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Unification of Generalized pre-regular closed Sets on Topological Spaces

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Abstract:
This paper intends to initiate a new type of generalized closed set in topological space with the theoretical application of generalized topological space. This newly defined set is a weaker form than the g_μ-closed set as well as $r\mu g$-closed set. Some phenomenal characterizations and results of newly defined sets are inculcated in a proper manner. The characteristics of normal spaces and regular spaces are achieved in the light of the generalized pre-regular closed set.

Keyword: $pr\mu g$-closed set. Pre-regular μ-$T_{1/2}$ space. $pr\mu g$-regular space. $pr\mu g$-normal space.

1. Introduction:

Csaszar [1] introduced the fundamental concept of generalized topology in the year 2002. Various researchers have been working in this field for the past few years [2, 3, 4, 5, 6]. It is observed from the literature that the generalized closed set which is initiated by Levine [7] is applied in the area of topological space as well as generalized topological space. Recent works on generalized closed sets may be found in [8, 9, 10, 11, 12]. Later on Palaniappan [13] defined a set that is weaker form than the generalized closed set and named it as regular generalized closed set in topological space. In the year 1997, Gnanambal [14] introduced another weaker concept of regular generalized closed set; it is called generalized pre-regular closed set. In this present treatise, our aim is to study the unification of generalized pre-regular closed set with the support of topological space and generalized topological space. In the present paper, our sole purpose is to bring forth the weaker form of $r\mu g$-closed set in generalized topological space using unification over it.

2. Preliminaries

Let X be a non-empty set and $expX$ be the power set of X. A class$\mu \subseteq expX$ is said to be generalized topology (briefly GT) [15] if $\phi \in \mu$ and the arbitrary union of element of μ belongs to μ. A set X with a GT μ on it is called a generalized topological space (briefly, GTS) and is denoted by (X,μ). For a GTS (X,μ), the elements of μ are called μ-open sets and the complement of μ-open sets are calledμ-closed sets. For $A \subseteq X$, $c_\mu(A)$ denotes the intersection of all μ-closed sets containing A, which is the smallest μ-closed set containing A; and $i_\mu(A)$ denotes the union of all μ-open sets contained in A and it is the largest μ-open set containedin A. In a topological space (X,T), if one takes $T = \mu$, then $i_\mu$becomes the usualinterior operator and others respectively.

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In the following, we present some preliminary ideas which will be played important role in the establishment of the results. Based on these concepts readers will easily understand the proofs of the newly introduced theorems.

2.1 Definition [7]

A set A is g-closed set iff$cl(A) \subseteq O$, whenever $A \subseteq O$ and O is an open set.

2.2 Definition [13]

A set A is regular generalized closed (r-g closed) set if and only if $cl(A) \subseteq U$,whenever $A \subseteq U$ and U is a regular open set.

2.3 Definition [14]

A subset A of a topological space (X, T) is called generalized pre-regular closed(briefly gpr-closed) set if $pclA \subset U$ whenever $A \subset U$ and U is regular open set in(X, T).

2.3 Definition

A subset A of a generalized topological space (X, μ) is called

i. [15] μ-preopen set if $A \subset i_\mu\left(c_\mu(A)\right)$.

ii. [15] μ-semiopen set if $A \subset c_\mu\left(i_\mu(A)\right)$.

iii. [2] μr-open set if $A = i_\mu\left(c_\mu(A)\right)$.

The complement of a μ-preopen (resp. μ-semiopen, μr-open) set is called μ-preclosed(resp. μ-semiclosed, μr-closed) set. The μ-pre interior of a subset A of a GTS (X, μ), denotes by $i_\pi(A)$, is defined by the union of all μ-preopen sets of (X, μ) contained in A and the μ-pre closure of a subset A of a GTS (X, μ), which is denoted by $c_\pi(A)$, isdefined by the intersection of all μ-preclosed sets of (X, μ) containing A.

2.5 Definition [16]

Let μ be GT on a topological space (X, T). Then, $A \subseteq X$ is called a generalized μ-closed (or simply $g\mu$-closed) set if $c_\mu(A) \subseteq U$, whenever $A \subseteq U$ and for an openset U in (X, T). The complement of a $g\mu$-closed set is called a generalized μ-open(or simply $g\mu$-open) set.

2.6 Definition [6]

Let μ be a GT on a topological space (X, T). Then, $A \subseteq X$ is called an regular μ-generalized closed (or simply an $r\mu g$-closed) set if $c_\mu(A) \subseteq U$, whenever$A \subseteq U \in RO(X)$. The complement of an $r\mu g$-closed set is called an $r\mu g$-open set.

2.7 Definition [4]

Let μ be a GT on a topological space (X, T). A subset A of X is called locallyμ-closed if $A = U \cap F$, where $U \in T$ and F is μ-closed set.

3. Some Properties of prg-Closed set

In this part, we invent the notion of $pr\mu g$-closed set and locally μ-pre-regular closed set. We established interrelationship between them. We have also shown the relationship of $pr\mu g$-closed set with some other sets. Furthermore we define pre regular μ-$T_{1/2}$ space and some related results.

3.1 Definition

Let μ be a GT on a topological space (X,T). Then $A \subset X$ is called pre-regular μ-ggeneralized closed (or simply $pr\mu g$-closed) set if $c_\pi(A) \subseteq U$, whenever $A \subseteq U$,for a regular open set U in (X,T).

3.2 Remark

Let μ be a GT on a topological space (X,T). Then every $pr\mu g$-closed set reduces to gpr-closed set if $\mu = T$.

3.3 Example

Let $X = \{a,b,c\}$and $T = \left\{X, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\right\}$. Then (X,T) is a topological space. Here $\mu = \left\{\phi, \{c\}, \{d\}, \{c,d\}\right\}$is a GT on X. Then regular open sets are $\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}$. If we consider $A = \{a\}$ and $U = \{a,b\}$, then $A \subset U$, where U is a regular open set in (X,T). Now $c_\pi(A) = \{a,b\} \subseteq U$. Hence, A is $pr\mu g$-closed set theirin.

3.4 Theorem

Every μ-closed set is a $pr\mu g$-closed set.

Proof: Let (X,μ) be a GTS on a topological space (X,T) and A be a μ-closed set. Then $c_\pi(A) = A$. Now let $A \subseteq U$ where U is a regular open set. Therefore $c_\mu(A) = A \subseteq U$. Since every μ-closed set is μ -preclosed set. Therefore, $c_\mu(A) \subseteq U$. Hence A is $pr\mu g$-closed set.

3.5 Remark

The following example shows that the converse of the Theorem 3.4 may not be true in general.

3.6 Example

Let $X = \{a,b,c,d\}$ and $T = \left\{X, \phi, \{a\}, \{b\}, \{a,b\}\right\}$. Then (X,T) is a topological space. If $\mu = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$then μ is a GT on X. Let $A = \{a,b\}$, then $A \subset X$, where X is a regular open set in (X,T). Now $c_\pi(A) = \{a,b\}$ is a $pr\mu g$-closed set, but $\{a,b\}$ is not a μ-closed set in (X,μ), since $c_\mu(A) \neq A$.

3.7 Theorem

Every μ-preclosed set is a $pr\mu g$-closed set.

Proof: Let μ be a GT on a topological space (X,T) and A be a μ-preclosed set. Then $c_\mu(A) = A$. Now let $A \subseteq U$, where U is a regular open set in (X,T). Then $c_\mu(A) = A \subseteq U$. Hence, A is $pr\mu g$-closed set.

3.8 Remark

The following example demonstrates that the converse of the above theorem is not true in general.

3.9 Example

Let $X = \{a,b,c,d\}$ and $T = \left\{X, \phi, \{a\}, \{b\}, \{a,b\}\right\}$. Then, (X,T) is a topological space. If $\mu = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ then μ is a GT on (X,T). Let $A = \{a,b\}$. Obviously, $A \subset X$ and X is a regular open set in (X,T). Now, $c_\pi(A) = \{a,b,d\} \subset X$. Hence,

$A = \{a, b\}$ is a $pr\mu g$-closed set, but $\{a, b\}$ is not a μ-closed set in (X, μ), since $c_\mu\left(i_\mu(A)\right) = \{a, b, d\} \not\subset A$.

3.10 Theorem
Every $g\mu$-closed set is a $pr\mu g$ -closed set.

Proof: Let μ be GT on a topological space (X, T) and A be $g\mu$-closed set. Now let $A \subseteq U$ where U is a regular open set in (X, T). Then U is also open set. So $c_\mu(A) \subseteq U$ as A is $g\mu$-closed set in (X, μ). Therefore, $c_\pi(A) \subset c_\mu(A) \subseteq U$ and hence A is a $pr\mu g$-closed set.

3.11 Remark
The next example is established to show that the Theorem (3.10) may not be true in general.

3.12 Example
Let $X = \{a, b, c, d\}$ and $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\mu = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. Then, (X, T) is a topological space and (X, μ) is a GT on X. It is easy to verify that $\{a, b\}$ is $pr\mu g$-closed set but not $g\mu$-closed set.

3.13 Theorem
Every $r\mu g$-closed set is a prg-closed set.

Proof: Let μ be a GT on a topological space (X, T) and A be an $r\mu g$-closed set. Now let $A \subseteq U$, where U is a regular open set in (X, T). Then $c_\pi(A) \subseteq U$ as A is an $r\mu g$-closed set. Therefore, $c_\pi(A) \subseteq c_\mu(A) \subseteq U$. Consequently, A is $pr\mu g$-closed set.

3.14 Remark
The following example shows the converse of Theorem (3.13) may not be true in general.

3.15 Example
Let$X = \{a, b, c, d\}$ and $T = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.Then (X, T) is a topological space. Here $\mu = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c, d\}, X\}$is a GT on (X, T). From this structure, we can verify that $A = \{a, b\}$ is $pr\mu g$-closed set but it is not an $rg\mu$-closed set.

3.16 Remark
Normally, $pr\mu g$-closed set is not closed under finite union and intersection which is verified in the following consecutive examples.

3.17 Example
Let$X = \{a, b, c, d, e\}$ and $T = \{X, \phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$. Then (X, T) is a topological space. Here $\mu = \{\phi, \{a, b\}, \{b, c, d, e\}, X\}$ is a GT on (X, T). It is easy to verify that $\{a\}, \{b\}$ are two $pr\mu g$-closed set in (X, T), but their union $\{a, b\}$ is not $pr\mu g$-closed set in (X, T).

3.18 Example
Let $X = \{a, b, c\}$ and $T = \{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. Then, (X, T) is a topologicalspace. Here $\mu = \{\varphi, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ is a GT on X. It is easy to verify that$\{a, b\}$ and $\{a, c\}$ are two $pr\mu g$-closed sets in (X, T), but their intersection $\{a\}$ is not $pr\mu g$-closed set in (X, T).

3.19 Theorem

Let μ be a GT on a topological space (X,T). If A is a $pr\mu g$-closed set then $c_\pi(A) - A$does not contain any non-empty regular closed set.

Proof: Let F be a regular closed subset of X such that $F \subseteq c_\pi(A) - A$. Then$F \subseteq c_\pi(A) \cap (X - A)$ and then $A \subset (X - F)$, where $(X - F)$ is a regular open set and A is an $pr\mu g$-closed set. Therefore, $c_\pi(A) \subset (X - F)$. Consequently, $F \subset \big(X - c_\pi(A)\big)$ .So $F \subset \big(X - c_\pi(A)\big) \cap c_\pi(A) = \varphi$. Thus $F = \varphi$. Therefore, $c_\pi(A) - A$ does notcontain any non-empty regular closed set.

3.20 Remark

The next example verifies that the reverse of Theorem (3.19) is not true in general.

3.21 Example

Consider that $X = \{a,b,c,d\}$ is equipped with the topology $T = \big\{X,\varphi,\{a\},\{b\},\{a,b\}\big\}$ and GT $\mu = \big\{\varphi,\{a\},\{b\},\{a,b\}\big\}$. The regular closed sets in T are $X,\varphi,\{a\},\{b\}$ only. Let $A = \{a\}$, then $c_\pi(A) - A = \{a,c,d\} - \{a\} = \{c,d\}$ does not contain anynon-empty regular closed set. A is not a $pr\mu g$-closed set because it does not satisfy the condition for the same.

This can be seen by finding the collection of regular open sets and applying μ-closure operator.

3.22 Theorem

Let μ be a GT on a topological space (X,T) and $A \subseteq B \subseteq c_\pi(A)$, where A is$pr\mu g$-closed set then B is also prμg-closed set.

Proof: Let $B \subseteq U$, where U is a regular open set in (X,T). Since A is a $pr\mu g$-closed set, then $A \subseteq U$ implies $c_\pi(A) \subseteq U$. Now since $B \subseteq c_\pi(A)$ then $c_\pi(B) \subseteq c_\pi\big(c_\pi(A)\big) = c_\pi(A) \subseteq U$. Hence, B is a $pr\mu g$-closed set.

3.23 Remark

If A and B are two $pr\mu g$-closed sets such that $A \subseteq B$, then B may not be a subsetof $c_\pi(A)$.

3.24 Example

Let $X = \{a,b,c,d\}$ and $T = \big\{X,\varphi,\{a\},\{b\},\{a,b\}\big\}$ then T is a topology on X.Let $\mu = \big\{\varphi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\big\}$, then (X,μ) is a GTS on(X,T). If we take $A = \{b\}$ and $B = \{b,c\}$ then $A \subset B$ and also A and B are both $pr\mu g$-closed set. But $c_\pi(A) = \{b,d\}$. Hence, $B \not\subset c_\pi(A)$.

3.25 Theorem

If A is a $pr\mu g$-closed subset of X and $A \subset B \subset c_\pi(A)$, then $c_\pi(B) - B$ contains nonon-empty regular closed set.

Proof: Let $A \subset B$. This implies that $c_\pi(A) \subset c_\pi(B)$. From the hypothesis, we have $B \subset c_\pi(A)$, which implies that $c_\pi(B) \subset c_\pi\big(c_\pi(A)\big)$, that is $c_\pi(B) \subset c_\pi(A)$.Then $c_\pi(A) = c_\pi(B)$. Also, $A \subset B$ implies $c_\pi(A) - A \supseteq c_\pi(B) - B$. Now A isa $pr\mu g$-closed set in (X,T). So, $c_\pi(A) - A$ contains no non-empty regular closedsubsets. Hence, $c_\pi(B) - B$ also does not contain any non-empty regular closedsubset.

3.26 Theorem

Let A be a subset of a topological space (X,T) with a GT μ on it. Then, A is a $pr\mu g$-closed set iff $rcl(\{x\}) \cap A \neq \varphi$, for every $x \in c_\pi(A)$.

Proof: Let A be a $pr\mu g$-closed set in X. Then, there exist a $x \in c_\pi(A)$ suchthat $rcl(\{x\}) \cap A = \varphi$, that is $A \subset X - rcl(\{x\})$ and $X - rcl(\{x\})$ is regularopen set in (X,T). Therefore, $c_\pi(A) \subset X - rcl(\{x\})$. Hence, $x \notin c_\pi(A)$, which is acontradiction. Hence, the result is established.

Conversely, let $rcl(\{x\}) \cap A \neq \varphi$, for every $x \in c_\pi(A)$ and U be any regular openset such that $A \subseteq U$. As $rcl(\{x\}) \cap A \neq \varphi$,so there exists$y \in rcl(\{x\}) \cap A$ and $y \in A \subseteq U$. Therefore, $\{x\} \cap U \neq \varphi$. Hence, $x \in U$n which implies $c_\pi(A) \subseteq U$.Therefore, A is a $pr\mu g$-closed set.

3.27 Theorem

Let (X,T) be a topological space and μ be a GT on X. Then, A is a $pr\mu g$-open set if and only if $F \subseteq i_\pi(A)$, whenever $F \subseteq A$ and F is a regular closed set in (X,T).

Proof: Let A be a $pr\mu g$-open set and $F \subseteq A$, where F is a regular closed set in (X,T).Then, $X - A$ is a$pr\mu g$-closed set containing $X - F$, where $X - F$ is a regular open se tin (X,T). Hence, $c_\pi(X - A) \subseteq X - F$, i.e, $X - i_\pi(A) \subseteq X - F$ and accordingly$F \subseteq i_\pi(A)$.
Conversely, Let $F \subseteq i_\pi(A)$, whenever $F \subseteq A$ for any regular closed set F. Let$X - A \subseteq U$, where U is a regular open set in (X,T). Then$X - U \subseteq A$ and $X - U$ isa regular closed set. Hence, by assumption $X - U \subseteq c_\pi(A)$, this implies $X - i_\pi(A) \subseteq U$.
Now $c_\pi(X - A) = X - i_\pi(A) \subseteq U$. Hence, $X - A$ is a $pr\mu g$-closed set. So A is a a$pr\mu g$-open set.

3.28 Theorem

Let μ be a GT on a topological space (X,T). If A is a $pr\mu g$-open set and $i_\pi(A) \subseteq B \subseteq A$, then B is a $pr\mu g$-open set.

Proof: Let A be a$pr\mu g$-open set and $i_\pi(A) \subseteq B \subseteq A$. i.e., $X - A \subseteq X - B \subseteq c_\pi(X - A)$ and $X - A$ is a prμg-closed set. Then, by theorem 3.22 $X - B$ is $pr\mu g$-closed set. Hence, B is a $pr\mu g$-open set.

3.29 Remark

Every regular open set in (X,T) needs not to be a $pr\mu g$-closed set which is confirmed by afterward example.

3.30 Example

Let $X = \{a,b,c,d\}$ with a topology $T = \{X, \varphi, \{a\}, \{b\}, \{a,b\}\}$ and the GT, $\mu = \{\varphi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. It is easy to verify that $\{a\}$ isa regular open set, but not a $pr\mu g$-closed set of X.

3.31 Theorem

Let μ be a GT on a topological space (X,T) such that $\mu \subseteq T$, then in a μ-extremally disconnected space every μ-regular open set is a $pr\mu g$-closed set.

Proof: Let A be a μ-regular open set. Then A is a regular open in (X,T). Sincein μ-extremally disconnected space, every μ-regular open set is a μ-closed set. So $c_\mu(A) = A$. Now $A \subseteq A$, where A is regular open in (X,T) implies $c_{\pi}(A) \subset c_{\mu}(A) = A$. Hence, A is a $pr\mu g$-closed set of X.

3.32 Definition

Let μ be a GT on a topological space (X,T). A subset A of X is a locally μ-preregularclosed set if $A = U \cap F$, where U is a regular open set in (X,T) and F is μ-preclosedset.

3.33 Proposition

Every μ-preclosed set is locally μ-preregular closed set.

Proof: Proof is obvious from the Definition 3.32.

3.34 Theorem

Let μ be a GT on a topological space (X,T). Then, A is μ-preclosed set if and onlyif it is $pr\mu g$-closed set and locally μ-preregular closed set.

Proof: Suppose that A is μ-preclosed set. Then by Theorem 3.7 it is $pr\mu g$-closed set and by the Proposition 3.32 it is also a locally μ-preregular closed set.

Conversely, let A be $pr\mu g$-closed set and also locally μ-preregular closed set. Then$A = U \cap F$, where U is a regular open set in (X,T) and F is μ-preclosed set. So we have $A \subseteq U$and $A \subseteq F$. Then, $c_{\pi}(A) \subseteq U$. Also, $c_{\pi}(A) \subseteq c_{\pi}(F) = F$, as F isμ-preclosed set. Therefore, $c_{\pi}(A) \subseteq U \cap F = A$. Hence, A is μ-preclosed set.

3.35 Theorem

Let μ be a GT on a topological space (X,T) such that $\mu \subseteq T$. Then every μ-nowhere dense set is a $pr\mu g$-closed set.

Proof: Let A be any μ-nowhere dense set in (X,μ), i.e, $i_{\mu}\left(c_{\mu}(A)\right) = \varphi$. Thatmeans there does not exist any μ-open set in between A and $c_{\mu}(A)$. Let U beany μ-regular open set, then U is a regular open set in (X,T). Let $A \subseteq U$ then$c_{\mu}(A) \subseteq U$. In other words, $c_{\mu}(A) \subseteq c_{\pi}(A) = U$. Hence, A is a $pr\mu g$-closed set.

3.36 Remark

The converse of the Theorem (3.35) is always not to be true. This is shown in the next example.

3.37 Example

Let $X = \{a,b,c,d\}$, $T = \{X,\varphi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$, then$(X,T)$ is a topological space. Let $\mu = \{\varphi,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$be the GT on (X,T). If we take $A = \{a,b\}$ then A is $pr\mu g$-closed set (since, $c_{\pi}(A) \subset U$, for all $A \subset U, U$ being a regular open set. But, $i_{\mu}(c_{\mu}(A)) = \{a,b\} \neq \phi$. Hence, A is not μ-nowhere dense set.

3.38 Theorem

In a hyperconnected space (X,T), every subset of X is a $pr\mu g$-closed set.

Proof: Let (X,T) be a hyperconnected space. Let A be any subset of X. We knowthat in hyperconnected space X and ϕ are the only regular open set in (X,T).Let $A \subseteq X$. Now $c_{\pi}(A) \subseteq c_{\pi}(X) = X$. Since A is an arbitrary set. Hence in ahyperconnected space (X,T), every subset of X is $pr\mu g$-closed set.

3.39 Theorem

Let μ be a GT on a topological space (X, T). If A is a regular open set as well as $pr\mu g$-closed set of X then A is μ-preclosed set.

Proof: Let A is a regular open subset as well as $pr\mu g$-closed subset of X. Therefore, $A \subseteq A$ implies $c_\pi(A) \subseteq A$. Also, $A \subseteq c_\pi(A)$. So, $c_\pi(A) = A$. Hence, A isμ-preclosed set.

3.40 Definition

A space (X, μ) is said to be a pre-regular $\mu - T_{1/2}$ space if every pre-regular μ-generalized closed set is a μ-preclosed set.

3.41 Theorem

Let μ be a GT on a topological space (X, T). Then the following conditions areequivalent:

(i) X is pre-regular μ-$T_{1/2}$.

(ii) Every singleton set of X is either regular closed set or μ-preopen set.

Proof: (i)⇒(ii): Assume that $x \in X$ and $\{x\}$ is not a regular closed set. Then$X - \{x\}$ is not regular open set and $X - \{x\}$ is trivially pre-regular μ-generalizedclosed set. Then by (i) it is μ-preclosed set. So that $\{x\}$ is μ-preopen set.

(ii)⇒(i): Let $A \subset X$ is pre-regular μ-generalized closed set. Let $x \in c_\pi(A)$. It is enough to show that $x \in A$. For these two cases may arise:

Case (1) : The set $\{x\}$ is a regular closed set. Then if $x \notin A$, then $c_\pi(A) - A$ containa regular closed set, which is a contradiction of Theorem 3.19, so that $x \in A$.

Case (2): The set $\{x\}$ is μ-preopen set. Since $x \in c_\pi(A)$, then $\{x\} \cap c_\pi(A) \neq \varphi$. Thus, $x \in A$.

So, in both the cases, $x \in A$, which implies that $c_\pi(A) \subset A$. Also, $A \subset c_\pi(A)$ isalways true, so A is a μ-preclosed set. Hence, X is pre-regular μ-$T_{1/2}$.

3.42 Theorem

Let μ be a GT on a topological space (X, T). Then the following conditions areequivalent:

(i) X is pre-regular μ-$T_{1.2}$.

(ii) Every μ-nowhere dense singleton set of X is regular closed set.

Proof: (i)⇒(ii): By Theorem 3.41, every singleton set of X is either regular closedset or μ-pre-open set. Since non-empty μ-nowhere dense sets cannot be μ-pre-openset at the same time, then every μ-nowhere dense singleton set of X is regularclosed set.

(ii)⇒(i): Since every singleton set is either μ-pre-open set or μ-nowhere dense set,then by (ii) every μ-nowhere dense singleton set of X is either μ-preopen set orregular closed set. Hence, by Theorem 3.40, X is a pre-regular μ-$T_{1/2}$ space.

4. Properties of $pr\mu g$-regular and $pr\mu g$-normal spaces

In this specific section, we introduce $pr\mu g$ -regular and $pr\mu g$ -normal spaces andcharacterize these two spaces. Also, we establish the relationship of these spacewith $pr\mu g$-open set.

4.1 Definition

Let (X, T) be a topological space and μ be a GT on X. Then (X, T) is said tobe $pr\mu g$-regular space if for each regular closed set F of X not containing x thereexist disjoint μ-preopen sets U and V such that $x \in U, F \subseteq V$.

4.2 Theorem

Let μ be a GT on a topological space (X,T). Then the following statements areequivalent:

(i) X is $pr\mu g$-regular space.

(ii) For each $x \in X$ and each regular open set U in (X,T) with $x \in U$, there existsμ-preopen set V such that $x \in V \subseteq c_{\pi}(V \ \subseteq U$.

(iii) For each regular closed set F of $X \cap \{c_{\pi}(V) : F \subseteq V, V$ is μ-preopen set$\} = F$.

(iv) For each $A \subseteq X$ and each regular open set U with $A \cap U \neq \varphi$, there exists μ-preopen set V such that $A \cap V \neq \varphi$ and $c_{\pi}(V) \subseteq U$.

(v) For each non-empty subset A of X and each regular closed subset F of X with$A \cap F = \varphi$, there exist μ-preopen sets V, W such that $A \cap V \neq \varphi, F \subseteq W$ and$W \cap V = \varphi$.

(vi) For each regular closed set F and $x \notin F$, there exists μ-preopen set U and a$pr\mu g$-open set V such that $x \in U, F \subseteq V$ and $U \cap V = \varphi$.

(vii) For each $A \subseteq X$ and each regular closed set F with $A \cap F = \varphi$, there exists μ-preopen set U and a $pr\mu g$-open set V such that $A \cap U \neq \varphi$, $F \subseteq V$ and$U \cap V = \varphi$.

Proof: (i)⇒(ii): Let $x \notin X - U$, where U is a regular open set in (X,T). Then there exists two disjoint μ-preopen sets G, V such that $X - U \subseteq G$ and $\in V$. Now as$G \cap U = \varphi$ then $V \subseteq X - G$. Therefore, $x \in V \subseteq c_{\pi}(V) \subseteq X - G \subseteq U$.

(ii)⇒(iii): Let $X - F$ is a regular open set with $x \in X - F$. Then by (ii), there exists μ-preopen set U such that $x \in U \subseteq c_{\pi}(U) \subseteq (X - F)$. So $F \subseteq X - c_{\pi}(U) = V$ which is μ-preopen set and $U \cap V = \varphi$ the $x \in c_{\pi}(V)$. Thus, $\cap \{c_{\pi}(V): F \subseteq V, V$ is μ-preopen set$\} \subseteq F$. Therefore, $\cap \{c_{\pi}(V): F \subseteq V, V$ is μ-preopen set$\} = F$.

(iii)⇒(iv): Let A be a subset of X such that U is a regular open set with $A \cap U \notin \varphi$. Let $x \in A \cap U$. Then $x \in (X - U)$. Hence, by (iii), there exists μ-preopen set Wsuch that $X - U \subseteq W$ and $x \notin c_{\pi}(W)$. Let $V = X - c_{\pi}(W)$ then $x \in V$ and V isμ-preopen set. Hence, $A \cap V \neq \varphi$. Now $V \subseteq X - W$ and $c_{\pi}(V) \subseteq X - W \subseteq U$.

(iv)⇒(v): Let F be a regular closed subset of X with $A \cap F = \varphi$. Then $X - F$ is regular a open set with $A \cap (X - F) \neq \varphi$ and hence by (iv), there exists μ-preopenset V such that $A \cap V \neq \varphi$ and $c_{\pi}(V) \subseteq X - F$. If we put $W = X - c_{\pi}(V)$, then$F \subseteq W$ and $W \cap V = \varphi$, where W is μ-preopen set.

(v)⇒(i): Let F be a regular closed set containing x. Then $F \cap \{x\} = \varphi$. Thus, by (v) there exist μ-preopen sets V, W such that $x \in V, F \subseteq W$ and $W \cap V = \varphi$.

(i)⇒(vi): By the definition of $pr\mu g$-regular space, one can easily verify this.

(vi)⇒(vii): Let $A \subseteq X$ and F be a regular closed set such that $A \cap F = \varphi$. Thenfor $a \in A$, $a \notin F$, hence by (vi), there exists μ-preopen set U and a $pr\mu g$-openset V such that $a \in U$, $F \subseteq V$ and $U \cap V = \varphi$. Therefore, $A \cap U \neq \varphi, F \subseteq V$ and$U \cap V = \varphi$.

(vii)⇒(i): Let $x \notin F$, where F is a regular closed set in (X,T). Since $\{x\} \cap F = \varphi$,by (vii), there exist μ-preopen set U and a $pr\mu g$-open set W such that $x \in U$,$F \subseteq W$ and $U \cap W = \varphi$. Then $\subseteq i\pi(W) = V$, by using Theorem 3.27. Hence, $V \cap U = \varphi$, where W is μ-preopen set.

4.3 Definition

Let (X,T) be a topological space and μ be a GT on X. Then (X,T) is said to be$pr\mu g$-normal space if for any two disjoint regular closed sets A and B there existtwo disjoint μ-preopen sets U and V such that $A \subseteq U$ and $B \subseteq V$.

4.4 Theorem

Let μ be a GT on a topological space (X,T). Then the following statements areequivalent:

(i) X is $pr\mu g$-normal space.
(ii) For any pair of disjoint regular closed sets A and B of X, there exists disjoint $pr\mu g$-open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.
(iii) For each regular closed set A and each regular open set B containing A, there exists a $pr\mu g$-open set U such that $A \subseteq U \subseteq c_\pi(U) \subseteq B$.
(iv) For each regular closed set A and each regular-open set B containing A, thereexist a μ-preopen set U such that $A \subseteq U \subseteq c_\pi(U) \subseteq B$.

Proof: (i)⇒(ii): Let A and B be two disjoint regular closed sets of X. Then by(i) there exist disjoint μ-preopen sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.Since every μ-pre-open sets are $pr\mu g$-open sets, then U and V are two disjoint$pr\mu g$-open sets X such that $A \subseteq U$ and $B \subseteq V$.
(ii)⇒(iii): Let A be a regular closed set and B be a regular open set containing$A$. Then A and $X - B$ are two disjoint regular closed sets. Therefore, by (ii) there exist two disjoint $pr\mu g$-open sets U and V of X such that $A \subseteq U$ and $X - B \subseteq V$.Since V is $pr\mu g$-open set and $X - B$ is a regular closed set with $X - B \subseteq V$, then by Theorem 3.27 $X - B \subseteq i_\pi(V)$. Hence, $c_\pi(X - V) = X - i_\pi(V) \subseteq B$. Thus, $A \subseteq U \subseteq c_\pi(U) \subseteq c_\pi(X - V) \subseteq B$.
(iii)⇒(i): Let A and B be two disjoint regular closed subsets of X. Then $X - B$ isregular open set containing regular closed set A. Then by (iii) there exist a $pr\mu g$-open set U such that $A \subseteq U \subseteq c_\pi(U) \subseteq X - B$. Thus $A \subseteq i_\pi(U), B \subseteq X - c_\pi(U)$and also $i_\pi(U)$ and $X - c_\pi(U)$ are two disjoint μ-preopen sets.
(iii)⇒(iv): Let A be a regular closed set and B be a regular-open set containing$A$. Then by (iii) there exists a $pr\mu g$-open set V such that $A \subseteq V \subseteq c_\pi(V) \subseteq B$.Since A is a regular closed set, then $A \subseteq i_\pi(V) = U$, by Theorem 3.27. Then, U is μ-preopen set and $A \subseteq U \subseteq c_\pi(U) \subseteq c_\pi(V) \subseteq B$
(iv)⇒(i): Let A and B be two disjoint regular closed set. Then $X - B$ is regularopen set and $A \subseteq X - B$. Then by (iv) there exist μ-preopen set U of X such that$A \subseteq U \subseteq c_\pi(U) \subseteq X - B$. Put $G = X - c_\pi(U)$, then $B \subseteq X - c_\pi(U)$. Therefore, V and G are two disjoint μ-preopen sets such that $B \subseteq G$ and $A \subseteq B$.

5. Conclusion:

In this article, we initiated a new type of generalized closed set in topological space with the application of generalized topological space. Some phenomenal characterizations and results of newly defined sets are inculcated in a proper manner. The characteristics of normal spaces and regular spaces are achieved in the light of a generalized pre-regular closed set. This research work can be extended through the operation approach on this same structure and other topological environments too.

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