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Pairwise Regularity and Normality Separation Axioms in Čech Fuzzy Soft Bi-Closure Spaces

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Abstract

In this paper, some new types of regularity axioms, namely pairwise quasi-regular, pairwise semi-regular, pairwise pseudo regular and pairwise regular are defined and studied in both Čech fuzzy soft bi-closure spaces (Čfs bicsp's) and their induced fuzzy soft bitopological spaces. We also study the relationships between them. We show that in all these types of axioms, the hereditary property is satisfied under closed Čfs bi-csubsp of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. Furthermore, we define some normality axioms, namely pairwise semi-normal, pairwise pseudo normal, pairwise normal and pairwise completely normal in both Čfs bicsp's and their induced fuzzy soft bitopological spaces, as well as their basic properties and the relationships between them are studied.

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Keywords: Fuzzy soft set, pairwise quasi-regular, pairwise semi-regular, pairwise pseudo regular, pairwise regular, pairwise semi-normal, pairwise pseudo normal, pairwise normal, and pairwise completely normal.

بديهيات الفصل المنتظمة والطبيعية الثنائية في فضاءات الاغلاق الثنائية الضبابية الناعمة تشيك

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الخلاصة

في هذا البحث، تم تعريف ودراسة بعض انواع بديهيات الانتظام الثنائية وهي، شبه المنتظمة الثنائية، نصف المنتظمة الثنائية،الزائفة المنتظمة الثنائية و المنتظمة الثنائية في كلا من فضاءات الاغلاق الثنائية الضبابية الناعمة تشيك والفضاءات الضبابية الناعمة ثنائية التبولوجي المشتقة منها و دراسة العلاقة فيما بينهم. علاوة على ذلك، قمنا بتعريف ودراسة انواع من بديهيات الفصل الطبيعية الثنائية وهي، شبه الطبيعية الثنائية، الطبيعية الثنائية و الطبيعية تماما الثنائية في كلا من فضاءات الاغلاق الثنائية الضبابية الناعمة تشيك والفضاءات الضبابية الناعمة ثنائية التبولوجي المشتقة منها وكذلك تمت دراسة العلاقة فيما بينهم.

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1. Introduction

The concept of a Fuzzy set is introduced by Zadeh [1]. Moldtsov [2] introduced the basic notions of the theory of soft sets and he presented the first results of the theory. The concept of fuzzy set and soft set are combined to establish a new concept named fuzzy soft set [3]. Tanay and Kandemir [4] introduced a concept of a topological structure based on fuzzy soft sets.

$\check{C}$ech [5] introduced the concept of $\check{C}$ech closure spaces. Mashhour and Ghanim [6] introduced a new concept of $\check{C}$ech fuzzy closure space. They replaced sets with fuzzy sets in the description of $\check{C}$ech closure space. Rao and Gowri [7] introduced the concept of biclosure space$(\mathcal{M}, \gamma_1, \gamma_2)$. Such space is equipped with two arbitrary $\check{C}$ech closure operators γ_1 and γ_2. Tapi and Navalakhe [8] introduced later the concept of fuzzy biclosure spaces. After the concept of soft theory appeared by Moldtsov [2], many authors used the principle of soft sets to introduce the concept of soft Čech closure spaces [9,10]. However, Gowri and Jegadeesan [11] introduced the concept of soft biČech closure spaces.

Majeed [12] recently established the definition of Čech fuzzy soft closure spaces, which were motivated by the concept of fuzzy soft set and fuzzy soft topology in Chang's sense [13]. Majeed and Maibed also studied the architecture of Čech fuzzy soft closure spaces including separation axioms and connectedness [14, 15, 16, 17]. As a generalization to $\check{C}$ech fuzzy soft closure space [12], the concept of $\check{C}$ech fuzzy soft bi-closure spaces ($\check{C}$fs bicsp's) is recently presented in [18] and some additional properties have been studied of $\check{C}$fs bicsp's in [19].

In the current work, some new kinds of pairwise regularity and normality in $\check{C}$fs bicsp's are introduced and studied. In section 3, regularity axioms are defined, namely pairwise quasi-regular, pairwise semi-regular, pairwise pseudo regular, and pairwise regular in both $\check{C}$fs bicsp's and their induced fuzzy soft bitopological spaces. The relationships between them are also studied. We show that in all these types of axioms, the hereditary property is satisfied under closed Čfs bi-csubsp of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. Finally, in Section 4, some normality axioms are introduced, namely pairwise semi-normal, pairwise pseudo normal, pairwise normal, and pairwise completely normal in both $\check{C}$fs bicsp's and their induced fuzzy soft bitopological spaces. The relationships between them are studied, and its basic properties are also discussed as in the previous section.

2. PRELIMINARIES

In this paper, the universe set is denoted by $\mathcal{M}$, the unit interval [0,1]is denoted by I and $\mathrm{I}_0 = (0,1]$, the set of parameters for $\mathcal{M}$ is represented by $\mathcal{D}$ and $\mathcal{S}$ will be an empty subset of $\mathcal{D}$. If $\mathcal{A}$ is a mapping from $\mathcal{M}$ into I, it is named a fuzzy set of $\mathcal{M}$ [1]. $I^{\mathcal{M}}$ stands for the family of all fuzzy sets of $\mathcal{M}$.

Definition 2.1 [20] A fuzzy soft set (fss) $\mathcal{A}_\mathcal{S}$ on the universe set $\mathcal{M}$ is a mapping from $\mathcal{D}$ to $I^{\mathcal{M}}$, that means $\mathcal{A}_\mathcal{S}: \mathcal{D} \to I^{\mathcal{M}}$, where $\mathcal{A}_\mathcal{S}(d) \neq \overline{0}$ if $d \in \mathcal{S} \subseteq \mathcal{D}$ and $\mathcal{A}_\mathcal{S}(d) = \overline{0}$ if $d \notin \mathcal{S}$, where $\overline{0}$ is the empty fuzzy set on $\mathcal{M}$. The family of all $fss's$ over $\mathcal{M}$ is denoted by $FS(\mathcal{M}, \mathcal{D})$.

Definition 2.2 [21] Let $\mathcal{A}_\mathcal{S}, \mu_\mathcal{B} \in FS(\mathcal{M}, \mathcal{D})$ so that we have the following:

1. $\mathcal{A}_\mathcal{S} \sqsubseteq \mu_\mathcal{B}$ iff $\mathcal{A}_\mathcal{S}(d) \leq \mu_\mathcal{B}(d)$, for all $d \in \mathcal{D}$.
2. $\mathcal{A}_\mathcal{S} = \mu_\mathcal{B}$ iff $\mathcal{A}_\mathcal{S} \sqsubseteq \mu_\mathcal{B}$ and $\mu_\mathcal{B} \sqsubseteq \mathcal{A}_\mathcal{S}$.
3. $\rho_{\mathcal{S} \cup \mathcal{B}} = \mathcal{A}_\mathcal{S} \sqcup \mu_\mathcal{B}$ iff $\rho_{\mathcal{S} \cup \mathcal{B}}(d) = \mathcal{A}_\mathcal{S}(d) \vee \mu_\mathcal{B}(d)$, for all $d \in \mathcal{D}$.
4. $\rho_{\mathcal{S} \cap \mathcal{B}} = \mathcal{A}_\mathcal{S} \sqcap \mu_\mathcal{B}$ iff $\rho_{\mathcal{S} \cap \mathcal{B}}(d) = \mathcal{A}_\mathcal{S}(d) \wedge \mu_\mathcal{B}(d)$, for all $d \in \mathcal{D}$.

5. The complement of $\mathcal{A}_{\mathcal{S}}$ is denoted by $\mathcal{A}_{\mathcal{S}}^{c}$ where $\mathcal{A}_{\mathcal{S}}^{c}(d) = \bar{1} - \mathcal{A}_{\mathcal{S}}(d)$, $\forall\ d \in \mathcal{D}$, where $\bar{1}(y) = 1\ \forall y \in \mathcal{M}$.
6. $\mathcal{A}_{\mathcal{S}}$ is called null fss, which is denoted by $\tilde{0}_{\mathcal{D}}$, if $\mathcal{A}_{\mathcal{S}}(d) = \bar{0}$, for all $d \in \mathcal{D}$.
7. $\mathcal{A}_{\mathcal{D}}$ is called universal fss, which is denoted by $\tilde{1}_{\mathcal{D}}$, if $\mathcal{A}_{\mathcal{D}}(d) = \bar{1}$, for all $d \in \mathcal{D}$.

Definition 2.3 [22] A $fss\ \mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M},\mathcal{D})$ is called fuzzy soft point (fs-point), denoted by x_t^s, if there exists $x \in \mathcal{M}$ and $s \in \mathcal{D}$ such that $\mathcal{A}_{\mathcal{S}}(s)(x) = t\ (0 < t \le 1)$ and $\bar{0}$ .Otherwise for all $y \in \mathcal{M} - \{x\}$, the fs-point x_t^s is said to belong to the $fss\ \mathcal{A}_{\mathcal{S}}$, denoted by $x_t^s\ \tilde{\in}\ \mathcal{A}_{\mathcal{S}}$, if for the element $x \in \mathcal{M}$, such that $t \le \mathcal{A}_{\mathcal{S}}(s)(x)$.

Definition 2.4 [21] Let $FS(\mathcal{M},\mathcal{D})$ and $FS(\mathcal{W},\mathcal{N})$ be families of all $fss's$ over $\mathcal{M}$ and $\mathcal{W}$, respectively. If $u: \mathcal{M} \to \mathcal{W}$ and $p: \mathcal{D} \to \mathcal{N}$ be two functions. Then, f_{up} is named fuzzy soft mapping from $FS(\mathcal{M},\mathcal{D})$ to $FS(\mathcal{W},\mathcal{N})$ and denoted by $f_{up}: FS(\mathcal{M},\mathcal{D}) \longrightarrow FS(\mathcal{W},\mathcal{N})$.
1. If $\mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M},\mathcal{D})$, then the image of $\mathcal{A}_{\mathcal{S}}$ under the fuzzy soft mapping f_{up} is the fss over $\mathcal{W}$ defined by $f_{up}(\mathcal{A}_{\mathcal{S}})$, where $\forall k \in p(s),\ \ for\ all\ y \in \mathcal{W}$.

$$f_{up}(\mathcal{A}_{\mathcal{S}})(k)(y) = \begin{cases} \vee_{u(x)=y}\ \left(\vee_{p(s)=k}\ \mathcal{A}_{\mathcal{S}}(s)\right)(x) & if\ x \in u^{-1}(y), \\ 0 & otherwise. \end{cases}$$

2. If $\mu_{\mathcal{B}} \in FS(\mathcal{W},\mathcal{N})$, then the pre-image of $\mu_{\mathcal{B}}$ under the fuzzy soft mapping f_{up} is the fss over $\mathcal{M}$ defined by $f_{up}^{-1}(\mu_{\mathcal{B}})$. where $\forall s \in p^{-1}(\mathcal{N}),\ for\ all\ x \in \mathcal{M}$.

$$f_{up}^{-1}(\mu_{\mathcal{B}})(s)(x) = \begin{cases} \mu_{\mathcal{B}}\big(p(s)\big)\big(u(x)\big) & for\ \ p(s) \in \mathcal{B}, \\ 0 & otherwise. \end{cases}$$

f_{up} *is* called surjective (respectively injective) if u and p are surjective (respectively injective), it is also said to be constant if u and p are constant.

Proposition 2.5 [14] Let $f_{up}: FS(\mathcal{M},\mathcal{D}) \longrightarrow FS(\mathcal{W},\mathcal{N})$ be a fuzzy soft mapping and let x_t^s be a fuzzy soft point in $\mathcal{M}$, then the image of x_t^s under the fuzzy soft mapping f_{up} is a fuzzy soft point in $\mathcal{W}$, which is defined as $f_{up}(x_t^s) = u(x)_t^{p(s)}$.

Proposition 2.6 [14] Let $f_{up}: FS(\mathcal{M},\mathcal{D}) \longrightarrow FS(\mathcal{W},\mathcal{N})$ be a bijective fuzzy soft mapping and let y_t^r be a fuzzy soft point in $\mathcal{W}$, then the inverse image of y_t^r under the fuzzy soft mapping f_{up} is a fuzzy soft point in $\mathcal{M}$, which is defined as $f_{up}^{-1}(y_t^r)$=x_t^s, $p(s) = r$ and $u(x) = y$.

Definition 2.7 [4] A triple $(\mathcal{M},\mathcal{T},\mathcal{D})$ is said to be a fuzzy soft topological space where $\mathcal{T}$ is the collection of $fss's$ over $\mathcal{M}$ such that.
1. $\tilde{0}_{\mathcal{D}}, \tilde{1}_{\mathcal{D}} \in \mathcal{T}$,
2. $\mathcal{A}_{\mathcal{S}}, \mu_{\mathcal{B}} \in \mathcal{T} \Longrightarrow \mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} \in \mathcal{T}$,
3. $(\mathcal{A}_{\mathcal{S}})_i \in \mathcal{T}\ \forall i \Longrightarrow \sqcup_{i \in J}\ (\mathcal{A}_{\mathcal{S}})_i \in \mathcal{T}$.

$\mathcal{T}$ is called a topology of $fss's$ on $\mathcal{M}$. Each member of $\mathcal{T}$ is called an $\mathcal{T}$-open fss $\mu_{\mathcal{B}}$ is called a $\mathcal{T}$-closed fss in $(\mathcal{M},\mathcal{T},\mathcal{D})$ if $\mu_{\mathcal{B}}^{c} \in \mathcal{T}$.

Definition 2.8 [23] A quadruple $(\mathcal{M},\mathcal{T}_1,\mathcal{T}_2,\mathcal{D})$ is said to be a fuzzy soft bi-topological space where $\mathcal{T}_1, \mathcal{T}_2$ are arbitrary fuzzy soft topologies on $\mathcal{M}$.

In the following, we recall the concept of Čfs bicsp and its fundamental properties for $i,j = 1,2$ where $i \ne j$. Otherwise , we will mention the value of i and j.

Definition 2.9 [18] A Čfs bicsp is a quadruple $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$, where $\mathcal{M}$ is a non-empty set and $\gamma_1,\gamma_2: FS(\mathcal{M},\mathcal{D}) \to FS(\mathcal{M},\mathcal{D})$ are two fuzzy soft closure operators on $\mathcal{M}$ which are correct according to the following axioms:

$(A_1)\ \gamma_i(\tilde{0}_{\mathcal{D}}) = \tilde{0}_{\mathcal{D}}$,
$(A_2)\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \gamma_i(\mathcal{A}_{\mathcal{S}})$ for all $\mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M},\mathcal{D})$,
$(A_3)\ \gamma_i(\mathcal{A}_{\mathcal{S}} \sqcup \mu_{\mathcal{B}}) = \gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcup \gamma_i(\mu_{\mathcal{B}})$ for all $\mathcal{A}_{\mathcal{S}}, \mu_{\mathcal{B}} \in FS(\mathcal{M},\mathcal{D})$.

Definition 2.10 [18] A $fss\ \mathcal{A}_{\mathcal{S}}$ of a Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is said to be γ_i-closed (γ_i-open) fss if $\gamma_i(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}$ ($\gamma_i(\mathcal{A}_{\mathcal{S}}^c)=\mathcal{A}_{\mathcal{S}}^c$) and it is called a closed fss if and only if $\gamma_i(\gamma_j(\mathcal{A}_{\mathcal{S}})) = \mathcal{A}_{\mathcal{S}}$. For $i,j = 1\ or\ 2$ where $i \neq j$. The complement of a closed fss is called an open fss.

Proposition 2.11 [18] Let $\mathcal{A}_{\mathcal{S}}$ be a fss of a Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$. Then,
1. A fuzzy soft set $\mathcal{A}_{\mathcal{S}}$ is a closed fss in $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ if and only if $\mathcal{A}_{\mathcal{S}}$ is γ_j-closed fss.
2. If $\mathcal{A}_{\mathcal{S}}$ is an open fss in $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$, then $\gamma_i(\gamma_j(\mathcal{A}_{\mathcal{S}}^c)) = \gamma_j(\gamma_i(\mathcal{A}_{\mathcal{S}}^c))$. For $i,j = 1\ or\ 2$ where $i \neq j$.

Lemma 2.12 [18] Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ be a Čfs bicsp and if $\{(\mathcal{A}_{\mathcal{S}})_\alpha : \alpha \in \Lambda\}$ is a family of $fss's$ over $\mathcal{M}$, then $\gamma_i(\sqcap_{\alpha\in\Lambda}(\mathcal{A}_{\mathcal{S}})_\alpha) \sqsubseteq \sqcap_{\alpha\in\Lambda}\gamma_i((\mathcal{A}_{\mathcal{S}})_\alpha)$.

Now, we need to introduce the following two definitions which we need in the sequel.
Definition 2.13 Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ be a Čfs bicsp, the induced fuzzy soft bitopological space (induced fs-bits) of $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$, is denoted by $(\mathcal{M},\mathcal{T}_{\gamma_1},\mathcal{T}_{\gamma_2},\mathcal{D})$ where $\mathcal{T}_{\gamma_i} = \{\mathcal{A}_{\mathcal{S}}^c : \gamma_i(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}\}$.

Definition 2.14 Let $(\mathcal{M},\mathcal{T}_{\gamma_1},\mathcal{T}_{\gamma_2},\mathcal{D})$ be the induced fs-bits of the Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ and $\mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M},\mathcal{D})$. If $\mathcal{A}_{\mathcal{S}} \in \mathcal{T}_{\gamma_i}$, then $\mathcal{A}_{\mathcal{S}}$ is called an $\mathcal{T}_{\gamma_i}$-open fss in $(\mathcal{M},\mathcal{T}_{\gamma_1},\mathcal{T}_{\gamma_2},\mathcal{D})$. The complement of an $\mathcal{T}_{\gamma_i}$-open $fss\ \mathcal{A}_{\mathcal{S}}$ is a $\mathcal{T}_{\gamma_i}$-closed fss and if $\mathcal{A}_{\mathcal{S}}$ is an $\mathcal{T}_{\gamma_i}$-open,then $\mathcal{A}_{\mathcal{S}}$ is called an open fss in $(\mathcal{M},\mathcal{T}_{\gamma_1},\mathcal{T}_{\gamma_2},\mathcal{D})$ for $i = 1,2$.

proposition 2.15 [19] Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ be Čfs bicsp and if $(\mathcal{M},\mathcal{T}_{\gamma_1},\mathcal{T}_{\gamma_2},\mathcal{D})$ is the induced fs-bits of $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$, then for any $\mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M},\mathcal{D})$,the following hold

$$\mathcal{T}_{\gamma_i}\text{-}int(\mathcal{A}_{\mathcal{S}}) \sqsubseteq Int_i(\mathcal{A}_{\mathcal{S}}) \sqsubseteq \mathcal{A}_{\mathcal{S}} \sqsubseteq \gamma_i(\mathcal{A}_{\mathcal{S}}) \sqsubseteq \mathcal{T}_{\gamma_i}\text{-}cl(\mathcal{A}_{\mathcal{S}}).$$

Definition 2.16 [18] Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ be a Čfs bicsp and $\mathcal{H} \subseteq \mathcal{M}$. The quadruple $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ is called a Čech fuzzy soft bi-closure subspace (Čfs bi-csubsp) of $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$, where $\gamma_{i_{\mathcal{H}}} : FS(\mathcal{H},\mathcal{D}) \to FS(\mathcal{H},\mathcal{D})$ which is defined by $\gamma_{i_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}}) = \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_i(\mathcal{A}_{\mathcal{S}})$ for all $\mathcal{A}_{\mathcal{S}} \in FS(\mathcal{H},\mathcal{D})$. The Čfs bi-csubsp $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ is said to be a closed (open) subspace if $\widetilde{\mathcal{H}}_{\mathcal{D}}$ is a closed (open) fss over $\mathcal{M}$.

Proposition 2.17 [18] Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ be a Čfs bicsp and $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$. If $\mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M},\mathcal{D})$, then $\mathcal{A}_{\mathcal{S}}$ is a closed fss over $\mathcal{H}$ if and only if $\gamma_{j_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}$.

Lemma 2.18 [19] Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ be a Čfs bicsp and $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ be a Čfs bi-csubsp of $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$. If $\mathcal{A}_{\mathcal{S}}$ is an γ_i-open fss of $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$, then $\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ is an $\gamma_{i_{\mathcal{H}}}$-open fss in $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$, for $i = 1\ or\ i = 2$.

Definition 2.19 [18] Let $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ and $(\mathcal{W},\gamma_1^*,\gamma_2^*,\mathcal{N})$ be two Čfs bicsp's. If $f_{up}(\gamma_i(\mathcal{A}_{\mathcal{S}})) \sqsubseteq \gamma_i^*(f_{up}(\mathcal{A}_{\mathcal{S}}))$, then a fuzzy soft mapping

$f_{up}: (\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \to (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is called a pairwise Čech fuzzy soft continuous ($P\check{C}$-*fs*-continuous) mapping for every $fss\ \mathcal{A}_{\mathcal{S}} \in FS(\mathcal{M}, \mathcal{D})$.

Theorem 2.20 [18] Let $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be two Čfs bicsp's. $f_{up}: (\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \to (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is a $P\check{C}$-*fs*-continuous mapping if and only if $\gamma_i(f_{up}^{-1}(\mu_{\mathcal{B}})) \sqsubseteq f_{up}^{-1}(\gamma_i^*(\mu_{\mathcal{B}}))$ for every $\mu_{\mathcal{B}} \in \mathcal{FS}(\mathcal{W}, \mathcal{N})$.

3. Pairwise Regularity in Čech Fuzzy Soft Bi-Closure Spaces

In this section, we define and study some new types of pairwise regularity axioms, namely pairwise quasi-regular, pairwise semi-regular, pairwise pseudo regular, and pairwise regular in both Čfs bicsp's and their induced fs-bits and we study the relationships between them. We also show that in all these types of axioms hereditary property satisfies under closed Čfs bi-csubsp of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ (see Theorems 3.4, 3.9, 3.16, and 3.21).

Definition 3.1 A Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be pairwise quasi-regular-Čfs bicsp (P-quasi-regular-Čfs bicsp), if for every fuzzy soft point x_t^s disjoint from a γ_i-closed $fss\ \rho_{\mathcal{C}}$ there exists an γ_j-open $fss\ \mathcal{A}_{\mathcal{S}}$ such that $x_t^s \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$.

Example 3.2 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$. Define fuzzy soft closure operators $\gamma_1, \gamma_2: \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}, \\ \{(s_1, x_1), (s_2, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, x_1), (s_2, x_1 \vee y_1)\}, \\ \{(s_1, y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, y_1)\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}, \\ \{(s_1, x_1), (s_2, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, x_1), (s_2, x_1 \vee y_1)\}, \\ \{(s_1, y_{0.5})\} & if\ \mathcal{A}_{\mathcal{S}} \in \{(s_1, y_{k_1}), 0 < k_1 < 0.5\}, \\ \{(s_1, y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \in \{(s_1, y_{k_1}), 0.5 \leq k_1 \leq 1\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

To show $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-quasi-regular-Čfs bicsp, we must find all γ_i-closed $fss's$ in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and all fuzzy soft points which are disjoint from these γ_i-closed $fss's$. Thus we have the following:

1. $\rho_{\mathcal{C}}$=$\{(s_1, x_1), (s_2, x_1 \vee y_1)\}$ is a γ_1-closed fss and $\{y_t^{s_1},\ t > 0\}$ be the set of all fuzzy soft points which are disjoint from $\rho_{\mathcal{C}}$. For any $t > 0$, there exists an γ_2-open $fss\ \mathcal{A}_{\mathcal{S}} = \{(s_1, y_1)\}$ such that $y_t^{s_1} \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_2(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. Similarly, $\rho_{\mathcal{C}}$=$\{(s_1, x_1), (s_2, x_1 \vee y_1)\}$ is a γ_2-closed fss and $\{y_t^{s_1},\ t > 0\}$ be the set of all fuzzy soft points which are disjoint from $\rho_{\mathcal{C}}$. For any $t > 0$, there exists an γ_1-open $fss\ \mathcal{A}_{\mathcal{S}} = \{(s_1, y_1)\}$ such that $y_t^{s_1} \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_1(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$.
2. $\rho_{\mathcal{C}}$=$\{(s_1, y_1)\}$ is a γ_1-closed fss and the fuzzy soft points which are disjoint from $\rho_{\mathcal{C}}$ are: $\{x_{t_1}^{s_1},\ t_1 > 0\}$, $\{x_{t_2}^{s_2},\ t_2 > 0\}$ and $\{y_{k_1}^{s_2}, k_1 > 0\}$. For all these fuzzy soft points there exists an γ_2-open $fss\ \mathcal{A}_{\mathcal{S}} = \{(s_1, x_1), (s_2, x_1 \vee y_1)\}$ such that $x_{t_1}^{s_1} \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_2(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$, $x_{t_2}^{s_2} \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_2(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$ and $y_{k_1}^{s_2} \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_2(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. Similarly, $\rho_{\mathcal{C}}$=$\{(s_1, y_1)\}$ is a γ_2-closed fss and the fuzzy soft points which are disjoint from $\rho_{\mathcal{C}}$ are: $\{x_{t_1}^{s_1},\ t_1 > 0\}$, $\{x_{t_2}^{s_2},\ t_2 > 0\}$ and $\{y_{k_1}^{s_2}, k_1 > 0\}$. For all these fuzzy soft points there exists an γ_1-open $fss\ \mathcal{A}_{\mathcal{S}} = \{(s_1, x_1), (s_2, x_1 \vee y_1)\}$ such that $x_{t_1}^{s_1} \tilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_1(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$,

$x_{t_2}^{s_2} \widetilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_1(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$ and $y_{k_1}^{s_2} \widetilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_1(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$ .Hence, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-quasi-regular-Čfs bicsp.

Lemma 3.3 Let $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ be a Čfs bicsp, $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ be a closed Čfs bi-csubsp of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and let $\mathcal{A}_{\mathcal{S}} \in \mathcal{FS}(\mathcal{M}, \mathcal{D})$. If $\mathcal{A}_{\mathcal{S}}$ is $\gamma_{i_{\mathcal{H}}}$-closed fss of $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$, then $\mathcal{A}_{\mathcal{S}}$ is γ_i-closed fss of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$, $i = 1$ or $i = 2$.

Proof: We prove the Lemma when $i = 1$ and the proof is similar for $i = 2$. Let $\mathcal{A}_{\mathcal{S}}$ be a $\gamma_{1_{\mathcal{H}}}$-closed fss over $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$, we must show $\mathcal{A}_{\mathcal{S}}$ is an γ_1-closed fss over $\mathcal{M}$, i.e, we must show $\gamma_1(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}$. Since $\gamma_{1_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}$, then $\widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_1(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}$. Since $\widetilde{\mathcal{H}}_{\mathcal{D}}$ is a closed fss in $\mathcal{M}$, then $\gamma_1(\widetilde{\mathcal{H}}_{\mathcal{D}}) = \widetilde{\mathcal{H}}_{\mathcal{D}}$. This implies $\gamma_1(\widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \mathcal{A}_{\mathcal{S}}) \sqsubseteq \gamma_1(\widetilde{\mathcal{H}}_{\mathcal{D}}) \sqcap \gamma_1(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$, then $\gamma_1(\mathcal{A}_{\mathcal{S}}) \sqsubseteq \mathcal{A}_{\mathcal{S}}$. On the other hand, from the definition of γ_1, $\mathcal{A}_{\mathcal{S}} \sqsubseteq \gamma_1(\mathcal{A}_{\mathcal{S}})$. Hence, $\gamma_1(\mathcal{A}_{\mathcal{S}}) = \mathcal{A}_{\mathcal{S}}$. Therefore, $\mathcal{A}_{\mathcal{S}}$ is an γ_1-closed fss over $\mathcal{M}$.

Theorem 3.4 Every closed Čfs bi-csubsp $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ of a P-quasi-regular-Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-quasi-regular -Čfs bi-csubsp.

Proof: Let x_t^s be a fuzzy soft point in $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ and $\rho_{\mathcal{C}}$ be a $\gamma_{i_{\mathcal{H}}}$-closed fss in $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ such that $x_t^h \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$, this implies $x_t^s \widetilde{\notin} \rho_{\mathcal{C}}$. By Lemma 3.3, we get $\rho_{\mathcal{C}}$ be a γ_i-closed fss in $\mathcal{M}$ does not contain x_t^s. But $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-quasi-regular-Čfs bicsp. This yield, there exists an γ_j-open fss $\mathcal{A}_{\mathcal{S}}$ such that $x_t^s \widetilde{\in} \mathcal{A}_{\mathcal{S}}$ and $\gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. From Lemma 2.18 $\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ is an $\gamma_{j_{\mathcal{H}}}$-open fss in $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ and $x_t^s \widetilde{\in} \mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$. That is mean we found an $\gamma_{j_{\mathcal{H}}}$-open fss $\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ in $\mathcal{H}$ contains x_t^s. Now, it remains only to show $\gamma_{j_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$.

$\gamma_{j_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqcap \rho_{\mathcal{C}} = \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_j(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqcap \rho_{\mathcal{C}}$ (BY Definition 2.16)

$\sqsubseteq \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\widetilde{\mathcal{H}}_{\mathcal{D}}) \sqcap \rho_{\mathcal{C}}$ (BY Lemma 2.12)

$= \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}}$

$= \tilde{0}_{\mathcal{D}}$.

Hence, $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ is a P-quasi-regular-Čfs bi- csubsp.

Definition 3.5 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be P-quasi-regular-fs-bits, if for every fuzzy soft point x_t^s disjoint from a τ_{γ_i}-closed fss $\rho_{\mathcal{C}}$ in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$, there exists an τ_{γ_j}-open fss $\mathcal{A}_{\mathcal{S}}$ in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ such that $x_t^s \widetilde{\in} \mathcal{A}_{\mathcal{S}}$ and τ_{γ_j}-$cl(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$.

Theorem 3.6 If $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-quasi-regular- fs-bits, then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is also a P-quasi-regular-Čfs bicsp.

Proof: Let x_t^s be a fuzzy soft point disjoint from a γ_i-closed fss $\rho_{\mathcal{C}}$ in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. That means $x_t^s \widetilde{\notin} \rho_{\mathcal{C}}$. Since $\rho_{\mathcal{C}}$ is a γ_i-closed fss in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. Then $\rho_{\mathcal{C}}$ is a τ_{γ_i}-closed fss in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$. But $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-quasi-regular-fs-bits. Therefore, it follows, there exists τ_{γ_j}-open fss $\mathcal{A}_{\mathcal{S}}$ such that $x_t^s \widetilde{\in} \mathcal{A}_{\mathcal{S}}$ and τ_{γ_j}-$cl(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. From Proposition 2.15, we get $\gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. Hence, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-quasi-regular-Čfs bicsp .

Definition 3.7 A Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is said to be pairwise semi-regular-Čfs bicsp (P-semi-regular- Čfs bicsp), if for every fuzzy soft point x_t^s disjoint from a γ_i-closed $\rho_{\mathcal{C}}$, there exists an γ_j-open fss $\mathcal{A}_{\mathcal{S}}$ such that $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $x_t^s \tilde{\notin} \gamma_j(\mathcal{A}_{\mathcal{S}})$.

Example 3.8 Let $\mathcal{M} = \{x,y\}$, $\mathcal{D} = \{s_1,s_2\}$. Define fuzzy soft closure operators $\gamma_1,\gamma_2: \mathcal{FS}(\mathcal{M},\mathcal{D}) \to \mathcal{FS}(\mathcal{M},\mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}\ , \\ \{(s_1, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, x_1 \vee y_1)\}, \\ \{(s_2, x_1 \vee y_1)\} & if \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_2, x_1 \vee y_1)\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}\ , \\ \{(s_1, x_{0.5} \vee y_{0.5})\} & if\ \mathcal{A}_{\mathcal{S}} \in \{x_{t_1}^{s_1}: 0 \le t_1 < 0.5\}, \\ \{(s_1, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \in \{x_{t_1}^{s_1}: 0.5 \le t_1 \le 1\}, \\ \{(s_1, x_{0.5} \vee y_{0.5})\} & if\ \mathcal{A}_{\mathcal{S}} \in \{y_{k_1}^{s_1}: 0 \le t_1 < 0.5\}, \\ \{(s_1, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \in \{y_{k_1}^{s_1}: 0.5 \le t_1 \le 1\}, \\ \gamma_2(x_{t_1}^{s_1}) \cup \gamma_2(y_{k_1}^{s_1}) & if\ \mathcal{A}_{\mathcal{S}} \in \{(s_1, x_{t_1} \vee y_{k_1}): t_1, k_1 \in I_0\}, \\ \{(s_2, x_1 \vee y_1)\} & if \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_2, x_1 \vee y_1)\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

To show that $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is P-semi-regular- Čfs bicsp, we must find all fuzzy soft points which is disjoint from a closed $fss's$ in $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$. Thus, we have the following:

1. $(\mathcal{A}_{\mathcal{S}})_1$=$\{(s_1, x_1 \vee y_1)\}$ is a γ_1-closed fss and the fuzzy soft points which are disjoint from $(\mathcal{A}_{\mathcal{S}})_1$ are: $\{x_{t_1}^{s_2}, t_1 > 0\}$ and $\{y_{t_2}^{s_2}, t_2 > 0\}$, there exists an γ_2-open fss $(\mathcal{A}_{\mathcal{S}})_1$ such that $(\mathcal{A}_{\mathcal{S}})_1 \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1$ and $x_{t_1}^{s_2}, y_{t_2}^{s_2} \tilde{\notin} \gamma_2(\mathcal{A}_{\mathcal{S}})_1$. Similarly, $(\mathcal{A}_{\mathcal{S}})_1$=$\{(s_1, x_1 \vee y_1)\}$ is a γ_2-closed fss and the fuzzy soft points which are disjoint from $(\mathcal{A}_{\mathcal{S}})_1$ are: $\{x_{t_1}^{s_2}, t_1 > 0\}$ and $\{y_{t_2}^{s_2}, t_2 > 0\}$, there exists an γ_1-open fss $(\mathcal{A}_{\mathcal{S}})_1$ such that $(\mathcal{A}_{\mathcal{S}})_1 \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1$ and $x_{t_1}^{s_2}, y_{t_2}^{s_2} \tilde{\notin} \gamma_1(\mathcal{A}_{\mathcal{S}})_1$.
2. $(\mathcal{A}_{\mathcal{S}})_2$=$\{(s_2, x_1 \vee y_1)\}$ is a γ_1-closed fss and the fuzzy soft points which are disjoint from $(\mathcal{A}_{\mathcal{S}})_2$ are: $\{x_{t_1}^{s_1}, t_1 > 0\}$ and $\{y_{t_2}^{s_1}, t_2 > 0\}$. For all these fuzzy soft points there exists an γ_2-open fss $(\mathcal{A}_{\mathcal{S}})_2$. Similarly, $(\mathcal{A}_{\mathcal{S}})_2$ is a γ_2-closed fss satisfied the required conditions of P-semi-regular- Čfs bicsp. Then $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is P-semi-regular- Čfs bicsp.

Theorem 3.9 Every closed Čfs bi-csubsp $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ of a P-semi-regular-Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is a P-semi-regular-Čfs bi- csubsp.

Proof: Let x_t^s be a fuzzy soft point in $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ and $\rho_{\mathcal{C}}$ be a $\gamma_{i_{\mathcal{H}}}$-closed fss in $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ such that $x_t^h \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$, this implies $x_t^s \tilde{\notin} \rho_{\mathcal{C}}$. By Lemma 3.3, we get $\rho_{\mathcal{C}}$ be a γ_i-closed fss in $\mathcal{M}$ does not contain x_t^s. But $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is a P-semi-regular-Čfs bicsp. Then, there exists an γ_j-open fss $\mathcal{A}_{\mathcal{S}}$ such that $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $x_t^s \tilde{\notin} \gamma_j(\mathcal{A}_{\mathcal{S}})$. Now, $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $\rho_{\mathcal{C}} \sqsubseteq \widetilde{\mathcal{H}}_{\mathcal{D}}$, this implies $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ which is an open fss from Lemma 2.18. Next, we must show $x_t^s \tilde{\notin} \gamma_{j_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}})$. Suppose, $x_t^s \tilde{\in} \gamma_{j_{\mathcal{H}}}(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) = \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_j(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqsubseteq \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\widetilde{\mathcal{H}}_{\mathcal{D}})$, it follows $x_t^s \tilde{\in} \gamma_j(\mathcal{A}_{\mathcal{S}})$ which is a contradiction with the hypothesis. Hence, $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ is a P-semi-regular-Čfs bi- csubsp.

The next example shows that the converse of Theorem 3.9 is not true in general.

Example 3.10: Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$. $\mathcal{H} = \{y\} \subseteq \mathcal{M}$. Define fuzzy soft closure operators $\gamma_1, \gamma_2: \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_\mathcal{S}) = \gamma_2(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}\ , \\ \{(s_1, y_1), (s_2, y_1)\} & if\ \mathcal{A}_\mathcal{S} \sqsubseteq \{(s_1, y_1), (s_2, y_1)\}, \\ \tilde{1}_\mathcal{D} & otherwise. \end{cases}$$

Then, it is clear that $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is not a P-semi-regular-Čfs bicsp, since there exists $\rho_\mathcal{C} = \{(s_1, y_1), (s_2, y_1)\}$ is a γ_1-closed fss and there exists a fuzzy soft point $x_{0.5}^{s_1}$ which is disjoint from $\rho_\mathcal{C}$ and there exists only $\mathcal{A}_\mathcal{S} = \tilde{1}_\mathcal{D}$ is γ_2-open fss such that $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$.However, $x_{0.5}^{s_1} \tilde{\in} \gamma_2(\mathcal{A}_\mathcal{S}) = \tilde{1}_\mathcal{D}$.

On the other hand, the closed Čfs bi-csubsp $(\mathcal{H}, \gamma_{1_\mathcal{H}}, \gamma_{2_\mathcal{H}}, \mathcal{D})$, where $\gamma_{1_\mathcal{H}}, \gamma_{2_\mathcal{H}}: \mathcal{FS}(\mathcal{H}, \mathcal{D}) \to \mathcal{FS}(\mathcal{H}, \mathcal{D})$ as follows:

$$\gamma_{1_\mathcal{H}}(\mathcal{A}_\mathcal{S}) = \gamma_{2_\mathcal{H}}(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}\ , \\ \tilde{1}_\mathcal{D} & otherwise \end{cases}$$

is a P-semi-regular-Čfs bi- csubsp.

Definition 3.11 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be a P-semi-regular-fs-bits, if for every fuzzy soft point x_t^s disjoint from a τ_{γ_i}-closed $fss\ \rho_\mathcal{C}$ in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$, there exists an τ_{γ_j}-open $fss\ \mathcal{A}_\mathcal{S}$ in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ such that $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$ and $x_t^s \tilde{\notin} \tau_{\gamma_j}\text{-}cl(\mathcal{A}_\mathcal{S})$.

Theorem 3.12 If $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-semi-regular-fs-bits, then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is also a P-semi-regular-Čfs bicsp.

Proof: Let x_t^s be a fuzzy soft point disjoint from a γ_i-closed $fss\ \rho_\mathcal{C}$ in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. That means $x_t^s \tilde{\notin} \rho_\mathcal{C}$. Since $\rho_\mathcal{C}$ is a γ_i-closed fss in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. Then $\rho_\mathcal{C}$ is a τ_{γ_i}-closed fss in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$. But $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-semi-regular-fs-bits. It follows, there exists τ_{γ_j}-open $fss\ \mathcal{A}_\mathcal{S}$ such that $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$ and $x_t^h \tilde{\notin} \tau_{\gamma_j}\text{-}cl(\mathcal{A}_\mathcal{S})$. From Proposition 2.15, we get $x_t^s \tilde{\notin} \gamma_i(\mathcal{A}_\mathcal{S})$. Hence, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-semi-regular-Čfs bicsp .

The next example shows that the converse of Theorem 3.12 is not true in general.

Example 3.13: Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$. Define fuzzy soft closure operators $\gamma_1, \gamma_2: \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}\ , \\ \{(s_1, x_{0.5} \vee y_{0.5})\} & if\ \mathcal{A}_\mathcal{S} \sqsubseteq \{(s_1, x_{0.5} \vee y_{0.5})\}, \\ \tilde{1}_\mathcal{D} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}, \\ \{(s_1, x_{0.4} \vee y_{0.4}), (s_2, x_1 \vee y_1)\} & if\ (\mathcal{A}_\mathcal{S})_1 \sqsubseteq \{(s_1, x_{0.4} \vee y_{0.4}), (s_2, x_1 \vee y_1)\}, \\ \{(s_1, x_{0.7} \vee y_{0.7})\} & if\ (\mathcal{A}_\mathcal{S})_1 \not\sqsubseteq \mathcal{A}_\mathcal{S} \sqsubseteq \{(s_1, x_{0.6} \vee y_{0.6})\}, \\ \{(s_1, x_{0.7} \vee y_{0.7}), (s_2, x_1 \vee y_1)\} & if\ (\mathcal{A}_\mathcal{S})_1 \not\sqsubseteq \mathcal{A}_\mathcal{S} \sqsubseteq \{(s_1, x_{0.6} \vee y_{0.6}), (s_2, x_1 \vee y_1)\}, \\ \tilde{1}_\mathcal{D} & otherwise. \end{cases}$$

Then, it is clear that $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-semi-regular-Čfs bicsp. However, the induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is not a P-semi-regular-fs-bits since there exists $\rho_\mathcal{C} = \{(s_1, x_{0.5} \vee y_{0.5})\}$ is a τ_{γ_1}-closed fss and there exists a fuzzy soft point $x_{0.5}^{s_2}$ which is disjoint from $\rho_\mathcal{C}$ such that for each $\mathcal{A}_{\mathcal{S}} = \{(s_1, x_{0.6} \vee y_{0.6})\}$ and $\tilde{1}_{\mathcal{D}}$ are τ_{γ_2}-open $fss's$ we have $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$ but $x_{0.5}^{s_2} \tilde{\notin} \mathcal{T}_{\gamma_2}$-$cl(\mathcal{A}_{\mathcal{S}}) = \tilde{1}_{\mathcal{D}}$.

Definition 3.14 A Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be pairwise pseudo-regular-Čfs bicsp (P-pseudo-regular-Čfs bicsp), if it is both P-quasi-regular-Čfs bicsp and P-semi-regular-Čfs bicsp.

Example 3.15 In Example 3.2, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-pseudo-regular- Čfs bicsp.

The next theorem follows directly from Theorem 3.4 and Theorem 3.9.

Theorem 3.16 Every closed Čfs bi-csubsp $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ of a P-pseudo-regular-Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-pseudo-regular-Čfs bi- csubsp.

Definition 3.17 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be a P-pseudo-regular-fs-bits, if it is both P-quasi-regular-fs-bits and P-semi-regular-fs-bits.

Theorem 3.18 If $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$is a P-pseudo-regular-fs-bits, then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is also a P-pseudo-regular-Čfs bicsp.

Proof: The proof directly follows from Theorem 3.6 and Theorem 3.12.

Definition 3.19 A Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be pairwise regular-Čfs bicsp (P-regular-Čfs bicsp), if for every fuzzy soft point x_t^s disjoint from a γ_i-closed $\rho_{\mathcal{C}}$, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for γ_i and γ_j respectively, such that $x_t^s \tilde{\in} \mathcal{A}_{\mathcal{S}}, \rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$.

Example 3.20 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$. Define fuzzy soft closure operators $\gamma_1, \gamma_2: \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}\ , \\ \{(s_1, y_1), (s_2, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, y_1), (s_2, x_1 \vee y_1)\}, \\ \{(s_1, x_1)\} & if \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, x_1)\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}\ , \\ \{(s_1, y_1), (s_2, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, y_1), (s_2, x_1 \vee y_{k_2}), k_2 \in I_0\}, \\ \{(s_1, x_{0.5})\} & if \mathcal{A}_{\mathcal{S}} \in \{(s_1, x_{t_1}), 0 \leq t_1 < 0.5\}, \\ \{(s_1, x_1)\} & if \mathcal{A}_{\mathcal{S}} \in \{(s_1, x_{t_1}), 0.5 \leq t_1 \leq 1\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

Then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-regular- Čfs bicsp.

Theorem 3.21 Every closed Čfs bi-csubsp $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ of a P-regular-Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-regular-Čfs bi-csubsp.

Proof: Let x_t^s be a fuzzy soft point in $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ and $\rho_{\mathcal{C}}$ be a $\gamma_{i_{\mathcal{H}}}$-closed fss in $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ such that $x_t^h \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$, this implies $x_t^s \tilde{\notin} \rho_{\mathcal{C}}$. By Lemma 3.3, we get $\rho_{\mathcal{C}}$ be a γ_i-closed fss in $\mathcal{M}$ does not contain x_t^s. But $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-regular-Čfs bicsp. Then, there exist open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ with respect to γ_i and γ_j such that $x_t^s \tilde{\in} \mathcal{A}_{\mathcal{S}}$, $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. Thus, we have $x_t^s \tilde{\in} \mathcal{A}_{\mathcal{S}} \sqcap \tilde{\mathcal{H}}_{\mathcal{D}}$ and $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}} \sqcap \tilde{\mathcal{H}}_{\mathcal{D}}$ and from Lemma

2.18, $\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ and $\mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ are open $fss's$ with respect to $\gamma_{i_{\mathcal{H}}}$ and $\gamma_{j_{\mathcal{H}}}$. Moreover, it is clear that $(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) \sqcap (\mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}) = \tilde{0}_{\mathcal{D}}$. Hence, $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ is a P-regular-Čfs bi-csubsp.

Definition 3.22 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be P-regular-fs-bits, if for every fuzzy soft point x_t^s disjoint from a τ_{γ_i}-closed $\rho_{\mathcal{C}}$, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for τ_{γ_i} and τ_{γ_j} respectively, such that $x_t^s \tilde{\in} \mathcal{A}_{\mathcal{S}}$, $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$, and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$.

Theorem 3.23 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is P-regular- fs-bits if and only if $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-regular-Čfs bicsp.

Proof: Let x_t^s be fuzzy soft point disjoint from a γ_i-closed fss $\rho_{\mathcal{C}}$ in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$, Since $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is P-regular-fs-bits, then there exist a disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ with respect to τ_{γ_i} and τ_{γ_j} such that $x_t^h \tilde{\in} \mathcal{A}_{\mathcal{S}}$, $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$, and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. From Proposition 2.15, we get $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ are open $fss's$ with respect to γ_i and γ_j such that $x_t^s \tilde{\in} \mathcal{A}_{\mathcal{S}}$, $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$, and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. Thus, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-regular-Čfs bicsp.

Conversely, let x_t^s be fuzzy soft point disjoint from a τ_{γ_i}-closed fss $\rho_{\mathcal{C}}$ in $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$. From Definition 2.13, this implies $\rho_{\mathcal{C}}$ is γ_i- closed fss in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. Since $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-regular-Čfs bicsp, then there exist a disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ with respect to γ_i and γ_j such that such that $x_t^h \tilde{\in} \mathcal{A}_{\mathcal{S}}$, $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$, and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. From Proposition 2.15, we get $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ are open $fss's$ with respect to τ_{γ_i} and τ_{γ_j} such that such that $x_t^h \tilde{\in} \mathcal{A}_{\mathcal{S}}$, $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$, and $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}} = \tilde{0}_{\mathcal{D}}$. Thus, $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-regular-fs-bits.

To study the topological property in P-regular-Čfs bicsp's we need first to define the notion of homeomorphism mappings between Čfs bicsp's and we give propositions about the image and inverse image of the fuzzy soft points in Čfs bicsp's.

Definition 3.24 Let $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be two Čfs bicsp's. A fuzzy soft mapping $f_{up}: (\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \to (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is said to be pairwise Čech fuzzy soft homeomorphism ($P\check{C}$-*fs*-homeomorphism) mapping, if and only if f_{up} is injective, surjective, $P\check{C}$-*fs*-continuous, and f_{up}^{-1} is $P\check{C}$-*fs*-continuous mapping.
The next example explains Definition 3.24.

Example 3.25 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$, $\mathcal{W} = \{z, w\}$, and $\mathcal{N} = \{n_1, n_2\}$ Define fuzzy soft closure operators $\gamma_1, \gamma_2: \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}, \\ \{(s_1, x_1), (s_2, x_1 \vee y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, x_1), (s_2, x_1 \vee y_1)\}, \\ \{(s_1, y_1)\} & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq \{(s_1, y_1)\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}\ , \\ \{(s_1, x_1), (s_2, x_1 \vee y_1)\} & if\ \mathcal{A}_\mathcal{S} \sqsubseteq \{(s_1, x_1), (s_2, x_1 \vee y_1)\}, \\ \{(s_1, y_{0.5})\} & if \mathcal{A}_\mathcal{S} \in \{(s_1, y_{k_1}), 0 < k_1 < 0.5\}, \\ \{(s_1, y_1)\} & if\ \mathcal{A}_\mathcal{S} \in \{(s_1, y_{k_1}), 0.5 \le k_1 \le 1\}, \\ \tilde{1}_\mathcal{D} & otherwise. \end{cases}$$

Define fuzzy soft closure operators $\gamma_1^*, \gamma_2^*: \mathcal{FS}(\mathcal{W}, \mathcal{N}) \to \mathcal{FS}(\mathcal{W}, \mathcal{N})$ as follows:

$$\gamma_1^*(\mathcal{A}_\mathcal{S}) = \gamma_2^*(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{N} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{N}\ , \\ \tilde{1}_\mathcal{N} & otherwise \end{cases}$$

Let $u: \mathcal{M} \to \mathcal{W}$ and $p: \mathcal{D} \to \mathcal{N}$ be two functions defined as $u(x) = z, u(y) = w$ and $p(s_1) = n_1, p(s_2) = n_2$. Then, it is clear that the fuzzy soft mapping $f_{up}: FS(\mathcal{M}, \mathcal{D}) \longrightarrow FS(\mathcal{W}, \mathcal{N})$ is $P\check{C}$-*fs*-homeomorphism.

Proposition 3.26 A fuzzy soft mapping $f_{up}:(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \to (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is $P\check{C}$-*fs*-homeomorphism mapping if and only if f_{up} is injective, surjective, $P\check{C}$-*fs*-continuous, and $\check{C}$-*fs*-open mapping.

Proof: The proof directly follows from the definition of $P\check{C}$-*fs*-homeomorphism mapping.

Theorem 3.27 The property of being P-regular-Čfs bicsp is a topological property.

Proof: Let $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be two Čfs bicsp's and let f_{up}: $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \to (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be a $P\check{C}$-*fs*-homeomorphism mapping and $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-regular-Čfs bicsp. We want to show $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is also a P-regular-Čfs bicsp. Let y_t^r be a fuzzy soft point in $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ and $\rho_\mathcal{C}$ be a γ_i^*-closed fss in $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ such that $y_t^r \sqcap \rho_\mathcal{C} = \tilde{0}_\mathcal{N}$. Since f_{up} is $P\check{C}$-*fs*-homeomorphism mapping, then $f_{up}^{-1}(y_t^r)$ is a fuzzy soft point in $\mathcal{M}$ and $f_{up}^{-1}(\ \rho_\mathcal{C})$ is a γ_i-closed fss in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ such that $f_{up}^{-1}(y_t^r) \sqcap f_{up}^{-1}(\ \rho_\mathcal{C}) = \tilde{0}_\mathcal{D}$. But $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-regular-Čfs bicsp this implies there exist disjoint open $fss's$ $\mathcal{A}_\mathcal{S}$ and $\mu_\mathcal{B}$ with respect to γ_i and γ_j such that $f_{up}^{-1}(y_t^r) \sqsubseteq \mathcal{A}_\mathcal{S}$ and $f_{up}^{-1}(\ \rho_\mathcal{C}) \sqsubseteq \mu_\mathcal{B}$. It follows, $f_{up}(f_{up}^{-1}(y_t^r)) \sqsubseteq f_{up}(\mathcal{A}_\mathcal{S})$ and $f_{up}(f_{up}^{-1}(\ \rho_\mathcal{C})) \sqsubseteq f_{up}(\ \mu_\mathcal{B})$. Since f_{up} is $P\check{C}$-*fs*-homeomorphism mapping, then f_{up} is $\check{C}$-*fs*-open mapping, this yields there exist open $fss's$ $f_{up}(\mathcal{A}_\mathcal{S})$ and $f_{up}(\ \mu_\mathcal{B})$ with respect to γ_i^* and γ_j^* such that $y_t^r \sqsubseteq f_{up}(\mathcal{A}_\mathcal{S})$ and $\rho_\mathcal{C} \sqsubseteq f_{up}(\ \mu_\mathcal{B})$. Moreover, $f_{up}(\mathcal{A}_\mathcal{S}) \sqcap f_{up}(\ \mu_\mathcal{B}) = \tilde{0}_\mathcal{N}$. Hence, $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is also a P-regular-Čfs bicsp. Similarly, if $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is a P-regular-Čfs bicsp, then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-regular-Čfs bicsp.

4 . Pairwise Normality in Čech Fuzzy Soft Bi-Closure Spaces

In this section, we define some pairwise normality axioms, namely pairwise semi-normal, pairwise pseudo normal, pairwise normal, and pairwise completely normal in both Čfs bicsp's and the induced fs-bits's. The relationships between them and their basic properties are studied as in the previous section.

Definition 4.1 A Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be pairwise semi normal-Čfs bicsp (P-semi normal-Čfs bicsp), if for each pair of disjoint closed $fss's$ $\rho_\mathcal{C}$ and η_E for γ_i and γ_j respectively, either there exists an γ_j-open fss $\mathcal{A}_\mathcal{S}$ such that $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$ and $\gamma_j(\mathcal{A}_\mathcal{S}) \sqcap \eta_E = \tilde{0}_\mathcal{D}$ or there exists an γ_i-open fss $\mu_\mathcal{B}$ such that $\eta_E \sqsubseteq \mu_\mathcal{B}$ and $\gamma_i(\ \mu_\mathcal{B}) \sqcap \rho_\mathcal{C} = \tilde{0}_\mathcal{D}$.
If both conditions hold, then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be P-pseudo normal-Čfs bi csp.

Example 4.2 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$, and let $(\mathcal{A}_\mathcal{S})_1$=$\{(s_1, x_1 \vee y_1)\}$, $(\mathcal{A}_\mathcal{S})_2$=$\{(s_2, x_1 \vee y_1)\}$, $(\mathcal{A}_\mathcal{S})_3 = \{(s_1, x_1), (s_2, x_1 \vee y_1)\}$. Define fuzzy soft closure operators $\gamma_1, \gamma_2: \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}, \\ (\mathcal{A}_\mathcal{S})_1 & if\ \mathcal{A}_\mathcal{S} \sqsubseteq (\mathcal{A}_\mathcal{S})_1, \\ (\mathcal{A}_\mathcal{S})_2 & if \mathcal{A}_\mathcal{S} \sqsubseteq (\mathcal{A}_\mathcal{S})_2, \\ \tilde{1}_\mathcal{D} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_\mathcal{S}) = \begin{cases} \tilde{0}_\mathcal{D} & if\ \mathcal{A}_\mathcal{S} = \tilde{0}_\mathcal{D}, \\ (\mathcal{A}_\mathcal{S})_1 & if\ \mathcal{A}_\mathcal{S} \sqsubseteq (\mathcal{A}_\mathcal{S})_1, \\ (\mathcal{A}_\mathcal{S})_2 & if \mathcal{A}_\mathcal{S} \sqsubseteq (\mathcal{A}_\mathcal{S})_2, \\ (\mathcal{A}_\mathcal{S})_3 & if \mathcal{A}_\mathcal{S} \sqsubset (\mathcal{A}_\mathcal{S})_3; \mathcal{A}_\mathcal{S} = \{(s_1, x_{t_1}), (s_2, x_{t_2} \vee y_1), t_1, t_2 \in I_0\}, \\ \tilde{1}_\mathcal{D} & otherwise. \end{cases}$$

Therefore, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-semi normal-Čfs bicsp. Since the only disjoint closed $fss's$ are $(\mathcal{A}_\mathcal{S})_1$ and $(\mathcal{A}_\mathcal{S})_2$ for γ_1 and γ_2 respectively and there exists an γ_2-open fss $(\mathcal{A}_\mathcal{S})_1$ such that $(\mathcal{A}_\mathcal{S})_1 \sqsubseteq (\mathcal{A}_\mathcal{S})_1$ and $\gamma_2(\mathcal{A}_\mathcal{S})_1 \sqcap (\mathcal{A}_\mathcal{S})_2$=$(\mathcal{A}_\mathcal{S})_1 \sqcap (\mathcal{A}_\mathcal{S})_2$=$\tilde{0}_\mathcal{D}$.

Next, we show that the hereditary property is satisfied under closed Čfs bi-csubsp.

Theorem 4.3 Every closed Čfs bi-csubsp $(\mathcal{H}, \gamma_{1_\mathcal{H}}, \gamma_{2_\mathcal{H}}, \mathcal{D})$ of a P-semi normal-Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-semi normal-Čfs bi- csubsp.

Proof: Let $\rho_\mathcal{C}$ and η_E be closed $fss's$ in $\gamma_{i_\mathcal{H}}$ and $\gamma_{j_\mathcal{H}}$ respectively such that $\rho_\mathcal{C} \sqcap \eta_E$=$\tilde{0}_\mathcal{D}$. Since $\widetilde{\mathcal{H}}_\mathcal{D}$ is closed fss in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$, then by Lemma 3.3, $\rho_\mathcal{C}$ and η_E are disjoint closed $fss's$ in γ_i and γ_j respectively. But $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P-semi normal-Čfs bicsp, it follows there exist an γ_j-open fss $\mathcal{A}_\mathcal{S}$ such that $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$ and $\gamma_j(\mathcal{A}_\mathcal{S}) \sqcap \eta_E$=$\tilde{0}_\mathcal{D}$. Since $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$, then $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S} \sqcap \widetilde{\mathcal{H}}_\mathcal{D}$ which is $\gamma_{j_\mathcal{H}}$-open fss in $(\mathcal{H}, \gamma_{1_\mathcal{H}}, \gamma_{2_\mathcal{H}}, \mathcal{D})$. And

$$\begin{aligned} \gamma_{j_\mathcal{H}}(\mathcal{A}_\mathcal{S} \sqcap \widetilde{\mathcal{H}}_\mathcal{D}) \sqcap \eta_E &= \widetilde{\mathcal{H}}_\mathcal{D} \sqcap \gamma_j(\mathcal{A}_\mathcal{S} \sqcap \widetilde{\mathcal{H}}_\mathcal{D}) \sqcap \eta_E \\ &\sqsubseteq \widetilde{\mathcal{H}}_\mathcal{D} \sqcap \gamma_j(\mathcal{A}_\mathcal{S}) \sqcap \gamma_j(\widetilde{\mathcal{H}}_\mathcal{D}) \sqcap \eta_E \\ &= \widetilde{\mathcal{H}}_\mathcal{D} \sqcap \gamma_j(\mathcal{A}_\mathcal{S}) \sqcap \eta_E \\ &= \tilde{0}_\mathcal{D}. \end{aligned}$$

Similarly, if there exists an γ_i-open fss $\mu_\mathcal{B}$ such that $\eta_E \sqsubseteq \mu_\mathcal{B}$ and $\gamma_i(\mu_\mathcal{B}) \sqcap \rho_\mathcal{C}$=$\tilde{0}_\mathcal{D}$. We have an $\gamma_{i_\mathcal{H}}$-open fss $\mu_\mathcal{B} \sqcap \widetilde{\mathcal{H}}_\mathcal{D}$ in $(\mathcal{H}, \gamma_{1_\mathcal{H}}, \gamma_{2_\mathcal{H}}, \mathcal{D})$ such that $\eta_E \sqsubseteq \mu_\mathcal{B} \sqcap \widetilde{\mathcal{H}}_\mathcal{D}$ and $\gamma_{i_\mathcal{H}}(\mu_\mathcal{B} \sqcap \widetilde{\mathcal{H}}_\mathcal{D}) \sqcap \rho_\mathcal{C}$=$\tilde{0}_\mathcal{D}$. Hence, $(\mathcal{H}, \gamma_{1_\mathcal{H}}, \gamma_{2_\mathcal{H}}, \mathcal{D})$ is a P-semi normal-Čfs bi-csubsp.

Definition 4.4 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be P-semi normal-fs-bits, if for each pair of disjoint closed $fss's$ $\rho_\mathcal{C}$ and η_E for τ_{γ_i} and τ_{γ_j}, respectively, either there exists an τ_{γ_j}-open fss $\mathcal{A}_\mathcal{S}$ such that $\rho_\mathcal{C} \sqsubseteq \mathcal{A}_\mathcal{S}$ and τ_{γ_j}-$cl(\mathcal{A}_\mathcal{S}) \sqcap \eta_E$=$\tilde{0}_\mathcal{D}$, or there exists an τ_{γ_i}-open fss $\mu_\mathcal{B}$ such that $\eta_E \sqsubseteq \mu_\mathcal{B}$ and τ_{γ_i}-$cl(\mu_\mathcal{B}) \sqcap \rho_\mathcal{C}$=$\tilde{0}_\mathcal{D}$.

Theorem 4.5 If $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-semi-normal-fs-bits, then $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is also a P-semi normal-Čfs bicsp.

Proof: Let $\rho_\mathcal{C}$ and η_E be disjoint closed $fss's$ in γ_i and γ_j respectively. So that $\rho_\mathcal{C}$ and η_E be disjoint closed $fss's$ in τ_{γ_i} and τ_{γ_j} respectively. By hypothesis, there exists an τ_{γ_j}-open fss

$\mathcal{A}_{\mathcal{S}}$ such that $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and τ_{γ_j}-$cl(\mathcal{A}_{\mathcal{S}}) \sqcap \eta_E = \tilde{0}_{\mathcal{D}}$, or there exists an τ_{γ_i}-open fss $\mu_{\mathcal{B}}$ such that $\eta_E \sqsubseteq \mu_{\mathcal{B}}$ and τ_{γ_i}-$cl(\mu_{\mathcal{B}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. From Proposition 2.15, we get either there exists an γ_j-open fss $\mathcal{A}_{\mathcal{S}}$ in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ such that $\rho_{\mathcal{C}} \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $\gamma_j(\mathcal{A}_{\mathcal{S}}) \sqcap \eta_E = \tilde{0}_{\mathcal{D}}$ or there exists an γ_i-open fss $\mu_{\mathcal{B}}$ in $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ such that $\eta_E \sqsubseteq \mu_{\mathcal{B}}$ and $\gamma_i(\mu_{\mathcal{B}}) \sqcap \rho_{\mathcal{C}} = \tilde{0}_{\mathcal{D}}$. Hence, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-semi normal-Čfs bicsp.

Definition 4.6 A Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be pairwise normal-Čfs bicsp (P-normal-Čfs bicsp), if for each pair of disjoint closed $fss's$ $\rho_{\mathcal{C}}$ and η_E for γ_i and γ_j respectively, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for γ_i and γ_j respectively, such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}}$.

Example 4.7 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$ and let $(\mathcal{A}_{\mathcal{S}})_i \in \mathcal{FS}(\mathcal{M}, \mathcal{D})$, $i = 1,2,3,4$, such that $(\mathcal{A}_{\mathcal{S}})_1 = \{(s_1, x_{0.5})\}, (\mathcal{A}_{\mathcal{S}})_2 = \{(s_2, x_{0.5})\}$,$(\mathcal{A}_{\mathcal{S}})_3 = \{(s_1, x_{0.5} \vee y_{0.5}), (s_2, x_1 \vee y_1)\}$ and $(\mathcal{A}_{\mathcal{S}})_4 = \{(s_1, x_1 \vee y_1), (s_2, x_{0.5} \vee y_{0.5})\}$. Define fuzzy soft closure operators $\mathcal{L}_1, \mathcal{L}_2 : \mathcal{FS}(\mathcal{M}, \mathcal{D}) \to \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_{\mathcal{S}}) = \gamma_2(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}\ , \\ (\mathcal{A}_{\mathcal{S}})_1 & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1, \\ (\mathcal{A}_{\mathcal{S}})_2 & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_2, \\ (\mathcal{A}_{\mathcal{S}})_1 \sqcup (\mathcal{A}_{\mathcal{S}})_2 & if \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1 \sqcup (\mathcal{A}_{\mathcal{S}})_2 \\ (\mathcal{A}_{\mathcal{S}})_3 & if \mathcal{A}_{\mathcal{S}} \in \left\{ \begin{matrix} (s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2}); \\ t_1, k_1 \leq 0.5, 0.5 < t_2, k_2 \leq 1 \end{matrix} \right\}, \\ (\mathcal{A}_{\mathcal{S}})_4 & if \mathcal{A}_{\mathcal{S}} \in \left\{ \begin{matrix} (s_1, x_{t_1} \vee y_{k_1}), (s_2, x_{t_2} \vee y_{k_2}); \\ t_2, k_2 \leq 0.5, 0.5 < t_1, k_1 \leq 1 \end{matrix} \right\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

Therefore, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-normal-Čfs bicsp. Since the only disjoint closed $fss's$ are $(\mathcal{A}_{\mathcal{S}})_1$, $(\mathcal{A}_{\mathcal{S}})_2$ for γ_1 and γ_2 respectively, and there exist disjoint γ_2-open fss $\mathcal{A}_{\mathcal{S}} = \{(s_1, x_{0.5} \vee y_{0.5})\}$ and γ_1-open fss $\mu_{\mathcal{B}} = \{(s_2, x_{0.5} \vee y_{0.5})\}$ such that $(\mathcal{A}_{\mathcal{S}})_1 \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $(\mathcal{A}_{\mathcal{S}})_2 \sqsubseteq \mu_{\mathcal{B}}$.

Theorem 4.8 Every closed Čfs bi-csubsp $(\mathcal{H}, \gamma_{1_{\mathcal{H}}}, \gamma_{2_{\mathcal{H}}}, \mathcal{D})$ of P-normal-Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is P -normal-Čfs bi- csubsp.

Proof: It is similar to the proof of Theorem 4.3.

Definition 4.9 The induced fs-bits $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ of a Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be P-normal-fs-bits, if for each pair of disjoint closed $fss's$ $\rho_{\mathcal{C}}$ and η_E for τ_{γ_i} and τ_{γ_j} respectively, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for τ_{γ_i} and τ_{γ_j} respectively, such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}}$.

Theorem 4.10 $(\mathcal{M}, \tau_{\gamma_1}, \tau_{\gamma_2}, \mathcal{D})$ is a P-normal-fs-bits if and only if $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-normal-Čfs bicsp.

Proof: The proof follows from the hypothesis and Proposition 2.15.

Theorem 4.11 Let $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be two Čfs bicsp's. If f_{up}: $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \to (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is $P\check{C}$-fs-continuous mapping, then $f_{up}^{-1}(\mu_{\mathcal{B}})$ is an γ_i-closed fss of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ for every γ_i^*-closed fss fuzzy soft set $\mu_{\mathcal{B}}$ of $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ $i = 1\ or\ 2$.

Proof: Let $\mu_{\mathcal{B}}$ be a γ_1^*-closed fss of $\mathcal{W}$. Therefore, from Theorem 2.20, we have $\gamma_1(f_{up}^{-1}(\mu_{\mathcal{B}})) \sqsubseteq f_{up}^{-1}(\gamma_1^*(\mu_{\mathcal{B}}))$. Since $\mu_{\mathcal{B}}$ is a γ_1^*-closed, we get $\gamma_1(f_{up}^{-1}(\mu_{\mathcal{B}})) \sqsubseteq f_{up}^{-1}(\mu_{\mathcal{B}})$. From Definition 2.9 part (A_2), $f_{up}^{-1}(\mu_{\mathcal{B}}) \sqsubseteq \gamma_1\left(f_{up}^{-1}(\mu_{\mathcal{B}})\right)$. This implies $\gamma_1\left(f_{up}^{-1}(\mu_{\mathcal{B}})\right) = f_{up}^{-1}(\mu_{\mathcal{B}})$. Hence, $f_{up}^{-1}(\mu_{\mathcal{B}})$ is a γ_1-closed fss of $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$. Similarly, when $i = 2$ we have $f_{up}^{-1}(\mu_{\mathcal{B}})$ is a γ_2- closed fss, for each $\mu_{\mathcal{B}}$ be a γ_2^*-closed fss.

Theorem 4.12 The property of being P-normal-Čfs bicsp is a topological property.

Proof: Let $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ and $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be two Čfs bicsp's and let $f_{up}:(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D}) \rightarrow (\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ be a $P\check{C}$-*fs*-homeomorphism mapping and $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-normal-Čfs bicsp. We want to show $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is also a P-normal-Čfs bicsp. Let $\rho_{\mathcal{C}}$ and η_E be disjoint closed $fss's$ in γ_i^* and γ_j^*, respectively. From hypothesis, f_{up} is $P\check{C}$-*fs*-homeomorphism mapping and from Theorem 4.11, we get $f_{up}^{-1}(\rho_{\mathcal{C}})$ and $f_{up}^{-1}(\eta_E)$ are closed $fss's$ in γ_i and γ_j respectively, such that $f_{up}^{-1}(\rho_{\mathcal{C}}) \sqcap f_{up}^{-1}(\eta_E) = \tilde{0}_{\mathcal{D}}$. But $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-normal-Čfs bicsp. This implies, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for γ_i and γ_j respectively, such that $f_{up}^{-1}(\rho_{\mathcal{C}}) \sqsubseteq \mu_{\mathcal{B}}$ and $f_{up}^{-1}(\eta_E) \sqsubseteq \mathcal{A}_{\mathcal{S}}$. It follows, $f_{up}(f_{up}^{-1}(\rho_{\mathcal{C}})) \sqsubseteq f_{up}(\mu_{\mathcal{B}})$ and $f_{up}(f_{up}^{-1}(\eta_E)) \sqsubseteq f_{up}(\mathcal{A}_{\mathcal{S}})$. Since f_{up} is $P\check{C}$-*fs*-homeomorphism mapping, then f_{up} is $\check{C}$-*fs*-open mapping, this yields there exist open $fss's$ $f_{up}(\mathcal{A}_{\mathcal{S}})$ and $f_{up}(\mu_{\mathcal{B}})$ in γ_i^* and γ_j^*, respectively, such that $\rho_{\mathcal{C}} \sqsubseteq f_{up}(\mu_{\mathcal{B}})$ and $\eta_E \sqsubseteq f_{up}(\mathcal{A}_{\mathcal{S}})$. Moreover, $f_{up}(\mathcal{A}_{\mathcal{S}}) \sqcap f_{up}(\mu_{\mathcal{B}}) = \tilde{0}_{\mathcal{N}}$. Hence, $(\mathcal{W}, \gamma_1^*, \gamma_2^*, \mathcal{N})$ is also a P-normal-Čfs bicsp.

Definition 4.13 A Čfs bicsp $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is said to be pairwise completely normal-Čfs bicsp (P-completely normal- Čfs bicsp), if for each pair of disjoint closed $fss's$ $\rho_{\mathcal{C}}$ and η_E for γ_i and γ_j respectively, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for γ_i and γ_j respectively, such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $\gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\mu_{\mathcal{B}}) = \tilde{0}_{\mathcal{D}}$.

Example 4.14 Let $\mathcal{M} = \{x, y\}$, $\mathcal{D} = \{s_1, s_2\}$, and let $(\mathcal{A}_{\mathcal{S}})_1 = \{(s_1, x_1 \vee y_1)\}$, $(\mathcal{A}_{\mathcal{S}})_2 = \{(s_2, x_1 \vee y_1)\}$, $(\mathcal{A}_{\mathcal{S}})_3 = \{(s_1, x_1 \vee y_1), (s_2, y_1)\}$. Define fuzzy soft closure operators $\gamma_1, \gamma_2 : \mathcal{FS}(\mathcal{M}, \mathcal{D}) \rightarrow \mathcal{FS}(\mathcal{M}, \mathcal{D})$ as follows:

$$\gamma_1(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{S}}, \\ (\mathcal{A}_{\mathcal{S}})_1 & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1, \\ (\mathcal{A}_{\mathcal{S}})_2 & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_2, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

And

$$\gamma_2(\mathcal{A}_{\mathcal{S}}) = \begin{cases} \tilde{0}_{\mathcal{D}} & if\ \mathcal{A}_{\mathcal{S}} = \tilde{0}_{\mathcal{D}}, \\ (\mathcal{A}_{\mathcal{S}})_1 & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1, \\ (\mathcal{A}_{\mathcal{S}})_2 & if\ \mathcal{A}_{\mathcal{S}} \sqsubseteq (\mathcal{A}_{\mathcal{S}})_2, \\ (\mathcal{A}_{\mathcal{S}})_3 & if\ \mathcal{A}_{\mathcal{S}} \sqsubset (\mathcal{A}_{\mathcal{S}})_3; \mathcal{A}_{\mathcal{S}} = \{(s_1, x_{t_1} \vee y_1), (s_2, y_{k_2}), t_1, k_2 \in I_0\}, \\ \tilde{1}_{\mathcal{D}} & otherwise. \end{cases}$$

Therefore, $(\mathcal{M}, \gamma_1, \gamma_2, \mathcal{D})$ is a P-completely normal-Čfs bicsp. Since the only disjoint closed $fss's$ are $(\mathcal{A}_{\mathcal{S}})_1$ and $(\mathcal{A}_{\mathcal{S}})_2$ for γ_1 and γ_2 respectively, and there exists disjoint γ_2-open fss $(\mathcal{A}_{\mathcal{S}})_1$ and γ_1-open $(\mathcal{A}_{\mathcal{S}})_2$, such that $(\mathcal{A}_{\mathcal{S}})_1 \sqsubseteq (\mathcal{A}_{\mathcal{S}})_1$, $(\mathcal{A}_{\mathcal{S}})_2 \sqsubseteq (\mathcal{A}_{\mathcal{S}})_2$ and $\gamma_1(\mathcal{A}_{\mathcal{S}})_1 \sqcap \gamma_2(\mathcal{A}_{\mathcal{S}})_2 = \tilde{0}_{\mathcal{S}}$.

Proposition 4.15 Every P-completely normal- Čfs bicsp is P-normal-Čfs bicsp.

Proof: Suppose $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is P-completely normal-Čfs bicsp and let $\rho_{\mathcal{C}}$, η_E be any disjoint closed $fss's$ in γ_i and γ_j respectively. From the hypothesis, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for γ_i and γ_j respectively, such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $\gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\mu_{\mathcal{B}})=\tilde{0}_{\mathcal{D}}$. By using (A_2) of Definition 2.9, we have $\mathcal{A}_{\mathcal{S}} \sqcap \mu_{\mathcal{B}}=\tilde{0}_{\mathcal{D}}$. Thus, $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is P-normal-Čfs bicsp.

Remark 4.16 The converse of the above proposition is not true, as in Example 4.7. Since $\gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\mu_{\mathcal{B}})=(\mathcal{A}_{\mathcal{S}})_3 \sqcap (\mathcal{A}_{\mathcal{S}})_4 = \{(s_1, x_{0.5} \vee y_{0.5}), (s_2, x_{0.5} \vee y_{0.5})\} \neq \tilde{0}_{\mathcal{D}}$.

Next, we show that the hereditary property is satisfied under closed Čfs bi-csubsp.

Theorem 4.17 Every closed Čfs bi-csubsp $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ of P-completely normal-Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is P-completely normal-Čfs bi-csubsp.

Proof: Let $\rho_{\mathcal{C}}$, η_E be any two disjoint closed $fss's$ in $\gamma_{i_{\mathcal{H}}}$ and $\gamma_{j_{\mathcal{H}}}$ respectively. Then, by Lemma 3.3, $\rho_{\mathcal{C}}$, η_E are disjoint closed $fss's$ in γ_i and γ_j, respectively. But $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is P-completely normal-Čfs bicsp, then there exist $\mathcal{A}_{\mathcal{S}}, \mu_{\mathcal{B}}$ disjoint open $fss's$ in γ_i and γ_j respectively such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $\gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\mu_{\mathcal{B}})=\tilde{0}_{\mathcal{D}}$. By Lemma 2.18, $\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ and $\mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ are open $fss's$ in $\gamma_{i_{\mathcal{H}}}$ and $\gamma_{j_{\mathcal{H}}}$ respectively, such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}$. To complete the proof, we must show $\gamma_{i_{\mathcal{H}}}\left(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}\right) \sqcap \gamma_{j_{\mathcal{H}}}\left(\mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}\right)=\tilde{0}_{\mathcal{D}}$.

Now,
$$\begin{aligned}\gamma_{i_{\mathcal{H}}}\left(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}\right) \sqcap \gamma_{j_{\mathcal{H}}}\left(\mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}\right) &= \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_i\left(\mathcal{A}_{\mathcal{S}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}\right) \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_j\left(\mu_{\mathcal{B}} \sqcap \widetilde{\mathcal{H}}_{\mathcal{D}}\right)\\ &\sqsubseteq \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_i\left(\widetilde{\mathcal{H}}_{\mathcal{D}}\right) \sqcap \gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j\left(\widetilde{\mathcal{H}}_{\mathcal{D}}\right) \sqcap \gamma_j(\mu_{\mathcal{B}})\\ &= \widetilde{\mathcal{H}}_{\mathcal{D}} \sqcap \gamma_i(\mathcal{A}_{\mathcal{S}}) \sqcap \gamma_j(\mu_{\mathcal{B}})\\ &= \tilde{0}_{\mathcal{D}}.\end{aligned}$$

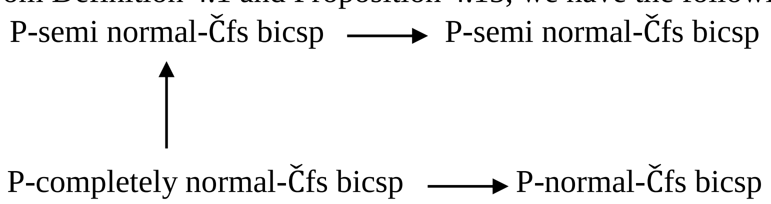
Hence, $(\mathcal{H},\gamma_{1_{\mathcal{H}}},\gamma_{2_{\mathcal{H}}},\mathcal{D})$ is P-completely normal-Čfs bi- csubsp.

Definition 4.18 The induced fs-bits $(\mathcal{M},\tau_{\gamma_1},\tau_{\gamma_2},\mathcal{D})$ of a Čfs bicsp $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is said to be P-completely normal-fs-bits, if for each pair of disjoint closed $fss's$ $\rho_{\mathcal{C}}$ and η_E for τ_{γ_i} and τ_{γ_j} respectively, there exist disjoint open $fss's$ $\mathcal{A}_{\mathcal{S}}$ and $\mu_{\mathcal{B}}$ for τ_{γ_i} and τ_{γ_j} respectively, such that $\rho_{\mathcal{C}} \sqsubseteq \mu_{\mathcal{B}}$ and $\eta_E \sqsubseteq \mathcal{A}_{\mathcal{S}}$ and $\tau_{\gamma_i}\text{-}cl(\mathcal{A}_{\mathcal{S}}) \sqcap \tau_{\gamma_j}\text{-}cl(\mu_{\mathcal{B}})=\tilde{0}_{\mathcal{D}}$.

Theorem 4.19 If $(\mathcal{M},\tau_{\gamma_1},\tau_{\gamma_2},\mathcal{D})$ is P-completely normal-fs-bits. Then, $(\mathcal{M},\gamma_1,\gamma_2,\mathcal{D})$ is also P-completely normal-Čfs bicsp.

Proof: The proof follows from the hypothesis and Proposition 2.15.

Remark 4.20: From Definition 4.1 and Proposition 4.15, we have the following diagram:



5. Conclusion

Researchers are highly interested in fuzzy soft sets. This work is more general than fuzzy and soft sets can be used in a variety of ways. In this paper, some new kinds of pairwise regularity and normality in Čech fuzzy soft bi-closure spaces and their induced fuzzy soft bitopological spaces have been introduced and studied as well as the relationships between them are also studied. We have been proved that hereditary property satisfies under closed Čech fuzzy soft bi-closure spaces in all of these kinds of axioms.

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