DOI: 10.24996/ijs.2022.63.2.28

ISSN: 0067-2904

On Strong Dual Rickart Modules

Enas Mustafa Kamil

College of pharmacy, AL-Kitab University, Kirkuk, Iraq

Received: 14/9/2021 Accepted: 24/10/2021

Abstract

Gangyong Lee, S. Tariq Rizvi, and Cosmin S. Roman studied Dual Rickart modules. The main purpose of this paper is to define strong dual Rickart module. Let *M* and *N* be *R*- modules , *M* is called *N*- strong dual Rickart module (or relatively sd-Rickart to *N*) which is denoted by *M* it is *N*-sd- Rickart if for every submodule *A* of *M* and every homomorphism $f \in$Hom (M , N) , f (A) is a direct summand of *N*. We prove that for an *R*- module *M* , if *R* is *M*-sd- Rickart , then every cyclic submodule of *M* is a direct summand . In particular, if *M* is projective, then *M* is *Z*-regular. We give various characterizations and basic properties of this type of modules.

Keywords: Strong dual Rickart modules, direct summands, semisimple module.

حول المقاسات الريكارتية الرديفة القوية

ايناس مصطفى كامل

كلية الصيدلة, جامعة الكتاب , كركوك , العراق.

الخلاصة:

درس جانجيونج لي و طارق و رومان المقاسات الريكارتية الرديفة. الهدف الرئيسي من هذا البحث هو تعريف المقاسات الريكارتية الرديفة القوية. ليكن *M* و *N* مقاسات معرفة على الحلقة *R* يسمى *M* مقاس ريكارتي رديف قوي مرتبط مع *N* و يرمز له بالرمز *N*–sd– مقاس ريكارتي اذا كان لكل مقاس جزئي *A* من *M* و لكل دالة *f*∈Hom (*M,N*) فأن *f*(*A*) جداء مباشر من *N*. نبرهن ان كل مقاس جزئي دائري من *M* هو جداء مباشر من *M* حيث ان R تكون M–sd– ريكارتي قوي. و بصورة خاصة اذا كان *M* مقاسا اسقاطيا فأن *M* يكون مقاسا منتظما من النمط Z–. كما نعطي بعض المكافئات و الخواص الاساسية لهذا النوع من المقاسات.

1. Introduction.

A module *M* is called dual Rickart module if for every $\varphi \in$ End (M) , then $Im\varphi = eM$, for some e^2 = e. Equivalently a module *M* is dual Rickart module if and only if for every $\varphi \in$End (M) , then $Im\varphi$ is a direct summand of *M* , See [1] , [2] . Some generalizations of dual Rickart modules and related concepts are recently introduced in [8], [9] and [10]. A module *M* is *N*-d-Rickart (or relatively d-Rickart to *N*) if for every homomorphism φ: $M \to N$, $Im\varphi$ is a direct summand of *N* , where *N* is any *R*-module , see [1]. In this paper , we define strong dual Rickart modules, A module *M* is called *N*- strong dual Rickart module (or relatively sd-

*Email: enas.m.kamil@uoalkitab.edu.iq

Rickart to N) which is denoted by M it is N-sd- Rickart if for every submodule A of M and every homomorphism $f \in$Hom (M , N) , f (A) is a direct summand of N. We also give some results on this type of modules.

In section two, we give properties for the relatively strong dual Rickart modules. For example, let M and N be R- modules and let A be a submodule of M. If M is N-sd- Rickart , then A is N-sd- Rickart.

In section three, we give various characterizations of relatively strong dual Rickart module, we show that if N and M modules, then M is N-sd- Rickart if and only if every short exact sequence

$$0 \longrightarrow f(A) \xrightarrow{\ i\ } N \xrightarrow{\ \pi\ } \frac{N}{f(A)} \longrightarrow 0$$

Splits , for every submodu where i is the inclusion map and π is the natural epimorphism.

2. Strong dual Rickart.

In this section, we investigate and study the notion of relatively strong dual Rickart modules, and we obtain some of fundamental properties, several relations between sd- Rickart modules, and other classes of modules are obtained in this section.

Definition (2.1): Let M and N be R- modules , M is called N- strong dual Rickart module (or relatively sd-Rickart to N) which is denoted by M it is N-sd- Rickart if for every submodule A of M and every homomorphism $f \in$Hom (M , N) , f (A) is a direct summand of N , we call M is sd- Rickart if M is M-sd- Rickart.

Remarks and example (2.2):

(1) An R- module M is semisimple if and only if M is sd- Rickart , that is sd-Rickart is reflexive relation if and only if M is semismple. In general, M is semisimple if and only if A is M-sd- Rickar, for every submodule A of M.

(2) Every sd-Rickart module is d-Rickart , the converse is not true in general , for example, Q as Z- module is d-Rickart, by [1] which is not sd- Rickart, because it is not semiseple.

(3) Z_n is not Z-sd-Rickart Z- modules , because there is a homomorphism φ:$Z_n \rightarrow Z$ defined by $\varphi(Z_n) = nZ$, n>1, which is not a direct summand of Z.

(4) If N is a semisimple, then M is N-sd- Rickart, for every R- module M.

(5) Note that M is Z_6-sd- Rickart for every R- module M, by (4). The converse is not true in general for example, Z_6 is not relatively -sd- Rickart to Z_{12} as Z- module, hence sd- Rickart is not symmetric property.

(6) Let M and N be R- modules with Hom(M , N) = 0 , then M is N-sd- Rickart. For example Hom(Q , Z) = 0 implies Q is Z-sd- Rickart. Also , Hom(Z_n , Z) = 0 implies Z_n is Z-sd- Rickart.

(7) One can easily show that 0 is M-sd- Rickart and M is 0-sd- Rickart , for every R- module M.

Recall that an R- module M is said to be coquasi- Dedekind if for every proper submodule A of M , Hom(M,A) =0.

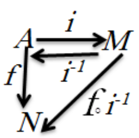
Equivalently, M is to be coquasi- Dedekind if every nonzero endomorphism of M is an epimorphism [3].

(8) If M is coquasi- Dedekind R- module, then M is A-sd- Rickart for every proper submodule A of M.

Now, we study the properties of the N- strong Dual Rickart modules.

Proposition (2.3): Let M and N be R- modules and let A be a submodule of M. If M is N-sd- Rickart , then A is N-sd- Rickart.

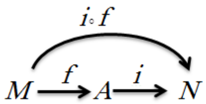
Proof: To show that A is N-sd- Rickart, let X be a submodule of A and let f : $A \rightarrow N$ be a homomorphism. Consider the diagram.



Where i is the inclusion map. Since M is N-sd- Rickart module , the $(f\circ i^{-1})(X) = f(X)$ is a direct summand of N. Thus A is N-sd- Rickart.

Proposition (2.4): Let M and N be R- modules and let A be a submodule of N. If M is N-sd-Rickart, then M is A-sd- Rickart.

Proof: Let X be a submodule of M and let $f : M\to A$ be a homomorphism, consider the following sequence.



Where i is the inclusion map. Since M is N-sd- Rickart , then $(i\circ f)(X) = f(X)$ is a direct summand of N. But $f(X)\leq A$, therefore $f(X)$ is a direct summand of A. Thus, M is A-sd- Rickart.

Proposition (2.5): Let M be an R- module and let N be any indecomposable R- module, if M is N-sd- Rickart , then either Hom $(M,N) = 0$ or every nonzero R-homomorphism $f : M\to N$ is an epimorphism.

Proof : Assume that M is N-sd- Rickart and Hom $(M,N) \neq 0$, let $f : M\to N$ be a nonzero homomorphism , then $f(M)$ is a direct summand of N. But N is an indecomposable therefore, f is an epimorphism.

Recall that an R- module M is called Z- regular if every cyclic (equivalently, every finitely generated) submodule of M is a projective and a direct summand of M, see [4].

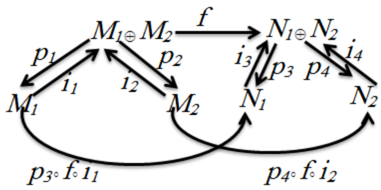
Proposition (2.6):Let M be an R- module such that R is M-sd-Rickart . Then every cyclic submodule of M is a direct summand . In particular , if M is priojective , then M is Z-regular.

Proof : Suppose that M is an R- module such that R is M-sd-Rickart and let $0\neq m \in M$. Define $f : R\to Rm$ by $f(r) = rm$, $r\in R$. Let i: $Rm\to M$ be the inclusion map. Consider the map $i\circ f : R\to M$. Clearly that $Im\ (i\circ f) = Rm$. Since R is M-sd- Rickart , then Rm is a direct summand of M. The last part is clear.

Recall that an R- module M is called distributive module if for all submodules A,B and C of M , $A\cap (B + C) = (A \cap B) + (A \cap C)$ for more details see [6].

Proposition (2.7): Let M_1 be N_1-sd- Rickart and M_2 be N_2-sd- Rickart. If $M_1\oplus M_2$ is distributive module, then $M_1\oplus M_2$ is $N_1\oplus N_2$- sd-Rickart module.

Proof: Let A be a submodule of $M_1\oplus M_2$ and let $f : M_1\oplus M_2\to N_1\oplus N_2$. Since $M_1\oplus M_2$ is distributive , then $A = (A\cap M_1)\oplus(A\cap M_2)$, we have to show that $f(A)$ is a direct summand of $N_1\oplus N_2$. Consider the following diagram.

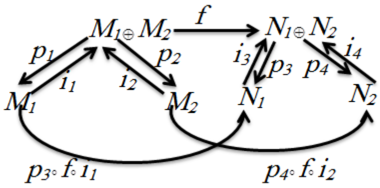


Where i_1, i_2, i_3 and i_4 are inclusion maps and p_1, p_2, p_3 and p_4 are projection maps. Since $A\cap M_1$ is a submodule of M_1 , $p_3\circ f\circ i_1 : M_1 \to N_1$ is R- homomrphiam and M_1 be N_1-sd- Rickart , then $(p_1\circ f\circ i_1)(A\cap M_1)$ is a direct summand of N_1. Similarly, $(p_4\circ f\circ i_2)(A\cap M_2)$ is a direct summand of N_2 , hence $(p_3\circ f\circ i_1)(A\cap M_1)\oplus(p_4\circ f\circ i_2)(A\cap M_2)$ is a direct summand of $N_1\oplus N_2$. Claim that $f(A) = (p_3\circ f\circ i_1)(A\cap M_1)\oplus(p_4\circ f\circ i_2)(A\cap M_2)$. To see this, let $a_1\in A\cap M_1$ and $a_2\in A\cap M_2$, then $(p_3\circ f\circ i_1)(a_1)+(p_4\circ f\circ i_2)(a_2) = (p_3\circ f)(a_1,0)+ (p_4\circ f)(a_2,0) = p_3(f(a_1), 0)+ p_4(0, f(a_2) = (f(a_1),0)+ (0, f(a_2))= (f(a_1), f(a_2))= f(a_1, a_2)$. Thus, $f(A)$ is a direct summand of $N_1\oplus N_2$.

Let M be an R- module. Recall that a submodule A of M is called a fully invariant if $f(A) \leq A$, for every $f \in End(M)$ and M is called duo module if every submodule of M is a fully invariant. See [7]

Proposition (2.8): Let M_1 be N_1-sd- Rickart and M_2 be N_2-sd- Rickart. If $M_1 \oplus M_2$ is duo module, then $M_1 \oplus M_2$ is $N_1 \oplus N_2$- sd-Rickart module.

Proof: Let A be a submodule of $M_1 \oplus M_2$ and let $f : M_1 \oplus M_2 \to N_1 \oplus N_2$. Since $M_1 \oplus M_2$ is duo module , then $A = (A \cap M_1) \oplus (A \cap M_2)$, we have to show that $f(A)$ is a direct summand of $N_1 \oplus N_2$. Consider the following diagram.



Where i_1, i_2, i_3 and i_4 are inclusion maps and p_1, p_2, p_3 and p_4 are projection maps. Since $A \cap M_1$ is a submodule of M_1 , $p_3 \circ f \circ i_1 : M_1 \to N_1$ is R- homomrphiam and M_1 be N_1-sd- Rickart , then $(p_1 \circ f \circ i_1)(A \cap M_1)$ is a direct summand of N_1. Similarly, $(p_4 \circ f \circ i_2)(A \cap M_2)$ is a direct summand of N_2 , hence $(p_3 \circ f \circ i_1)(A \cap M_1) \oplus (p_4 \circ f \circ i_2)(A \cap M_2)$ is a direct summand of $N_1 \oplus N_2$. One can easily show that $f(A)$ is a direct summand of $N_1 \oplus N_2$.

3. Characterizations of strong dual Rickart modules.

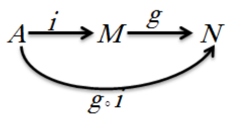
In this section, we give various characterizations of strong dual Rickart modules. We also obtain a characterization for an arbitrary direct sum of relatively sd-Rickart modules.

We start this section by the following proposition.

Proposition (3.1): Let M and N be R- modules , then M is N-sd- Rickart if and only if for every submodule A of M and every monomorphism f: $A \to M$ and homomorphism $g : M \to N$,$(g \circ f)(A)$ is a direct summand of N.

Proof: ($\Rightarrow$) Let f: $A \to M$ be a monomorphusm and let $g : M \to N$ be a homomorphism , where A is a submodule of M. Since $f(A)$ is a submodule of M and M is N-sd- Rickart, then $g(f(A)) = (g \circ f)(A)$ is a direct summand of N.

($\Leftarrow$) Let A be a submodule of M and let $g : M \to N$ be a homomorphism , we have to show that $g(A)$ is a direct summand of N. Consider the following diagram.



Where i is the in r assumption, $g(A)$ is a direct summand of N.

Proposition (3.2 R- modules , then M is N-sd- Rickart if and only if every short exact sequence

$$0 \longrightarrow f(A) \xrightarrow{i} N \xrightarrow{\pi} \frac{N}{f(A)} \longrightarrow 0$$

splits , for every subn where i is the inclusion map and π is the natural epimorphism.

Proof: Assume that M is N-sd- Rickart, it is clear that $f(A)$ is a direct summand of N , for each submodule A of M. Thus we get the result. By the same way, we can prove the converse.

Theorem (3.3): Let M and N be R- modules, the following statements are equivalent.

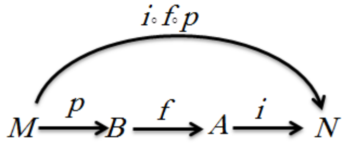
(1) M is N-sd- Rickart.

(2) For each direct summand B of M and a submodule A of N, B is A-sd- Rickart

(3) For each direct summand B of M, if L is a submodule of N with $\varphi \in$ Hom (M , L) , then $\varphi(X)$ is a direct summand of L , for every submodule X of B.

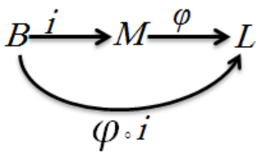
(4) For every submodule L of N, M is L-sd- Rickart.

Proof: (1)$\Rightarrow$(2) Suppose that M is N-sd- Rickart, let B be a direct summand of M and A be a submodule of N. To show that B is A-sd- Rickart , let X be a submodule of B and let $f : B \rightarrow A$ be an R- homomorphism. Consider the following sequence.



Where i is the inclusion map and p is the projection map. Since X is a submodule of M and M is N-sd- Rickart, then $(i \circ f \circ p)(X) = (f \circ p)(X) = f(X)$ is a direct summand of N, hence $f(X)$ is a direct summand of A. Thus , we get the result.

(2)$\Rightarrow$(3) Let B be a direct summand of M and let L be a submodule of N , we have to show that $\varphi(X)$ is a direct summand of L , for every submodule X of B and every $\varphi \in$ Hom (M , L). By (2) we get B is L-sd- Rickart. Consider the following sequence.



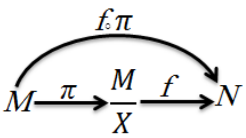
Where i is the inclusion map. Clearly that $\varphi(X)$ is a direct summand of L.

(3)$\Rightarrow$(4) Let L be a submodule of N and let $f : M \rightarrow L$ be a homomorphism , we have to prove that $f(X)$ is a direct summand of L , for every submodule X of M. Take $B = M$ and $\varphi = f$ and apply (3) , we get the result.

(4)$\Rightarrow$(1) Clear.

Proposition (3.4): Let M and N be R- modules , then M is N-sd- Rickart if and only if $\frac{M}{X}$ is N-sd- Rickart , for every submodule X of M.

Proof: ($\Rightarrow$) Suppose that M is N-sd-Rickart, let $\frac{A}{X}$ be a submodule of $\frac{M}{X}$ and let $f : \frac{M}{X} \rightarrow N$ be an R- homomorphism. Consider the following sequence.



Where π is the natural epimorphism. Since M is N-sd- Rickart , then $(f \circ \pi)(A)$ is a direct summand of N , so we get the result.

($\Leftarrow$) It is clear by taking $X = 0$.

Corollary (3.5): Let M and N be R- modules , then M is N-sd- Rickart if and only if $\frac{M}{A}$ is K-sd- Rickart, for each submodules A of M and K of N.

Proof: Let M be N-sd- Rickart and let A be a sunmodule of M , hence $\frac{M}{A}$ is N-sd- Rickart , be proposition (3.4), hence $\frac{M}{A}$ is K-sd- Rickart, for each submodule K of N, by proposition (2.4). For the converse , take $A = 0$ and $K = N$.

Theorem (3.6): The following statements are equivalent for a ring R.

(1) M is R-sd- Rickart, for every R- module M.

(2) M is R-sd- Rickart, for every free (projective) R- module M.

(3) R is R-sd- Rickart .

(4) R *is* semisimple ring.

Proof: Clear.

In the next result, we present conditions under which M_i is $\overset{n}{\underset{j=1}{\oplus}} M_j$- sd- Rickart.

Theorem (3.7): Let $\{Mi\}_{i=1}^{n}$ be a family of R- modules such that M_i is M_j- projective for all $i > j$. Then N is $\overset{n}{\underset{j=1}{\oplus}} M_j$ – sd- Rickart if and only if N is M_j -sd- Rickart , for any R- module N and all $j = 1, 2, \ldots, n$

Proof: The necessity it follows from theorem (3.3). Conversely, suppose that N is M_j -sd- Rickart for all $j = 1, 2, \ldots, n$ and M_i is M_j- projective for all $i > j$, $j = 1, 2, \ldots, n$. We will prove that, N is $\overset{n}{\underset{j=1}{\oplus}} M_j$ -sd- Rickart by induction on n.

Start with $n = 2$. Suppose that N is M_j -sd- Rickart for $j = 1, 2$ and M_2 is M_1 – projective. Let A be a submodule of N, and let $\varphi = (\pi_1 \circ \varphi, \pi_2 \circ \varphi)$ be any homomorphism from N to $M_1 \oplus M_2$, where π_j is the projection map from $M_1 \oplus M_2$ to M_j for $j = 1,2$. We want to prove that $\varphi(A)$ is a direct summand of $M_1 \oplus M_2$. Since $(\pi_2 \circ \varphi)(A)$ is a direct summand of M_2 as N is M_2 -sd- Rickart, then $(\pi_2 \circ \varphi)(A)$ is M_1 – projective. We have also $M_1 + \varphi(A) = M_1 \oplus (\pi_2 \circ \varphi)(A)$ is a direct summand of $M_1 \oplus M_2$, then there exists $L \leq \varphi(A)$ such that $M_1 + \varphi(A) = M_1 \oplus L$, by Lemma 4.47 in [5], we have $\varphi(A) = (M_1 \cap \varphi(A)) \oplus L$. In addition, $(\pi_1 \circ \varphi)(A)$ is a direct summand of M_1 because N is M_1 -sd- Rickart. Hence $\varphi(A) = M_1 \oplus L$ is a direct summand of $M_1 \oplus M_2$. Therefore , N is $M_1 \oplus M_2$-sd- Rickart .

Now assume that N is $\overset{n}{\underset{i=1}{\oplus}} M_j$ – sd- Rickart, we need to show that N is $\overset{n}{\underset{i=1}{\oplus}} M_j \oplus M_{n+1}$– sd- Rickart . Note that M_{n+1} is $\overset{n}{\underset{i=1}{\oplus}} M_j$- projective. Also, since N is $\overset{n}{\underset{i=1}{\oplus}} M_j$-sd-Rickart and N is M_{n+1}-sd-Rickart, so by similar arguments in the previous case for n = 2, N is $\overset{n+1}{\underset{i=1}{\oplus}} M_i$-sd-Rickart.

References

[1] G. Lee,S.T. Riziv and C. S. Roman, " Dual Rickart modules," *Comm in Algebra* , vol. 39, no. 11, pp. 4036-4058, 2011.

[2] J. L. Garcia, " Properties of direct summand of modules", *Comm. Algebra*, vol. 17, pp. 73-92, 1989.

[3] S.Yaseen , "Coquasi-Dedekind modules" , Phd. D. Thesis , University of Baghdad ,2003.

[4] A. G. Naoum and N.S.AL-Mothafar ," Anote on Z- regular modules", *Dirasat* , vol. 22, pp. 25-33, 1995.

[5] S. H. Mohamed, B. J.Muller, " *Continuous and Discrete modules", London Math. Soc. Lecture Note Ser No. 147, Cambridge University Press*, 1990.

[6] V. Erdogdu," Distributive Modules", *Can. Math. Bull*, pp. 248-254, 1987.

[7] C. Ozcan, A. Harmanci and P. F. Smith, "Duo Modules", *Glasgow Math. Journal Trust*, vol. 48, pp. 533-545, 2006.

[8] I. M. Ali , T. Younis, "On Closed Rickart Modules", *Iraqi Journal of Science*, vol. 57, no. 4, pp. 2746-2753 ,2017.

[9] B. H.Al-Bahrani, M. Q. Rahman, "On Y-Closed Dual Rickart Modules", *Iraqi Journal of Science*, vol. 62, no. 4, pp. 1226-1230, 2021.

[10] R.Tribak, "On Weak Dual Rickart Modules and Dual Baer Modules", *Comm in Algebra*, 2015.