\tau(t))\right)^{\gamma}. \tag{3.6}$$

From equation (3.1) and (3.2) we obtain:

$$0 = (\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta} + \sum_{i=1}^{n} q_i(t)\, y^{\gamma}\, \delta_i(t) + a^{\gamma}(\xi(\tau(t))(\omega^{\Delta}(\tau(t)))^{\gamma})^{\Delta}$$
$$+ a^{\gamma} \sum_{i=1}^{n} q_i(\tau(t))\, y^{\gamma}\, \delta_i(\tau(t)),$$

$$0 \geq (\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta} + \sum_{i=1}^{n} Q(t) y^{\gamma}\, \delta_i(t) + a^{\gamma}(\xi(\tau(t))(\omega^{\Delta}(\tau(t)))^{\gamma})^{\Delta}$$
$$+ a^{\gamma} \sum_{i=1}^{n} Q(\tau(t))\, y^{\gamma}\, \delta_i(\tau(t))$$

$$(\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta} + a^{\gamma}\,(\xi(\tau(t))(\omega^{\Delta}(\tau(t)))^{\gamma})^{\Delta} + Q(t) \sum_{i=1}^{n} (y^{\gamma}(\delta_i(t)$$
$$+ a^{\gamma} y^{\gamma}(\delta_i(\tau(t))) \leq 0$$

$$(\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta} + a^{\gamma}[\xi(\tau(t))(\omega^{\Delta}(\tau(t)))^{\gamma}]^{\Delta}$$
$$+ Q(t) \sum_{i=1}^{n} \left[y^{\gamma}(\delta_i(t)) + p^{\gamma}(\delta_i(t)) y^{\gamma}(\delta_i(\tau(t))\right] \leq 0.$$

Let $\delta(t) = \min\{\delta_i(t), i = 1, 2 \dots, n\}$, then by using (3.6) the last inequality reduce to

$$[\xi(t)(\omega^{\Delta}(t))^{\gamma}]^{\Delta} + a^{\gamma}[\xi(\tau(t))(\omega^{\Delta}(\tau(t)))^{\gamma}]^{\Delta} + \lambda Q(t) \sum_{i=1}^{n} \omega^{\gamma}(\delta_i(t)) \leq 0$$

$$[\xi(t)(\omega^{\Delta}(t))^{\gamma}]^{\Delta} + a^{\gamma}[\xi(\tau(t))(\omega^{\Delta}(\tau(t)))^{\gamma}]^{\Delta} + n\lambda Q(t)\omega^{\gamma}(\delta(t)) \leq 0,$$

$$n\lambda Q(t)\omega^{\gamma}(\delta(t)) \leq -(\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta} - a^{\gamma}\left[\xi(\tau(t))\left(\omega^{\Delta}(\tau(t))\right)^{\gamma}\right]^{\Delta} \tag{3.7}$$

Since $\omega(t)$ is nondecreasing, we can find a constant $k > 0$, such that $\omega(t) \geq k,\ t \geq t_2 \geq t_1$, consequently, it follows that by integrating (3.7) from t_2 to t we get

$$n\lambda k^{\gamma} \int_{t_2}^{t} Q(s)\,\Delta s \leq -\int_{t_2}^{t} [\xi(s)\,\omega^{\Delta}(s))^{\gamma}]^{\Delta} \Delta s - a^{\gamma} \int_{t_2}^{t} \left[\xi(\tau(s))\left(\omega^{\Delta}(\tau(s))\right)^{\gamma}\right]^{\Delta} \Delta s,$$
$$\leq \xi(t_2)(\omega^{\Delta}(t_2))^{\gamma} + a^{\gamma}\xi(\tau(t_2))[\omega^{\Delta}(\tau(t_2))]^{\gamma}.$$

as $t \to \infty$, the last inequality leads to $\int_{t_2}^{\infty} Q(s)\,\Delta s < \infty$, which is a contradiction with $(\mathbf{H_4})$.

Theorem $\mathbf{3.3}$. Let $(\mathbf{H_1}) - (\mathbf{H_4})$, $(\mathbf{3.5})$ hold and $\xi^{\Delta}(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$. If $\tau(t) \geq t$, $\delta(t) = \min\{\delta_i(t),\ i = 1, 2 \dots, n\}$ **and** $\delta(t) < t$. Furthermore,

$$\limsup_{t\to\infty} \int_{\delta(t)}^{t} \frac{Q(s)\delta^{\gamma}(s)}{\xi(\delta(s))} \Delta s > \frac{1}{n\lambda c^{\gamma}} (1 + a^{\gamma}),\ \ \text{for } c \in (0, 1). \qquad (3.8)$$

Then every solution of equation (3.1) oscillates.

Proof. Assume without loss of generality equation (3.1) has eventually positive solution $y(t)$, that is $y(t) > 0, y(\tau(t)) > 0,\ y(\delta_i(t)) > 0,\ i = 1, 2, \dots, n$, proceeding as in the proof of Theorem 3.2, we conclude that (3.3) and (3.7) hold.
Letting $x(t) = \xi(t)(\omega^{\Delta}(t))^{\gamma}$, then (3.7) becomes

$$x^{\Delta}(t) + a^{\gamma}\left[x(\tau(t))\right]^{\Delta} + n\lambda Q(t)\omega^{\gamma}(\delta(t)) \leq 0, \qquad (3.9)$$

for $t \geq t_2$. Since $\xi^{\Delta}(t) > 0$ and $\omega(t)$ is nondecreasing, obviously we can see that $\omega^{\Delta\Delta}(t) \leq 0$ for $t \geq t_2$. Thus $\omega^{\Delta}(t)$ is positive nonincreasing function on $[t_2, \infty)_{\mathbb{T}}$. Consequently, $t \geq t_2$ it is clear that

$$\omega(t) = \omega(t_2) + \int_{t_2}^{t} \omega^{\Delta}(s)\Delta s \geq \int_{t_2}^{t} \omega^{\Delta}(s)\Delta s \geq (t - t_2)\omega^{\Delta}(t).$$

Then for $t_3 \geq \frac{t_2}{1-c}$, $c \in (0, 1)$it follows that:

$$\omega(t) \geq (t - t_2)\omega^{\Delta}(t) > ct\omega^{\Delta}(t), \qquad t \geq t_3.$$

Substituting the last inequality in (3.9) to obtain

$$x^{\Delta}(t) + a^{\gamma}\left[x(\tau(t))\right]^{\Delta} + n\lambda c^{\gamma} \frac{Q(t)\delta^{\gamma}(t)}{\xi(\delta(t))} x(\delta(t)) < 0. \qquad (3.10)$$

Since $x(t)$ is nonincreasing, then integrating the last inequality from $\delta(t)$ to t becomes

$$x(t) - x(\delta(t)) + a^{\gamma}x(\tau(t)) - a^{\gamma}x(\tau(\delta(t)) + n\lambda c^{\gamma}\, x(\delta(t)) \int_{\delta(t)}^{t} \frac{Q(s)\delta^{\gamma}(s)}{\xi(\delta(s))} < 0, \quad (3.11)$$

$$x(t) + a^{\gamma}x(\tau(t)) + x(\delta(t)) \left[n\lambda c^{\gamma} \int_{\delta(t)}^{t} \frac{Q(s)\delta^{\gamma}(s)}{\xi(\delta(s))} \Delta s - 1 - a^{\gamma} \right] < 0\ ,$$

$$\frac{x(t)}{x(\delta(t))} + a^{\gamma}\frac{x(\tau(t))}{x(\delta(t))} + n\lambda c^{\gamma} \int_{\delta(t)}^{t} \frac{Q(s)\delta^{\gamma}(s)}{\xi(\delta(s))} \Delta s - 1 - a^{\gamma} < 0.\ t \geq t_3$$

This contradicts with $(\mathbf{3.8})$.Thus every solution of equation (3.1) oscillates. The proof is complete.

Remark 3.1 A conclusion similar to the Theorem 3.3 can be introduced using the terms $\tau(t) < t,\ \delta(t) > t, \tau(\delta(t)) > t$.

Theorem 3.5. Assume that$(\mathbf{H_1})$– $(\mathbf{H_4})$ hold, $\xi^{\Delta}(t) \geq 0,\ \tau(t) \geq t$. Furthermore,

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t}\frac{Q(s)\delta^{\gamma}(s)}{\xi(\delta(s))}\Delta s>0. \qquad (3.12)$$

Where $\delta(t)=\min\{\delta_i(t), i=1,2\ldots,n$.Then every solution of equation (3.1) oscillates.

Proof. Suppose to the contrary that equation (3.1) has a nonoscillatory solution $y(t)$.

We may assume without loss of generality that $y(t)>0, y(\tau(t))>0, y(\delta(t))>0$, for all $t\geq t_0$. Set $\omega(t)=y(t)+p(t)y(\tau(t))$, in view of equation (3.1) and (H_2) we have,

$$[\xi(t)\,(\omega^{\Delta}(t))^{\gamma}]^{\Delta}+\sum_{i=1}^{n}Q(t)y^{\gamma}(\delta_i(t))\leq 0, \qquad (3.13)$$

for all $t>t_0$ and so $\xi(t)(\omega^{\Delta}(t))^{\gamma}$is eventually decreasing function. Then from Lemma 3.1 it follows $\xi(t)(\omega^{\Delta}(t))^{\gamma}$ is eventually non-negative. Therefore, we see that for some $t_1\geq t_0$,

$$\omega(t)>0\,,\omega^{\Delta}(t)\geq 0\,,(\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta}\leq 0,\ \ t\geq t_1. \qquad (3.14)$$

This implies that

$$y(t)=\omega(t)-p(t)y(\tau(t))=\omega(t)-p(t)[\omega(\tau(t))-p(\tau(t))y(\tau(\tau(t)))]$$
$$\geq\omega(t)-\ p(t)\omega(\tau(t))\geq(\ 1-a)\omega(t).$$

Then for $t\geq t_2\geq t_1$ we have

$$y^{\gamma}(\delta_i(t))\geq(1-\ a)^{\gamma}\omega^{\gamma}(\delta_i(t)))\geq(1-\ a)^{\gamma}\omega^{\gamma}(\delta(t))).$$

From (3.13) and the last inequality we obtain

$$[\xi(t)(\omega^{\Delta}(t))^{\gamma}]^{\Delta}+nQ(t)(1-\ a)^{\gamma}\omega^{\gamma}(\delta(t))\leq 0,\ \ t\geq t_2. \qquad (3.15)$$

From $\xi^{\Delta}(t)\geq 0$ and (3.13) we can verify that $\omega^{\Delta\Delta}(t)\leq 0$ for $t\geq t_2$ and then $\omega^{\Delta}(t)$is positive and nonincreasing. Using this, we have

$$\omega(t)=\omega(t_2)+\int_{t_2}^{t}\omega^{\Delta}(s)\Delta s\geq\int_{t_2}^{t}\omega^{\Delta}(s)\Delta s\geq(t-t_2)\omega^{\Delta}(t).$$

Hence for $t_3\geq ct_2$, $c\in(0,1)$ it follows the last inequality leads to

$$\omega(t)\geq(t-t_2)\omega^{\Delta}(t)>c\,t\,\omega^{\Delta}(t).$$
$$\omega(\delta(t))>c\,\delta(t)\omega^{\Delta}(\delta(t)),\ \ t\geq t_3.$$

Substituting the last inequality in (3.15) to obtain

$$[\xi(t)(\omega^{\Delta}(t))^{\gamma}]^{\Delta}+nQ(t)(1-\ a)^{\gamma}(\ c\,\delta(t))^{\gamma}\left(\omega^{\Delta}(\delta(t))\right)^{\gamma}<0.$$

Let $w(t)=\xi(t)\,(\omega^{\Delta}(t))^{\gamma}$, that is $(\omega^{\Delta}(t))^{\gamma}=\frac{w(t)}{\xi(t)}$.

We have from the last inequality

$$w^{\Delta}(t)+Q(t)(1-a)^{\gamma}\frac{w(\delta(t)))}{\xi(\delta(t)))}(\ c\delta(t))^{\gamma}<0.$$

It is obvious that $w^{\Delta}(t)\leq 0.$ Integrating the last inequality from $\delta(t)$ to t for t sufficiently large to get

$$0>w(t)-w(\delta(t))+(c(1-\ a))^{\gamma}w(\delta(t))\int_{\delta(t)}^{t}\frac{Q(s)}{\xi(\delta(s))}(\delta(s))^{\gamma}\Delta s$$
$$=w(t)+w(\delta(t))\left((c(1-\ a))^{\gamma}\int_{\delta(t)}^{t}\frac{Q(s)}{\xi(\delta(s))}(\delta(s))^{\gamma}\Delta s-1\right)$$

$$\geq w(t) + w(t)\left((c(1-\ a))^{\gamma} \int_{\delta(t)}^{t} \frac{Q(s)}{\xi(\delta(s))} (\delta(s))^{\gamma}\, \Delta s - 1\right)$$
$$(c(1-\ a))^{\gamma} w(t) \int_{\delta(t)}^{t} \frac{Q(s)}{\xi(\delta(s))} (\delta(s)))^{\gamma}\, \Delta s < 0.$$

As $t \to \infty$, by (3.12) the last inequality leads to a contradiction. That means the proof hasbeen completed.

Theorem 3.6. Let $\delta(t) \leq \tau(t) < t,\ t \in [t_0, \infty)_{\mathbb{T}}$ and $\xi^{\Delta}(t) \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$. If $(\mathrm{H_1}) - (\mathrm{H_4})$ hold and there exists a positive rd-continuous Δ differentiable function $\alpha(t)$, such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} [\alpha(s) Q_1(s) - \frac{\xi(\tau(t))((\alpha^{\Delta}(s))^{+})^{2}}{4(c\delta(t))^{\gamma-1}\alpha(s)}]\, \Delta s = \infty,\ \ c \in (0, \infty). \qquad (3.16)$$

Where $(\alpha^{\Delta}(s))^{+} = \max\{0, \alpha^{\Delta}(s)\}$ and $Q_1(t) = Q(t)(1 - p(\delta(t)))^{\gamma}$.

Then every solution of equation (3.1) oscillates on $[t_0, \infty)_{\mathbb{T}}$.

Proof*:* Suppose that equation (3.1) possess eventually positive solution, then as in Theorem 3.3, we can get (3.14) and (3.15). Consequently, (3.15) can be rewritten

$$\alpha(t) \frac{(\xi(t)\, \omega^{\Delta}(t))^{\gamma}\,)^{\Delta}}{\omega^{\gamma}(\delta(t))} + \frac{\alpha(t)\, Q(t)}{\omega^{\gamma}(\delta(t))} -\ \alpha(t)\big(p(\delta(t)\big)^{\gamma} \leq 0\,,\ \ \text{for all}\ \ t \geq t_1 \geq t_0.$$

Define:

$$W(t) = \alpha(t) \frac{\xi(t)(\omega^{\Delta}(t))^{\gamma}}{\omega^{\gamma}(\delta(t))}, \quad t \geq t_1 \qquad (3.17)$$

Then $W(t) > 0$ and by using the preliminaries in section 2, one can obtain

$$W^{\Delta}(t) = (\xi(\omega^{\Delta})^{\gamma})^{\sigma} \left[\frac{\alpha(t)}{\omega^{\gamma}(\delta(t))}\right]^{\Delta} + \frac{\alpha(t)}{\omega^{\gamma}(\delta(t))}\ ((\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta}$$

$$W^{\Delta}(t) = \frac{\alpha(t)((\xi(t)(\omega^{\Delta}(t))^{\gamma})^{\Delta}}{\omega^{\gamma}(\delta(t))} + (\xi(\omega^{\Delta})^{\gamma})^{\sigma} \frac{\omega^{\gamma}(\delta(t))\alpha^{\Delta}(t) - \alpha(t)\left(\omega^{\gamma}(\delta(t))\right)^{\Delta}}{\omega^{\gamma}(\delta(t))\omega^{\gamma}(\delta(\sigma(t))}\,. \qquad (3.18)$$

On simplification (3.18) and (3.14) leads to

$$W^{\Delta}(t) \leq -\alpha(t) Q_1(t) + \frac{\alpha^{\Delta}(t) W(t)^{\sigma}}{\alpha(t)^{\sigma}} - \left[\frac{\alpha(t)\ \ (\xi((\omega)^{\Delta})^{\gamma}\,)^{\sigma}\left(\omega^{\gamma}(\delta(t))\right)^{\Delta}}{\omega^{\gamma}(\delta(t))\omega^{\gamma}(\delta(\sigma(t)))}\right]. \quad (3.19)$$

Since $\left(\omega^{\gamma}(t)\right)^{\Delta} = \frac{\omega^{\gamma}(\sigma(t)) - \omega^{\gamma}(t)}{\mu(t)}$, thus by using the inequality in [10], to obtain

$$\left(\boldsymbol{\omega}^{\gamma}(\boldsymbol{\delta}(\boldsymbol{t}))\right)^{\Delta}=\frac{\boldsymbol{\omega}^{\gamma}\left(\boldsymbol{\sigma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)\right)-\boldsymbol{\omega}^{\gamma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}{\boldsymbol{\mu}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)} \geq \frac{\boldsymbol{\gamma}\boldsymbol{\omega}^{\gamma-1}(\boldsymbol{\delta}(\boldsymbol{t}))}{\boldsymbol{\mu}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}[\boldsymbol{\omega}^{\gamma}(\boldsymbol{\sigma}(\boldsymbol{\delta}(\boldsymbol{t})))-\boldsymbol{\omega}^{\gamma}(\boldsymbol{\delta}(\boldsymbol{t}))]$$
$$\geq \boldsymbol{\gamma}\boldsymbol{\omega}^{\gamma-1}(\boldsymbol{\delta}(\boldsymbol{t}))\,\boldsymbol{\omega}^{\Delta}(\boldsymbol{\delta}(\boldsymbol{t}))). \qquad (\mathbf{3.20})$$

So it follows from (3.19) and (3.20) that

$$\boldsymbol{W}^{\Delta}(\boldsymbol{t}) \leq -\boldsymbol{\alpha}(\boldsymbol{t})\boldsymbol{Q}_{\mathbf{1}}(\boldsymbol{t})+\frac{\boldsymbol{\alpha}^{\Delta}(\boldsymbol{t})\boldsymbol{W}(\boldsymbol{t})^{\sigma}}{\boldsymbol{\alpha}(\boldsymbol{t})^{\sigma}}$$
$$-\left[\frac{\boldsymbol{\alpha}(\boldsymbol{t})(\boldsymbol{\xi}((\boldsymbol{\omega})^{\Delta})^{\gamma})^{\sigma}\boldsymbol{\gamma}\boldsymbol{\omega}^{\gamma-1}(\boldsymbol{\delta}(\boldsymbol{t}))\,\boldsymbol{\omega}^{\Delta}(\boldsymbol{\delta}(\boldsymbol{t})))}{\boldsymbol{\omega}^{\gamma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)\boldsymbol{\omega}^{\gamma}\left(\boldsymbol{\delta}(\boldsymbol{\sigma}(\boldsymbol{t}))\right)}\right] \qquad (\mathbf{3.21})$$

It follows from Theorem (3.4), when$\boldsymbol{\xi}^{\Delta}(\boldsymbol{t}) \geq \mathbf{0}$, and $(\mathbf{3.13})$ there exists $\boldsymbol{t} \geq \frac{t_2}{1-c}$,for $\boldsymbol{c} \in (\mathbf{0},\mathbf{1})$ .So that for $\boldsymbol{t} \in [\boldsymbol{t}_{\mathbf{3}},\infty)$ we have, $\boldsymbol{\omega}(\boldsymbol{t}) \geq \boldsymbol{ct}\boldsymbol{\omega}^{\Delta}(\boldsymbol{t})$ and so $\boldsymbol{\omega}^{\gamma-1}(\boldsymbol{t}) \geq (\boldsymbol{ct}))^{\gamma-1}(\boldsymbol{\omega}^{\Delta}(\boldsymbol{t}))^{\gamma-1}$.

Due to (3.13), since$(\boldsymbol{\xi}(\boldsymbol{t})\,\boldsymbol{\omega}^{\Delta}(\boldsymbol{t}))^{\gamma})^{\Delta} < \mathbf{0}$, we have

$$\boldsymbol{\xi}(\boldsymbol{t})\left(\boldsymbol{\omega}^{\Delta}(\boldsymbol{t})\right)^{\gamma} > \boldsymbol{\xi}\left(\boldsymbol{\sigma}(\boldsymbol{t})\right)(\boldsymbol{\omega}^{\Delta}(\boldsymbol{\sigma}(\boldsymbol{t})))^{\gamma}. \qquad (\mathbf{3.22})$$

So it follows from (3.22) and (3.13) that

$$\boldsymbol{\omega}^{\gamma-1}(\boldsymbol{\delta}(\boldsymbol{t})\boldsymbol{\omega}^{\Delta}((\boldsymbol{\delta}(\boldsymbol{t})) \geq \boldsymbol{\gamma}\left(\boldsymbol{c}\boldsymbol{\delta}(\boldsymbol{t})\right)^{\gamma-1}(\boldsymbol{\omega}^{\Delta}(\boldsymbol{\delta}(\boldsymbol{t})))^{\gamma}$$
$$\geq \boldsymbol{\gamma}\left(\boldsymbol{c}\boldsymbol{\delta}(\boldsymbol{t})\right)^{\gamma-1}\frac{\boldsymbol{\xi}\left(\boldsymbol{\sigma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)\right)}{\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}\left(\boldsymbol{\omega}^{\Delta}\left(\boldsymbol{\delta}(\boldsymbol{\sigma}(\boldsymbol{t}))\right)\right)^{\gamma}$$
$$\geq \boldsymbol{\gamma}\left(\boldsymbol{c}\boldsymbol{\delta}(\boldsymbol{t})\right)^{\gamma-1}\frac{\boldsymbol{\xi}\left(\boldsymbol{\sigma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)\right)\left(\boldsymbol{\omega}^{\Delta}\left(\boldsymbol{\sigma}(\boldsymbol{t})\right)\right)^{\gamma}}{\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}, \boldsymbol{t}_{\mathbf{3}} \geq \boldsymbol{c}\boldsymbol{t}_{\mathbf{2}}$$

$$\left(\boldsymbol{\omega}^{\gamma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)\right)^{\Delta} \geq \boldsymbol{\gamma}\left(\boldsymbol{c}\boldsymbol{\delta}(\boldsymbol{t})\right)^{\gamma-1}\frac{\boldsymbol{\xi}\left(\boldsymbol{\sigma}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)\right)\left(\boldsymbol{\omega}^{\Delta}\left(\boldsymbol{\sigma}(\boldsymbol{t})\right)\right)^{\gamma}}{\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}. \qquad (\mathbf{3.23})$$

Using (3.22) and (3.17), then (3.19) becomes

$$\boldsymbol{W}^{\Delta}(\boldsymbol{t}) \leq -\boldsymbol{Q}(\boldsymbol{t})\boldsymbol{\alpha}(\boldsymbol{t})+\frac{\boldsymbol{\alpha}^{\Delta}(\boldsymbol{t})\boldsymbol{W}(\boldsymbol{t})^{\sigma}}{\boldsymbol{\alpha}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}-\frac{\left(\boldsymbol{a}\boldsymbol{\delta}(\boldsymbol{t})\right)^{\gamma-1}\boldsymbol{\alpha}(\boldsymbol{t})(\boldsymbol{W}(\boldsymbol{t})^{\sigma})^{2}}{(\boldsymbol{\alpha}(\boldsymbol{t})^{\sigma})^{2}\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}. \qquad (\mathbf{3.24})$$

Using the fact that $\boldsymbol{x}-\boldsymbol{m}\boldsymbol{x}^{2} \leq \frac{1}{4m}$, the inequality (3.23) become

$$\boldsymbol{W}^{\Delta}(\boldsymbol{t}) \leq -\boldsymbol{Q}_{\mathbf{1}}(\boldsymbol{t})\boldsymbol{\alpha}(\boldsymbol{t})+\frac{\left(\boldsymbol{\alpha}^{\Delta}(\boldsymbol{t})\right)^{+}}{\boldsymbol{\alpha}(\boldsymbol{\sigma}(\boldsymbol{t}))}\left[\boldsymbol{W}(\boldsymbol{t})^{\sigma}-\frac{(\boldsymbol{a}\boldsymbol{\delta}(\boldsymbol{t}))^{\gamma-1}\boldsymbol{\alpha}(\boldsymbol{t})(\boldsymbol{W}(\boldsymbol{t})^{\sigma})^{2}}{\left(\left(\boldsymbol{\alpha}^{\Delta}(\boldsymbol{t})\right)^{+}\right)\boldsymbol{\alpha}^{\sigma}\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}\right]$$

$$\boldsymbol{W}^{\Delta}(\boldsymbol{t}) \leq -\boldsymbol{Q}_{\mathbf{1}}(\boldsymbol{t})\boldsymbol{\alpha}(\boldsymbol{t})+\frac{\left(\left(\boldsymbol{\alpha}^{\Delta}(\boldsymbol{t})\right)^{+}\right)\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}{\mathbf{4}\left(\boldsymbol{c}\boldsymbol{\delta}(\boldsymbol{t})\right)^{\gamma-1}\boldsymbol{\alpha}(\boldsymbol{t})}$$
$$=-\left[\boldsymbol{\gamma}\,\boldsymbol{Q}_{\mathbf{1}}(\boldsymbol{t})\boldsymbol{\alpha}(\boldsymbol{t})-\frac{\left(\left(\boldsymbol{\alpha}^{\Delta}(\boldsymbol{t})\right)^{+}\right)^{2}\boldsymbol{\xi}\left(\boldsymbol{\delta}(\boldsymbol{t})\right)}{\mathbf{4}(\boldsymbol{a}\boldsymbol{\delta}(\boldsymbol{t}))^{\gamma-1}\boldsymbol{\alpha}(\boldsymbol{t})}\right].$$

Then

$$W^{\Delta}(t) \leq -\left[\gamma\, Q_1(t)\alpha(t) - \frac{\left(\left(\alpha^{\Delta}(t)\right)^{+}\right)^2 \xi(\delta(t))}{4\left(c\delta(t)\right)^{\gamma-1}\alpha(t)}\right].$$

Integrating the last inequality from t_3 to t we obtain

$$-W(t_3) \leq W(t) - W(t_3) \leq -\int_{t_3}^{t}[\ Q_1(s)\alpha(s) - \frac{\left(\left(\alpha^{\Delta}(s)\right)^{+}\right)^2 \xi(\delta(s))}{4\left(c\delta(t)\right)^{\gamma-1}\alpha(s)}]\,\Delta s.$$

Hence

$$\int_{t_3}^{t}[Q_1(s)\alpha(s) - \frac{\left(\left(\alpha^{\Delta}(s)\right)^{+}\right)^2 \xi(\delta(s))}{4(c\delta(t))^{\gamma-1}\alpha(s)}]\,\Delta s \leq W(t_3)$$

for all large t, which is a contradiction to (3.16). Then $y(t)$ is oscillates solution of equation (3.1) on$[t_0, \infty)_{\mathbb{T}}$.

4. Examples

In this section, we give some examples which illustrate our main results

Example4.1Consider the following dynamic equation:

$$\left(\frac{1}{2t}((y(t) + e^{-2t}y(t+\frac{1}{5}))^{\Delta})^3\right)^{\Delta} + \sum_{i=1}^{n} K_i t\, y\left(t - \frac{1}{5i}\right) = 0, t \in \mathbb{T}. \quad (4.1)$$

Where $\mathbb{T} = \left[\frac{1}{2}, \infty\right)$is a time scale, $(t) = \frac{1}{2t}$,$p(t) = e^{-2t} < 1$,$\tau(t) = t + \frac{1}{5}$,

$$q_i(t) = K_i t\,, K_i \geq 2i \ \text{and}\delta_i(t) = t - \frac{1}{5i}, \qquad \delta(t) = \min_{t \geq t_1} \delta_i(t) = t - \frac{1}{5},$$

$i = 1, 2, \ldots, n, \gamma\ = 3$.Clearly, $\mathrm{H_1} - \mathrm{H_4}$are hold for every $t \geq \frac{1}{2}$. And

$$Q(t) = \min\left\{\sum_{i=1}^{n} q_i(t), \sum_{i=1}^{n} q_i\left(t + \frac{1}{5}\right)\right\} = \min\left\{\sum_{i=1}^{n} K_i t, \sum_{i=1}^{n} K_i\left(t + \frac{1}{5}\right)\right\} = \sum_{i=1}^{n} K_i t\,.$$

$$\int_{t_0}^{\infty} Q(s)\Delta s = \int_{\frac{1}{2}}^{\infty} \sum_{i=1}^{n} K_i s\,\Delta s = \sum_{i=1}^{n} K_i \int_{\frac{1}{2}}^{\infty} s\,\Delta s = \infty.$$

Then $(\mathrm{H_4})$ is holds for every $t \geq \frac{1}{2}$. Recall equation (3.5). There exists $\lambda > 0$, such that

$$v^{\gamma} + u^{\gamma} \geq \lambda(u + v)^{\gamma}, u, v \in R,\ \ uv > 0.$$

Let $u =\ y(t),\ v = e^{-2t}y(t + \frac{1}{5})$, Choose$\lambda = \frac{\lambda_a}{(1+a)^{\gamma}} > 0$then$\lambda = \frac{\varepsilon}{8} > 0\ , \varepsilon >\ 0$. By (3.6) $y^3(t) + (e^{-2t}y(t + \frac{1}{5}))^3\ \geq \frac{\varepsilon}{8}\ (y(t) + e^{-2t}y(t + \frac{1}{5}))^3$. Henceall condition of Theorem 3.2

is satisfiesed therefore, according toTheorem 3.2 every solution of equation (3.1) is oscillatory.

Example 4.2.Consider the following dynamic equation

$$\left(\left(\left(\boldsymbol{y}(\boldsymbol{t})+(\boldsymbol{e}^{-t}+\mathbf{1})\boldsymbol{y}\left(\boldsymbol{t}+\frac{\mathbf{1}}{\mathbf{2}}\right)\right)^{\Delta}\right)^{3}\right)^{\Delta}+\sum_{i=1}^{n}\frac{K_i}{2t-2}\,y\left(t-\frac{1}{2i}\right)=\mathbf{0},t\in\mathbb{T},\quad(\mathbf{4.2})$$

here $\mathbb{T}=[h,\infty),h>1$ is a time scale, $\xi(t)=1,\ \tau(t)=\ t+\frac{1}{2}\ ,\ p(t)=1-e^{-t}\leq 1-e^{-h}<1$ and $q_i(t)=\frac{K_i}{2t-2}\ ,K_i\geq 7^i,\delta_i(t)=t-\frac{1}{2i}\ ,\delta(t)=\min_{t\geq t_1}\delta_i(t)=t-\frac{1}{2}\ ,i=1,2,\ldots,n$. In this case $\gamma=3,\lambda=2\ ,n=2\ ,c=\ \frac{3}{4}$. Clearly $\mathbf{H_1}-\mathbf{H_3}$ are hold for every $t\geq h$.

$Q(t)=min\left\{\sum_{i=1}^{n}q_i(t)\,,\sum_{i=1}^{n}q_i\left(t+\frac{1}{2}\right)\right\}=\min\left\{\sum_{i=1}^{n}\frac{K_i}{2t-2},\sum_{i=1}^{n}\frac{K_i}{2t-1}\right\}=\sum_{i=1}^{n}\frac{K_i}{2t-1},$

then (3.8) becomes

$$\frac{1}{n\,\lambda c^{\gamma}}\,(1+a^{\gamma})=\frac{1}{2(2)(\frac{3}{4})^{3}}\left(1+(1)^{3}\right)=\frac{32}{27}=1.18.$$

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t}\frac{Q(s)\delta^{\gamma}(s)}{\xi(\delta(s))}\Delta s=\limsup_{t\to\infty}\int_{t-\frac{1}{2}}^{t}\sum_{i=1}^{2}\frac{\frac{K_i}{2(s-\frac{1}{2})}(s-\frac{1}{2})^{3}}{1}\Delta s$$

$$=\limsup_{t\to\infty}\sum_{i=1}^{n}\frac{K_i}{2}\int_{t-\frac{1}{2}}^{t}\left(s-\frac{1}{2}\right)^{2}\Delta s=\sum_{i=1}^{2}\frac{K_i}{2}\limsup_{t\to\infty}\int_{t-\frac{1}{2}}^{t}(s^{2}-s+\frac{1}{4})\,\Delta s$$

$$\geq\sum_{i=1}^{n}\frac{K_i}{2}\limsup_{t\to\infty}\int_{t-\frac{1}{2}}^{t}\frac{1}{4}\Delta s\geq\sum_{i=1}^{2}\frac{K_i}{8}\limsup_{t\to\infty}\int_{t-\frac{1}{2}}^{t}\Delta s=\sum_{i=1}^{2}\frac{K_i}{16}\geq\mathbf{1.18}.$$

Hence all condition 3.3 satisfies therefore, according toTheorem 3.3 every solution of equation $(\mathbf{3.1})$ is oscillatory

Example4.3.Considerthe following dynamic equation:

$$(\ t^{\alpha}((y(t)+\frac{t-\frac{3}{4}}{t}\ y(t+\ 1))^{\Delta})^{\frac{1}{3}})^{\Delta}+\sum_{i=1}^{n}k_i\left(\ t-\frac{1}{2}\right)^{\alpha}y\left(t-\frac{i}{2}\right)=0,\ \ t\in\mathbb{T}.\qquad(\mathbf{4.3})$$

Where the time scale $\mathbb{T}=[e,\infty),\ e\geq 1,\ \xi(t)=t,\ p(t)=\frac{t-\frac{3}{4}}{t}\ \leq 1,\tau(t)=t+1$

$q_i(t)=\ k_i\left(t-\frac{1}{2}\right)^{\alpha},\alpha>0,k_i\geq 2^{i},\ \delta_i(t)=t-\frac{i}{2},\delta(t)=\min_{t\geq e}\delta_i(t),\ i=1,2,\ldots,\mathrm{n}$

Clearly $\mathbf{H_1}-\mathbf{H_3}$are hold such that$\xi^{\Delta}(t)=t^{\alpha}$, for every $t\geq e$.

In this case$\gamma=3,=,\ c=\frac{3}{4}\ ,\alpha=\frac{2}{3}$ and

$$Q(t)=\min\left\{\sum_{i=1}^{n}q_i(t)\,,\sum_{i=1}^{n}q_i(t+1)\right\}=\min\left\{\sum_{i=1}^{n}k_i(t-\frac{1}{2})^{\alpha},\sum_{i=1}^{n}k_i\left(t+\frac{1}{2}\right)^{\alpha}\right\}$$

$$=\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{k_i}\left(\mathrm{t}-\frac{1}{2}\right)^{\alpha}$$

Then (3.12) becomes

$$\limsup_{t\to\infty}\int_{\delta(t)}^{t}\frac{Q(s)\left(1-\ p(\delta(s))\right)^{\gamma}}{\xi\left(\delta(s)\right)}\left(c\delta(s)\right)^{\gamma}\Delta s$$

$$= \limsup_{t\to\infty} \int_{t-\frac{1}{2}}^{t} \frac{\sum_{i=1}^{n} k_i \left(s-\frac{1}{2}\right)^{\frac{2}{3}} \left(\frac{\frac{3}{4}}{s-\frac{1}{2}}\right)^{\frac{1}{3}}}{\left(s-\frac{1}{2}\right)^{\frac{2}{3}}} \left(\frac{3}{4}\left(s-\frac{1}{2}\right)\right)^{\frac{1}{3}} \Delta s$$

$$= \limsup_{t\to\infty} \sum_{i=1}^{n} k_i \left(\frac{3}{4}\right)^{\frac{2}{3}} \int_{t-\frac{1}{2}}^{t} \Delta s = \sum_{i=1}^{n} k_i \left(\frac{3}{4}\right)^{\frac{2}{3}} \left(\frac{1}{2}\right) \geq \frac{1}{e}, k_i \geq 2^i, i = 1,2,\dots,n.$$

Therefore, according to Theorem 3.5 every solution of equation (3.1) is oscillatory.

Example4.4. Consider the following second-order neutral non-linear delay dynamic equation

$$((t-\frac{1}{2})^2((y(t)+\frac{t-\frac{3}{4}}{2t}y(t+\frac{1}{4}))^{\Delta})^3)^{\Delta} + \sum_{i=1}^{n} K_i t^{\beta} y\left(t-\frac{1}{4i}\right) = 0, t \in \mathbb{T}. \qquad (4.4)$$

Where $\mathbb{T} = [2,\infty)$is a time scale,$\xi(t) = (t-\frac{1}{2})^2$,$p(t) = \frac{t-\frac{3}{4}}{2t} \leq 1$,$\tau(t) = t+\frac{1}{2}$ $q_i(t) = K_i t^{\beta}, \beta > 0, K_i \geq 3i, \delta_i(t) = t-\frac{1}{4i}, \delta(t) = \min\delta_i(t) = t-\frac{1}{4}$, for$i = 1,2,\dots,n$. In this case $\gamma = 3$, $\lambda = 2$,$n = 2$, $c = \frac{1}{2}, \beta = \frac{2}{3}$. Clearly $\mathbf{H_1} - \mathbf{H_3}$ are hold for every $t \geq 1$, $\alpha(t) = t$ and $(\alpha^{\Delta}(s))^+ = \max\{0, \alpha^{\Delta}(s)\} = \{0,1\} = 1$,

$Q(t) = \min\left\{\sum_{i=1}^{n} q_i(t), \sum_{i=1}^{n} q_i\left(t-\frac{1}{4}\right)\right\} = \min\left\{\sum_{i=1}^{n} K_i t^{\frac{2}{3}}, \sum_{i=1}^{n} K_i (t-\frac{1}{4})^{\frac{2}{3}}\right\} = \sum_{i=1}^{n} K_i (t-\frac{1}{4})^{\frac{2}{3}}$. Then

$$Q_1(t) = Q(t)\left(1-p(\delta(t))\right)^{\gamma} = \sum_{i=1}^{n} K_i (t-\frac{1}{4})^{\frac{2}{3}} \left(\frac{t+\frac{1}{4}}{2(t-\frac{1}{4})}\right)^{\frac{1}{3}} = \frac{1}{\sqrt[3]{2}} \sum_{i=1}^{n} K_i (t-\frac{1}{4})^{\frac{1}{3}} \left(t+\frac{1}{4}\right)^{\frac{1}{3}}.$$

We will apply Theorem 3.6.

$$\limsup_{t\to\infty} \int_{t_0}^{t} [\alpha(s) Q_1(s) - \frac{\xi(\tau(t))((\alpha^{\Delta}(s))^+)^2}{4(c\delta(t))^{\gamma-1}\alpha(s)}] \Delta s =$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{\sqrt[3]{2}} \sum_{i=1}^{n} K_i (t-\frac{1}{4})^{\frac{2}{3}} \left(t+\frac{1}{4}\right)^{\frac{1}{3}} - \frac{\left(s-\frac{1}{4}\right)^2}{4\left(\frac{1}{2}\right)^2 \left(s-\frac{1}{4}\right)^2 s}) \Delta s =$$

$$\frac{1}{\sqrt[3]{2}} \sum_{i=1}^{n} K_i \limsup_{t\to\infty} \int_{t_0}^{t} (t^2 - \frac{1}{16})^{\frac{1}{3}} - \frac{1}{s}) \Delta s \leq \frac{1}{\sqrt[3]{2}} \sum_{i=1}^{n} K_i \limsup_{t\to\infty} \int_{t_0}^{t} (t^2 - \frac{1}{16})^{\frac{1}{3}}) \Delta s = \infty$$

$K_i \geq 3i$, $i = 1,2,\dots,n$.

Therefore, according to Theorem 3.6.Every solution of equation(3.1) is oscillatory.

5. Conclusions

Some conditions that guarantee the oscillation of all solutions of second order half linear neutral dynamic equation obtained. The equation studied in this paper is a generalization of the equation used in [2] and [8]. The results extracted and obtained are an improvement of the corresponding results in the two sources mentioned. Some illustrative examples of the obtained results are given.

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