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The dual notions of semi-essential submodules and semi-uniform modules

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Abstract
The purpose of this paper is to introduce dual notions of two known concepts which are semi-essential submodules and semi-uniform modules. We call these concepts; cosemi-essential submodules and cosemi-uniform modules respectively. Also, we verify that these concepts form generalizations of two well-known classes; coessential submodules and couniform modules respectively. Some conditions are considered to obtain the equivalence between cosemi-uniform and couniform. Furthermore, the relationships of cosemi-uniform module with other related concepts are studied, and some conditional characterizations of cosemi-uniform modules are investigated.

Keywords: Semi-essential submodules, P-small submodules, Cosemi-essential submodules, Cosemi-uniform modules.

المقاسات الرديفة للمقاسات الجزئية شبه الجوهرية والمقاسات شبه المنتظمة

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الخلاصة
إن الهدف من هذا البحث هو إيجاد ردائف لمقاسات معروفة هي المقاسات الجزئية شبه الجوهرية والمقاسات شبه المنتظمة التي قدمت من قبل كل من مجباس وعبدالله في بحثهما عام 2009. أطلقنا عليهما اسم المقاسات الجزئية الرديفة للمقاسات الجزئية شبه الجوهرية والمقاسات الرديفة للمقاسات شبه المنتظمة . كذلك بيَنا بأن هذين المفهومين أكبر من مفهومين معروفين آخرين هما المقاسات الجزئية الرديفة للمقاسات الجزئية الجوهرية (coessential submodules) والمقاسات الرديفة للمقاسات المنتظمة (couniform modules) على التوالي.

برهنا عدد من القضايا التي تتعلق بالخصائص المهمة لهذين الصنفين، كما تم البرهنة على عدد من القضايا حول التكافؤ المشروط بين المقاسات الرديفة للمقاسات المنتظمة و المقاسات الرديفة للمقاسات شبه المنتظمة، إضافة الى ذلك درسنا علاقة المقاسات الرديفة للمقاسات شبه المنتظمة ببعض المقاسات الأخرى.

1. Introduction

Throughout this article, all rings are commutative with non-zero identity, and all modules are unitary left R-modules. "A submodule V of an R-module U is called essential (denoted by $V\leq_e U$), provided that for each non-zero submodule L of U, $V\cap L\neq 0$ [1]". In 2009 Mijbass and Abdullah introduced the class of semi-essential submodule as a generalization of essential submodule. "A submodule P of U is called prime, if whenever $ru \in P$ for $r \in R$ and $u \in U$, then either $u \in P$ or $r \in (P:U)$ [2], where $(P:U)=\{r\in R|\ rU\subseteq P\}$, and a non-zero submodule V of U is said to be semi-essential, if $V \cap P \neq 0$ for each non-zero prime submodule P of U [3]. A non-zero module U is called uniform, if every non-zero submodule V of U is essential; that is $V\cap L\neq 0$ for every non-zero submodule L of U

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[1], and a non-zero R-module U is called semi-uniform if every non-zero submodule of U is semi-essential [3]. A submodule V of U is called small (denoted by V≪U), if V+W≠U for every proper submodule W of U [1]". Hadi and Ibrahiem introduced the class of P-small submodule, where "a proper submodule V of an R-module U is called P-small (simply $V \ll_P U$), if V+P≠U for every prime submodule P of U [4]".

The latter definition motivates us to construct the class of cosemi-essential submodules and cosemi-uniform module as dual notions of semi-essential submodules and semi-uniform module respectively. A non-zero submodule A is said to be cosemi-essential submodule of V in U if $\frac{V}{A}$ is a P-small submodule of $\frac{U}{A}$. A non-zero R-module U is called cosemi-uniform, if every proper submodule V of U is either zero or there exists a proper submodule S of V such that $\frac{V}{S} \ll_P \frac{U}{S}$.

This article consists of four sections; in section 2; we dualize the notion of semi-essential submodule; named it cosemi-essential submodule, we dualize some main properties of semi-essential submodules which appeared in [5] and [3]; see the results (2.10), (2.11), (2.12) and (2.13). Also, more other useful results are investigated, for example; we give condition under which cosemi-essential and coessential be equivalent, see proposition (2.8), where "a submodule S is called coessential submodule of V in U (denoted by $S \leq_{ce} V$ in U), if $\frac{V}{S} \ll \frac{U}{S}$ [6]". Section 3; is devoted to introduce a class of modules named cosemi-uniform module as a dual notion of semi-uniform modules, various properties of cosemi-uniform modules are given. This class of modules contains properly the class of couniform modules, "where a non-zero module U is called couniform, if every proper submodule V of U is either (0) or there exists a proper submodule S of V such that $\frac{V}{S} \ll \frac{U}{S}$ [6]"; that is for each proper submodule V of U, either V = (0) or there exists a proper submodule S of V such that S is coessential submodule of V in U. We investigate some conditions under which the class of cosemi-uniform coincides with the class of couniform modules; see propositions (3.8), (3.10), (3.11) and (3.12). Furthermore, in corollary (3.13), we verify that in the category of rings there is no difference between the two concepts cosemi-uniform and couniform. In section 4; we study the relationships of cosemi-uniform with other related concepts such as hollow, Pr-hollow and epiform modules, where an R-module U is called epiform, if every nonzero homomorphism f: U $\longrightarrow \frac{U}{K}$ with K a proper submodule of U is an epimorphism [6], and a non-zero R-module U is called hollow if every proper submodule of U is small [7]. A non-zero module U is called Pr-hollow, if every prime submodule of U is small [7]". The concept of Pr-hollow modules is a generalization of hollow modules. "A non-zero R-module U is called semi-uniform, if every non-zero submodule of U is semi-essential [3]". The hollow module is a dual notion of the uniform module; we will see that Pr-hollow module is a dual notion of semi-uniform module. Also we give some characterizations of cosemi-uniform modules under certain conditions; see theorems (4.11), (4.12), (4.13).

2. Cosemi-essential Submodules

This section is devoted to introduce the class of cosemi-essential submodules as a dual notion of semi-essential submodules. Before that we need to recall the following definition.

Definition (2.1): [4]

"A proper submodule V of an R-module U is called P-small (simply $V \ll_P U$), if V+P≠U for every prime submodule P of U".

This definition together with the concept of coessential submodule has been a motivation to introduce the following new concept.

Definition (2.2): A submodule A of an R-module U is called cosemi-essential submodules of V in U (simply $A \leq_{cosm} V$ in U), if $\frac{V}{A} \ll_P \frac{U}{A}$. An ideal I of a ring R is cosemi-essential of J in R if I is a cosemi-essential R-submodule of J in R.

Examples and Remarks (2.3):

1. For the Z-module Z_4, $(\bar{0}) \leq$ cosm $(\bar{2})$ in Z_4, since $\frac{(\bar{2})}{(\bar{0})} \ll_P \frac{Z_4}{(\bar{0})}$. In fact the only prime submodule of $\frac{Z_4}{(\bar{0})}$ is $\frac{(\bar{2})}{(\bar{0})}$, and $\frac{(\bar{2})}{(\bar{0})} + \frac{(\bar{2})}{(\bar{0})} \neq \frac{Z_4}{(\bar{0})}$.
2. $(\bar{0}) \nleq$cosm $(\bar{2})$ in Z_6, since there exists a prime submodule $(\bar{3})$ of Z_6 such that $\frac{(\bar{2})}{(\bar{0})} + \frac{(\bar{3})}{(\bar{0})} = \frac{Z_6}{(\bar{0})}$.

3. For any submodules A and B of U such that A⊆B⊆U; if $A\leq_{ce}B$ in U, then $A\leq_{cosm}B$ in U. We think the converse is not true in general, but we don't have example verifying that.

Proof (3): It is clear, since every small submodule is P-small.

4. A submodule V of an R-module U is P-small if and only if $(0)\leq_{cosm}V$ in U.

Proof (4): Since $V\ll_P U$, then $\frac{V}{(0)}\ll_P\frac{U}{(0)}$, that is $(0)\leq_{cosm}V$ in U. The converse is clear.

5. Consider the Z-module of the rational numbers Q. Note that every submodule V of Q is P-small [4], so according to (2.3)(4), we get $(0)\leq_{cosm}V$ in Q for every submodule V of Q.
6. For any submodule V of U, $V\leq_{cosm}V$ in U, since $\frac{V}{V}\ll_P\frac{U}{V}$. In fact since $\frac{V}{V}\cong(0)$ and $\frac{V}{V}+\frac{W}{V}\neq\frac{U}{V}$ for each prime submodule $\frac{W}{V}$ of $\frac{U}{V}$.
7. Consider the set of rational number Q. Since every submodule of $\frac{Q}{Z}$ is a P-small [4, Prop.(1.3)], then $Z\leq_{cosm}L$ in Q, for every submodule L of Q.
8. If U is an R-module such that A≤ B≤ V≤ U, and $B\leq_{cosm}V$ in U, then $A\leq_{cosm}V$ in U.

Proof (8): Let A≤ B, and $B\leq_{cosm}V$ in U, so $\frac{V}{B}\ll_P\frac{U}{B}$. This implies that $\frac{V/B}{B/A}\ll_P\frac{U/B}{B/A}$ [4, Prop.(1.3)], hence $\frac{V}{A}\ll_P\frac{U}{A}$, thus $A\leq_{cosm}V$ in U.

9. A semisimple module has no cosemi-essential submodule.

In the following property we prove the transitive property of cosemi-essential submodules by using the class of coessential submodule.

Proposition (2.4): For any chain of submodules A≤ B≤ C≤ U of an R-module U, if $A\leq_{cosm}B$ in U and $B\leq_{ce}C$ in U, then $A\leq_{cosm}C$ in U.

Proof: Suppose there exists a prime submodule $\frac{P}{A}$ of $\frac{U}{A}$ such that $\frac{C}{A}+\frac{P}{A}=\frac{U}{A}$, then C+P = U. This implies that $\frac{C}{B}+\frac{P+B}{B}=\frac{U}{B}$. Since $B\leq_{ce}C$ in U, therefore $\frac{P+B}{B}=\frac{U}{B}$, and so that P+B = U. Hence $\frac{P}{A}+\frac{B}{A}=\frac{U}{A}$. But $A\leq_{cosm}B$ in U, so we have a contradiction, therefore $A\leq_{cosm}C$ in U.

For the converse of the proposition (2.4), we have the following, before that we need the following lemma.

Lemma (2.5): [4]

"Let U and U_1 be R-modules, and f: U→ U_1 be R-homomorphism. If $A\ll_P U$, then $f(A)\ll_P U_1$".

Proposition (2.6): For the chain of submodules A≤ B≤ C≤ U of an R-module U. If $A\leq_{cosm}C$ in U, then $A\leq_{cosm}B$ in U and $B\leq_{cosm}C$ in U.

Proof: Since $A\leq_{cosm}C$ in U, then $\frac{C}{A}\ll_P\frac{U}{A}$. But $\frac{B}{A}\subseteq\frac{C}{A}$, thus $\frac{B}{A}\ll_P\frac{U}{A}$ [4, Remark (1.2)(3)]. That is $A\leq_{cosm}B$ in U. Now, define h: $\frac{U}{A}\to\frac{U}{B}$ by h(u+A) = u+B ∀u+A∈ $\frac{U}{A}$. It is clear that h is an epimorphism, therefore $h(\frac{C}{A})=\frac{C}{B}$. On the other hand, by lemma (2.5) we get $h(\frac{C}{A})\ll_P\frac{U}{B}$ thus $\frac{C}{B}\ll_P\frac{U}{B}$. That is $B\leq_{cosm}C$ in U.

We need the following lemma.

Lemma (2.7): [4, Cor. (1.10)]

"Let U be an R-module, and A be a finitely generated proper submodule of U, then $A\ll_P U$ if and only if A≪U".

Proposition (2.8): Let U be an R-module and A⊆ B⊆ U such that B is finitely generated, then $A\leq_{cosm}B$ in U if and only if $A\leq_{ce}B$ in U.

Proof: Assume that A is a cosemi-essential submodule of B in U, then $\frac{B}{A}\ll_P\frac{U}{A}$. Since B is a finitely generated of U, so clearly $\frac{B}{A}$ is also finitely generated. By lemma (2.7), $\frac{B}{A}\ll\frac{U}{A}$, that is A is a coessential submodule of B in A. The converse is clear.

Proposition (2.9): For any chain of submodules A⊆B⊆U, the following statements are satisfied:

1. If $A\leq_{cosm}B$ in U, then $B\ll_P U$, provided that A contained properly in any prime submodule of U.
2. If B=A+T and $T\ll_P U$, then $A\leq_{cosm}B$ in U.

Proof:

1. Let X be a prime submodule of U such that B+X=U. By assumption A⊊X, so that $\frac{X}{A}$ is a proper submodule of $\frac{U}{A}$, and by [2, Prop.(3.8)], $\frac{X}{A}$ is a prime submodule of $\frac{U}{A}$. But $A\leq_{cosm}B$ in U, therefore $\frac{B}{A}+\frac{X}{A}=\frac{U}{A}$ which is a contradiction. This implies that B+X≠U, hence $B\ll_P U$.
2. Let $\frac{V}{A}$ be a prime submodule of U. Assume that $\frac{B}{A}+\frac{V}{A}=\frac{U}{A}$, then B+V=U. By assumption A+T=B, then A+T+V=U. Since A⊆V, so that T+V=U which is a contradiction since $T\ll_P U$ and V is a prime submodule of U. Thus $\frac{B}{A}+\frac{V}{A}\neq\frac{U}{A}$ for every prime submodule $\frac{V}{A}$ of $\frac{U}{A}$, and hence $A\leq_{cosm}B$ in U.

The following proposition shows that the quotient of cosemi-essential submodule is cosemi-essential.

Proposition (2.10): For any submodules A and B of an R-module U, if $A\leq_{cosm}B$ in U then $\frac{A}{L}\leq_{cosm}\frac{B}{L}$ in $\frac{U}{L}$ for every submodule L of A.

Proof: Assume that $A\leq_{cosm}B$ in U, then $\frac{B}{A}\ll_P\frac{U}{A}$. For every submodule L of A, we have $\frac{A}{L}\leq\frac{B}{L}\leq\frac{U}{L}$, and by lemma (2.5), $\frac{B/L}{A/L}\ll_P\frac{U/L}{A/L}$. That is $\frac{A}{L}\leq_{cosm}\frac{B}{L}$ in $\frac{U}{L}$, and we are done.

Under a certain condition, we can generalize proposition (2.10), as the following proposition shows.

Proposition (2.11): Let f: U→ U′ be an R-homomorphism, where U and U′ be R-modules. If $A\leq_{cosm}B$ in U, then $f(A)\leq_{cosm}f(B)$ in U′, provided that A⊆ rad(U), where rad(U) is the prime radical of U.

Proof: Assume that $A\leq_{cosm}B$ in U, and let $\frac{L'}{f(A)}$ be a prime submodule of $\frac{U'}{f(A)}$ such that $\frac{f(B)}{f(A)}+\frac{L'}{f(A)}=\frac{U'}{f(A)}$. Then f(B)+ L′ = U′. We claim that B+ f^{-1}(L′) = U. To see that, let u∈U, then f(u)∈ U′= f(B) + L′, so ∃t∈L′ and ∃b∈B such that f(u) = f(b) + t. This implies that f(u-b)=t, hence u-b= f^{-1}(t), that is u∈B+ f^{-1}(L′). Now, since L′ is a prime submodule of U, then f^{-1}(L′) is also prime submodule of U [2, Prop.(3.8)]. On the other hand, A⊆ rad(U), so by definition of rad(U) we have A⊆ f^{-1}(L′). Now, $\frac{B}{A}+\frac{f^{-1}(L)}{A}=\frac{U}{A}$. Again by [2, prop.(3.8)] $\frac{f^{-1}(L)}{A}$ is a prime submodule of $\frac{U}{A}$. But $A\leq_{cosm}B$ in U, therefore $\frac{f^{-1}(L)}{A}=\frac{U}{A}$ which is a contradiction since $\frac{f^{-1}(L)}{A}$ is proper, thus $f(A)\leq_{cosm}f(B)$ in U′.

"Recall that an R- module U is called multiplication if every submodule V of U can be written as the form V=IU for some ideal I of R [8]".

Theorem (2.12): Let U be a finitely generated faithful and multiplication module. Then $A\leq_{cosm}B$ in R if and only if $AU\leq_{cosm}BU$ in U, for all ideals A and B of R.

Proof: Assume that $A\leq_{cosm}B$ in R, and we have to show that $\frac{BU}{AU}\ll_P\frac{U}{AU}$. If that is not true, then there exists a prime submodule $\frac{V}{AU}$ of $\frac{U}{AU}$ such that $\frac{BU}{AU}+\frac{V}{AU}=\frac{U}{AU}$, hence BU+V=U. By [2, Prop.(3.8)], V is a prime submodule of U, and since U is multiplication, so there exists a prime ideal C of R such that V=CU [8, Cor.(2.11)]. Now, (B+C)U=RU. Since U is finitely generated, faithful and multiplication, so B+C=R [8, Th.(3.1)]. This implies that $\frac{B}{A}+\frac{C}{A}=\frac{R}{A}$. But $A\leq_{cosm}B$ in R, thus $\frac{C}{A}=\frac{R}{A}$ which is a contradiction since $\frac{C}{A}$ is proper. Therefore $\frac{BU}{AU}\ll_P\frac{U}{AU}$, that is $AU\leq_{cosm}BU$ in U. Conversely, we have to show that $\frac{B}{A}\ll_P\frac{R}{A}$. Assume that $A\nleq_{cosm}B$ in R, so there exists a prime ideal $\frac{C}{A}$ of $\frac{R}{A}$ such that:

$$\frac{B}{A}+\frac{C}{A}=\frac{R}{A}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(1)$$

Hence B+C=R, this implies that (B+C)U=RU, that is:

$$BU+CU=RU=U\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(2)$$

Note that CU≠RU since otherwise, we get C=R which is a contradiction. Moreover, since U is a faithful and multiplication module, then CU is a prime ideal of R [8, Cor.(2.11)]. Now by using (2), we can write (1) as follows:

$$\frac{BU}{AU}+\frac{CU}{AU}=\frac{U}{AU}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\quad(3)$$

But $AU\leq_{cosm}BU$ in U, then $\frac{BU}{AU}\ll_P\frac{U}{AU}$. Thus by using (3) we get $\frac{CU}{AU}=\frac{U}{AU}$ which is a contradiction, since $\frac{CU}{AU}$ is proper submodule of $\frac{U}{AU}$. Thus $\frac{B}{A}\ll_P\frac{R}{A}$, that is $A\leq_{cosm}B$ in R.

"Recall that a submodule A of an R-module U is called a supplement of B, where B is a submodule of U; if A is a minimal with the property B+A=U [1]".

Proposition (2.13): Let A be a supplement of B in an R-module U. If there exists a submodule T of B such that U=A+T, then T $\leq_{cosm}$ B in U.

Proof: In order to prove T $\leq_{cosm}$ B in U, we must show that $\frac{B}{T} \ll_P \frac{U}{T}$. Assume that $\frac{L}{T}$ be a prime submodule of $\frac{U}{T}$ such that $\frac{B}{T}+\frac{L}{T}=\frac{U}{T}$. Then U=B+L=B+(U∩L). But U=A+T, then U=B+((A+T)∩L). Since T⊆L, so by Modular Law U=(B+(A∩L)+T), hence U=B+(A∩L). Since A is a supplement of B in U, therefore A∩L=A, hence A⊆L. Thus U=L+T=L. This implies that $\frac{L}{T}=\frac{U}{T}$, which is a contradiction since $\frac{L}{T}$ is a proper submodule, thus T $\leq_{cosm}$ B in U.

3. Cosemi-uniform modules

Following [6], Hadi and Ahmed introduced a dual notion for uniform modules named couniform modules, where "a non-zero module U is said to be couniform, if every proper submodule V of U is either zero or there exists S $\subsetneqq$ V such that $\frac{V}{S} \ll \frac{U}{S}$ ".

In this section we introduce a dual notion of semi-uniform modules, which is analogue of couniform module, named a cosemi-uniform module.

Definition (3.1): A non-zero R-module U is called cosemi-uniform, if every proper submodule V of U is either zero or there exists a proper submodule S of V such that $\frac{V}{S} \ll_P \frac{U}{S}$. A ring R is called cosemi-uniform if R is a cosemi-uniform R-module.

Examples and Remarks (3.2):

1. It is clear that every couniform module is cosemi-uniform. The converse is not true in general as we will see later; see example (4,8).

2. The Z-module Z is a cosemi-uniform module, since Z is a couniform module [6].

3. According to remark (2.3)(9), A semisimple module U is not cosemi-uniform module. In fact every non-zero proper submodule V of U doesn't have a proper submodule S such that $\frac{V}{S} \ll_P \frac{U}{S}$, since V is a direct summand of U and the only small submodule of U is (0).

4. Every chained module is cosemi-uniform module. In fact if A is a proper submodule of U, then A is either A=(0) or A≠(0). If A≠(0), then $\frac{A}{(0)} \ll_P \frac{U}{(0)}$. In particular, Z_8 is cosemi-uniform Z-module. Note that Z_8 is also couniform [6].

5. Q as Z-module is a cosemi-uniform module, "where Q is the field of rational numbers". Since in Q every proper submodule is P-small, and the result follows by lemma (2.5).

6. Every simple module is a cosemi-uniform module.

7. An epimorphic image of cosemi-uniform module need not be cosemi-uniform module, for example; consider the natural epimorphism π: Z→ $\frac{Z}{10Z}$ from Z-module Z to the quotient $\frac{Z}{10Z}$. Note that Z is cosemi-uniform module, while $\pi(Z)=\frac{Z}{10Z}\simeq Z_{10}$, and by (3), Z_{10} is not cosemi-uniform since it is semisimple

The following proposition is about the direct summand of cosemi-uniform modules.

Proposition (3.3): Let $U=U_1\oplus U_2$ be R-module, where U_1 and U_2 are R-modules. If U is cosemi-uniform, then U_1 and U_2 are cosemi-uniform modules.

Proof: Let $0 \neq L \subsetneqq U_1$, so L$\subsetneqq$U. But U is a cosemi-uniform module, then there exists a proper submodule S_1 of L such that:

$$\frac{L}{S_1} \ll_P \frac{U}{S_1}=\frac{U_1\oplus U_2}{S_1}=\frac{U_1}{S_1}\oplus\frac{S_1+U_2}{S_1}$$

Therefore:

$$\frac{L}{S_1} \ll_P \frac{U_1}{S_1}\oplus\frac{S_1+U_2}{S_1} \quad\ldots\ldots\ldots\ldots (1)$$

The step (1) can be written as follows:

$$\frac{L}{S_1}\oplus(0) \ll_P \frac{U_1}{S_1}\oplus\frac{S\oplus U_2}{S_1} \quad\ldots\ldots\ldots\ldots (2)$$

But $\frac{L}{S_1}\subseteq\frac{U_1}{S_1}$ so from (2) and by [4, Remark (1.2)(4)], we conclude the following:

$\frac{L}{S_1} \ll_P \frac{U_1}{S_1}$

Thus U_1 is a cosemi-uniform module. In similar way we can prove that U_2 is a cosemi-uniform module.

Remark (3.4): The direct sum of cosemi-uniform module may not be cosemi-uniform, for example both of Z_2 and Z_3 are cosemi-uniform module, but the direct sum of them is isomorphic to Z_6 which is not cosemi-uniform since Z_6 is a semisimple module, see remark (3.2)(3).

The following theorem gives the hereditary of the cosemi-uniform property.

Theorem (3.5): Any finitely generated faithful and multiplication R-module U is cosemi-uniform if and only if R is a cosemi-uniform ring.

Proof: **⇒)** Assume that U is a cosemi-uniform R-module, and let A be a non-zero proper ideal of R. To find a non-zero proper ideal A_1 of R such that such that $\frac{A}{A_1} \ll_P \frac{R}{A_1}$. Put AU= V, then V is a non-zero proper submodule of U. Since U is a cosemi-uniform module, then there exists a submodule V_1 of V such that such that:

$$\frac{V}{V_1} \ll_P \frac{U}{V_1} \quad \ldots\ldots (1)$$

Now, since U is multiplication then U=RU, and since $V_1 \leq V \leq U$, thus $V_1 = A_1U$ for some ideal A_1 of R [8]. So we can put (1) as follows:

$$\frac{AU}{A_1U} \ll_P \frac{RU}{A_1U} \quad \ldots\ldots (2)$$

We claim that $\frac{A}{A_1} \ll_P \frac{R}{A_1}$. If that is not true, then there exists a prime ideal $\frac{C}{A_1}$ of $\frac{R}{A_1}$ such that $\frac{A}{A_1} + \frac{C}{A_1} = \frac{R}{A_1}$, that is:

$$\frac{A+C}{A_1} = \frac{R}{A_1} \quad \ldots\ldots (3)$$

This implies that $\frac{(A+C)U}{A_1U} = \frac{AU+CU}{A_1U} = \frac{AU}{A_1U} + \frac{CU}{A_1U} = \frac{RU}{A_1U}$, hence:

$$\frac{V}{V_1} + \frac{CU}{V_1} = \frac{U}{V_1} \quad \ldots\ldots (4)$$

Since $\frac{C}{A_1}$ is a prime ideal of $\frac{R}{A_1}$, so C is also prime ideal of R [2]. Note that CU≠U, in fact if CU=U, and since U is finitely generated faithful and multiplication then C=R, which is a contradiction since C is a proper ideal of R. Thus CU is a prime submodule of U [8]. Again by [2], $\frac{CU}{V_1}$ is a prime submodule of $\frac{U}{V_1}$. This implies that $\frac{V}{V_1} \ll_P \frac{U}{V_1}$ which is a contradiction, thus $\frac{A}{A_1} \ll_P \frac{R}{A_1}$.

⇐) Let V be a non-zero proper submodule of U, we have to find a non-zero proper submodule W of U such that $\frac{V}{W} \ll_P \frac{U}{W}$. By assumption there exists an ideal A of R such that V = AU [8]. Since U is a finitely generated multiplication module, then A is a non-zero proper ideal of R [8]. But R is a cosemi-couniform ring, so ∃ B⪇A such that $\frac{A}{B} \ll_P \frac{R}{B}$. It is clear that BU≠AU. We claim that $\frac{AU}{BU} \ll_P \frac{RU}{BU}$. If not, then there exists a prime submodule $\frac{V}{BU}$ of $\frac{U}{BU}$ such that:

$$\frac{AU}{BU} + \frac{V}{BU} = \frac{U}{BU} \quad \ldots\ldots (1)$$

Hence:

$$AU + V = U \quad \ldots\ldots (2)$$

Since U is a multiplication, so there exists a prime ideal C of R such that V=CU [8, Cor.(2.11)]. From (2) we get AU + CU = U, hence (A+C) U=U. Since U is a finitely generated faithful and multiplication, therefore A+C=R [8]. This implies that $\frac{A}{B} + \frac{C}{B} = \frac{R}{B}$. By [2], $\frac{C}{B}$ is a prime ideal of $\frac{R}{B}$. So $\frac{A}{B}$ is not P-small submodule of $\frac{R}{B}$ which is a contradiction, therefore $\frac{AU}{BU} \ll_P \frac{RM}{BU}$. Put W=BU, then $\frac{V}{W} \ll_P \frac{U}{W}$, thus U is cosemi-uniform.

The concept of fully essential submodules was appeared in [5], where "a non-zero module U is called fully essential if every non-zero semi-essential submodule of U is essential". We give analogue concept named fully coessential.

Definition (3.6): A non-zero module U is called fully coessential, if every non-zero proper cosemi-essential submodule of U is coessential. That is for every $0\neq W\lneqq V\leq U$, if $W\leq_{cosm}V$ in U, then $W\leq_{ce}V$ in U. A ring R is called fully co essential if R is fully coessential R-module.

Examples (3.7):

1. Every couniform module is fully coessential.
2. Z is fully coessential module, since Z is a couniform module [6].
3. Since every hollow module is couniform [6], so every hollow module is fully coessential.
4. Z_4 is a fully coessential module, see example (2.3)(1).

As we mentioned in remark (3.2)(1) that every couniform is cosemi-uniform, and we need some information to verify the converse is not true in general; see example (4.8). However, the converse is true under the class of fully coessential modules as the following proposition shows.

Proposition (3.8): Let U be a fully coessential module. Then U is a couniform module if and only if U is cosemi-uniform.

Proof: The necessity is clear. Conversely, if U is a cosemi-uniform module then every proper submodule V of U is either zero or there exists a proper submodule S of V such that $S\leq_{cosm}V$ in U. since U is a fully coessential module, then $S\leq_{ce}V$ in U, hence we are through.

"Recall that an R- module U is called fully prime if every proper submodule of U is a prime [5]". The authors in [5] proved that for any fully prime module; the class of uniform modules coincides with the class of semi-uniform module. As a dual of this statement we give the following, before that we need to give the following useful lemma which can be easily proved.

Lemma (3.9): If U is a fully prime R-module, then $\frac{U}{V}$ is fully prime module for every submodule V of U.

Proposition (3.10): Let U be a fully prime module. Then U is a couniform module if and only if U is cosemi-uniform module.

Proof: The necessity follows by remark (3.2)(1). For the sufficiency; suppose that U is a cosemi-uniform module, and let V be a proper submodule of U. If V is zero, then we are done. If $V\neq 0$, and since U is a cosemi-uniform module, then there exists a proper submodule S of U such that $\frac{V}{S}\ll_P \frac{U}{S}$, i.e $\frac{V}{S}+\frac{P}{S}\neq\frac{U}{S}$ for each prime submodule $\frac{P}{S}$ of $\frac{U}{S}$. But U is a fully prime module, so by lemma (3.9), $\frac{U}{S}$ is also fully prime module, that is all submodules of $\frac{U}{S}$ are prime submodules. Thus $\frac{V}{S}\ll\frac{U}{S}$, so that U is a couniform module.

Proposition (3.11): For any multiplication R-modules; the class of cosemi-uniform coincides with the class of couniform modules.

Proof: Assume that U is a cosemi-uniform, and let V be a non-zero proper submodule of U, then there exists a proper submodule S of V such that $\frac{V}{S}\ll_P\frac{U}{S}$. Since U is multiplication and it is well-known that the quotient of multiplication is also multiplication, then $\frac{V}{S}\ll\frac{U}{S}$ [4, Prop.(1.4)]. That is U is a couniform module. The converse is clear.

As consequence of proposition (3.11), we have the following.

Corollary (3.12): Any ring is a couniform if and only if it is a cosemi-uniform ring.

Proposition (3.13): Let U be a finitely generated R-module, then U is a cosemi-uniform module if and only if U is a couniform.

Proof: Suppose that U is a cosemi-uniform module. Because in a finitely generated module there is no different between small and P-small submodule [4, Prop.(1.4)], thus we are done.

"An R-module U is called Noetherian, if every submodule of U is finitely generated [1, P.55]". The following proposition follows directly by proposition (3.13).

Corollary (3.14): Let U be a Noetherian R-module, then U is a cosemi-uniform module if and only if U is couniform.

4. Cosemi-uniform module and related concepts

This section devoted to study the relationships between cosemi-uniform module and other well-known related concepts such as hollow and Pr-hollow and epiform module.

Lemma (4.1): [4]

"If every prime submodule of an R-module U is small, then every proper submodule of U is P-small".

Proposition (4.2): Every Pr-hollow module is cosemi-uniform module

Proof: Let U be a Pr-hollow module, and let V≤ U. If V=0, then we are done. If V≠0, then by lemma (4.1), $V \ll_P U$. This implies that $\frac{V}{W} \ll_P \frac{U}{W}$ for every proper submodule W of V [4, Prop. (1.3)]. That is U is a cosemi-uniform module.

The converse is not true in general as the following example shows.

Examples (4.3): Consider the Z-module Z. By example (3.2)(2), Z is a couniform module. On the other hand, Z is not Pr-hollow [7].

However, if U is fully prime then the convers is true, as the following proposition shows.

Proposition (4.4): If an R-module U is a fully prime and cosemi-uniform module, then U is Pr-hollow.

Proof: Let V be a non-zero prime submodule of U. Suppose the converse; that is V is not small submodule of U, so there exists a proper submodule K of U such that V+K=U. Since U is a cosemi-uniform module, then there exists a proper submodule S of V such that $\frac{V}{S} \ll_P \frac{U}{S}$. It is clear that S≠0, since if S=0, then V≪U which is a contradiction. Since V+K=U, then $\frac{V+K+S}{S} = \frac{U}{S}$. This implies that $\frac{V}{S} + \frac{S+K}{S} = \frac{U}{S}$. By assumption $\frac{S+K}{S}$ is a prime submodule, so we get a contradiction, thus V is a small submodule of U, and hence U is Pr-hollow.

Theorem (4.5): Assume that U is a fully prime module, the following statements are equivalent.

1. U is a couniform module.
2. U is cosemi-uniform module.
3. U is a Pr-hollow module.
4. U is a hollow module.

Proof:

(1) ⇒ (2): By remark (3.2)(1).

(2) ⇒ (1): Proposition (3.10)

(2) ⇒ (3): By proposition (4.4).

(3) ⇒ (4): Let V be a proper submodule of U, if V is not small in U, then there exists a submodule L such that V+L=U. Since U is a fully prime module, then L is prime. But U is a Pr-hollow, then L≪ U, and so V=U which is a contradiction. Therefore V≪U. That is U is a hollow module.

(4) ⇒ (1): It is clear.

"Recall that a non-zero module U is called coquasi-Dedekind if every proper submodule of U is coquasi-invertible [9]. A proper submodule V of U is called coquasi-invertible, if $Hom_R(U,V) = 0$ [9]". We need the following lemma.

Lemma (4.6): [6, Th.(2.9)]

If an R-module U is epiform, then U is a couniform and coquasi-Dedekind module.

Corollary (4.7): Every epiform module is cosemi-uniform.

Proof: By remark (3.2)(1) and lemma (4.6).

In the following example we verify that the class of couniform modules is contained properly in the class of cosemi-uniform modules.

Example (4.8): We mentioned in example (3.2)(5), that Q is a cosemi-uniform module. But we can easily show that Q is not couniform. In fact, if we consider the negation of the lemma (4.6), which is: if an R-module U is not couniform module or not coquasi-Dedekind, then U is not epiform. Note that Q is coquasi-Dedekind and not epiform module [6, P.247], thus according to lemma (4.6), Q must be not couniform module.

Theorem (4.9): Let U be a multiplication module. Consider the following statements:

1. U is a hollow module.
2. U is a Pr-hollow module.
3. U is cosemi-uniform module.
4. U is a couniform module.

Then **(1)⇔ (2)⇒(3)⇒(4)**.

Proof:

(1) ⇒ (2): It is clear.

(2) ⇒ (1): Let V be a proper submodule of U. If V is not small submodule of U, then there exists a submodule W of U such that V+W=U. Since U is a multiplication module, so there exists maximal

(hence prime) submodule P of U such that W⊆ P [8, Th.(2.5)]. This implies that V+P=U. Since U is Pr-hollow, and P≪ U, so V=U which is a contradiction. Thus V≪ U and so U is a hollow module.
(**2**) ⇒ **(3):** By proposition (4.2).
(**3**) ⇒ **(4):** Since U is a multiplication module, so by proposition (3.11), U is couniform**.**

It is well known that if U is a finitely generated, then every proper submodule contained in a maximal submodule of U. By using this fact instead of [8, Th.(2.5)], and in similar way of the proof of theorem (4.9), we have the following.

Theorem (4.10): Assume that U is a finitely generated R-module, consider the following statements:
1. U is a hollow module.
2. U is a Pr-hollow module.
3. U is a cosemi-uniform module.
4. U is a couniform module.
Then **(1)⇔ (2)⇔(3)⇒(4)**.

"Following [9], an R-module U is called self-projective if for every submodule V of U, any homomorphism f: U→$\frac{U}{V}$ can be lifted to a homomorphism g: U→U".

Theorem (4.11): For any self-projective R-module U, the following statements are equivalent:
1. U is an epiform module.
2. U is a coquasi-Dedekind module.
3. U is a cosemi-uniform and coquasi-Dedekind module.

Proof:
(1) ⇒ **(3):** It follows by lemma (4.6) and remark (3.2)(1).
(**3**) ⇒ **(2):** It is clear.
(2) ⇒ **(1):** Since U is self-projective and coquasi-Dedekind, then U is an epiform module **[**6, Prop. (2.11)].

Following [9, Th.(1.2.16)], Yasseen proved that if U is self-projective module with J(End(U)) = 0, then a submodule V of U is small if and only if V is coquasi-invertible, where J(End(U)) is the Jacobson radical of the endomorphism of the ring R. For that reason, we can easily prove the following.

Theorem (4.12): If U is self-projective R-module U with the property J(End(U))=0, then the following statements are equivalent:
1. U is an epiform module.
2. U is a hollow module.
3. U is a coquasi-Dedekind module.
4. U is a cosemi-uniform and coquasi-Dedekind module.

Proof:
(1) ⇒ **(2):** [6].
(2) ⇒ **(3):** [9, Th.(1.2.16)].
(4) ⇒ **(3):** It is clear.
(4) ⇔ **(1):** Since U is self-projective module and coquasi Dedekind, then U is an epiform module [6, Prop.(2.11)].

Proposition (4.13): For any multiplication R-module U, the following statements are equivalent:
1. U is an epiform module.
2. U is a coquasi-Dedekind module.
3. U is a couniform and coquasi-Dedekind module.

Proof:
(1)⇒ **(2):** By lemma (4.6).
(**2**) ⇒ **(1):** Since U is coquasi-Dedekind, so every proper submodule V of U is quasi invertible [9, Th.(1.2.13)]. But U is multiplication, then V is corational [9, Th.(1.2.7)], hence U is epiform [6].
(1) ⇒ **(3):** By lemma (4.6).
(**3**) ⇒ **(2):** It is clear.

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