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New Path of Popularized Homogeneous Balance Method and Travelling Wave Solutions of a Nonlinear Klein-Gordon Equation

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Abstract

The aim of this paper is to obtain a set of traveling wave solutions for klein – Gorden equation with kerr law non-linearity. More precisely, we apply a new path of popularized homogeneous balance (HB) method in terms of using linear auxiliary equations to find the results of non-linear klein-Gorden equation, which is a fundamental approach to determine competent solutions. The solutions are achieved as the integration of exponential, hyperbolic, trigonometric and rational functions. Besides, some of the solutions are demonstrated by the3D graphics.

Keywords: Klein–Gorden equation, Traveling wave, Homogeneous balance method, NLEEs.

مسار جديد لاحتساب طريقة التوازن المتجانس وحلول الموجات المتنقلة لمعادلة كلاين–جوردون اللاخطية

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الخلاصة

الهدف من هذا البحث هو الحصول على مجموعة من حلول الموجات المتنقلة لمعادلة كلاين–جوردن مع قانون كير غير الخطية. بتعبير أدق ، نطبق مسارًا جديدًا لطريقة التوازن المتجانس الشائعة (HB) ويتم ذلك باستخدام المعادلات المساعدة الخطية للعثور على نتائج معادلة كلاين–جوردن غير الخطية ، وهي طريقة أساسية لتحديد الحلول المختصة بهذه الطريقة. يتم تحقيق الحلول من خلال تكامل الوظائف الأسية والقطعية والمثلثية والكسرية. إلى جانب ذلك ، يتم عرض بعض الحلول بواسطة رسومات ثلاثية الأبعاد.

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1. Introduction

Nonlinear evolution equations (NLEEs) have been used extensively to describe many physical, biology, chemistry, plasma physics, optical fibres and solid-state phenomenon that occur naturally or unnaturally, so they have been studied by many authors, see for instance [1-5]. Studying travelling waves is a key point to understanding the mechanism of any physical problem described by nonlinear evolution. Many PDEs do not have the exact solution, and thus require applying numerical methods to compute approximate solutions. Therefore, it becomes crucial to find the exact solution for the travelling wave problem which plays a vitally important role in understanding such physical problems.

Finding an exact solution for travelling waves has widely been studied in the literature, and many researchers have proposed different methods. One of the first algebraic methods for calculating the exact travelling wave solutions is the homogeneous balance method. This method has been proposed by Fan [6] and later developed by other researchers [7-11]. According to Fan's study [6], there is a connection between the HB method, Wiess, Tabor, Carnevale (WTC) method and Clarkson, Kruskal(CK) method.

Recently, an enormous number of studies have used the homogeneous balance (HB) method to determine a nonlinear transformation and exact solutions [12],[13] and auto Bäcklund transformations [6], [14]. Additionally, the HD method has also been used in the reductions of nonlinear partial differential equations (PDEs) in mathematical physics. According to Wang et al.[15],[16] and Abdel-Rady et al.[17-19], the HD methods have been introduced to calculate new exact traveling wave solutions of particular nonlinear partial differential equations. In 2014, El-Wakil et al. [14] suggested using the homogeneous balance method to obtain new exact solitary wave solutions of generalized shallow water wave equation and generalized variable coefficient 2D KdV equation. Many other methods such as Hirota's method, Bäcklund and Darboux transformation, Painlevé expansions, Jacobi elliptic function, Extended tanh-function method, F-expansion method and extended F-expansion method have been proposed and developed for calculating exact travelling wave solutions of nonlinear evolutions equations in which each method has its own advantages and drawbacks, also there are many methods could be applied for solving nonlinear partial differential equations by kamal implemented, such as the sine collocation method, which used to solve the fisher's reaction-diffusion equation, the polynomial deferential method, the cubic B-spline method, the Runge-kutta method and so on.

In this article, the explanation of the enhanced new path popularized HB technique is given in section 2. Also, this technique is to search the advanced and wide-spectral soliton solutions for the Klain-Gorden equation, and some of the important graphical depictions of the solutions obtained are given in the same section. Finally, in the last section, the conclusion is given.

2. Application of New Path Popularized HB method

In this portion, the procedure of reliable treatment of the Klain –Gorden equation is explored with the help of new path popularized HB method [20] .

$$U_{tt} - \alpha^2 U_{xx} + \beta U - \gamma U^2 = 0 \tag{2.1.1}$$

Where α, β and γ are arbitrary constants. Now we use the travelling wave $\xi = kx - \omega t$ to transform the non-linear into O.D.E as follows:

$$(\omega^2 - \alpha^2 k^2) G'' + \beta G - \gamma G^2 = 0 , \tag{2.1.2}$$

where k remains as a constant and ω stands to wave speed.

Operating the complex transformation

$$\Theta(t, x) = G(x, t) e^{iv(t,x)} , \ v(x, t) = \eta x - ct. \tag{2.1.3}$$

Where c and η are constants. Now we look for the solutions of (2.1.2) in the form

$$\mathrm{G} = a_0 + \sum_{\mathrm{i}=1}^{\mathrm{m}}\left(\mathrm{a_i}\varphi^{\mathrm{i}} + b_i\varphi^{-i}\right). \tag{2.1.4}$$

Where $\mathrm{a_i}, b_i$ are constants and φ satisfies the equation [21]

$$\varphi' = \ln(\rho)\,(\mu + v\varphi + \lambda\varphi^2). \tag{2.1.5}$$

Where μ, v, and λ are real numbers, and we are balancing G'' with G^2 in (2.1.2) gives $m + 2 = 2m$, so that $m = 2$.

Thus, the new path popularized HB method admits the following solution

$$\mathrm{G} = a_0 + \sum_{\mathrm{i}=1}^{\mathrm{m}}(\mathrm{a_i}\varphi^{\mathrm{i}} + b_i\varphi^{-i}), \mathrm{G} = \mathrm{a_0} + \mathrm{a_1}\varphi + \mathrm{a_2}\varphi^2 + \mathrm{b_1}\varphi^{-1} + \mathrm{b_2}\varphi^{-2}. \tag{2.1.6}$$

By substituting Eq. (2.1.5), (2.1.6) in (2.1.2), with derivatives of G, and equating the coefficients of φ^{i} to zero, we obtain a set of equations, solving the system of algebraic equation by using Maple software, we obtain:

$Case\ 1{:}\ if\ \ a_0 = 0\,, a_1 = \dfrac{-6\lambda\beta}{\gamma}\ , a_2 = \dfrac{-6\lambda^2\beta}{\gamma}\ , b_1 = 0, \quad b_2 = 0, \alpha = \pm\dfrac{\sqrt{\beta}}{ln\rho\ \ k}\ , \mu = 0.$

1) When $v^2 - 4\mu\lambda < 0\ with\ \ \lambda \neq 0.$

$$\Theta_1 = \frac{3\beta}{2\gamma}\left[1 + tan_\rho{}^2\left(\frac{i}{2}\right)\xi\right]e^{i(\eta x - ct)}.$$

$$\Theta_2 = \frac{3\beta}{2\gamma}\left[1 + cot_\rho{}^2\left(\frac{i}{2}\right)\xi\right]e^{i(\eta x - ct)}.$$

$$\Theta_3 = \frac{-3\beta}{\gamma}\left[\left(-1 + \frac{i}{2}\left(tan_\rho\left(\frac{i}{4}\right)\xi - cot_\rho\left(\frac{i}{4}\right)\xi\right)\right) + \frac{1}{2}\left(-1 + \frac{i}{2}\left(tan_\rho\left(\frac{i}{4}\right)\xi - cot_\rho\left(\frac{i}{4}\right)\xi\right)\right)^2\right]e^{i(\eta x - ct)}.$$

2) $When\ (v^2 - 4\mu\lambda) > 0\ with\ \lambda \neq 0.$

$$\Theta_4 = \frac{3\beta}{2\gamma}\ [\left(1 - tanh_\rho{}^2\left(\tfrac{1}{2}\right)\xi\right)e^{i(\eta x - ct)}.$$

$$\Theta_5 = \frac{-3\beta}{\gamma}\ [\left(-1 + coth_\rho\left(\tfrac{1}{2}\right)\xi\right) + \tfrac{1}{2}\left(-1 + coth_\rho\left(\tfrac{1}{2}\right)\xi\right)^2]e^{i(\eta x - ct)}.$$

$$\Theta_6 = \frac{-3\beta}{\gamma}\left[\left(-1 + \frac{1}{2}(tanh_\rho\left(\frac{1}{4}\right)\xi - coth_\rho\left(\frac{1}{4}\right)\xi\right) + \frac{1}{2}\left(-1 + \frac{1}{2}(tanh_\rho\left(\frac{1}{4}\right)\xi - coth_\rho\left(\frac{1}{4}\right)\xi\right)^2\right]e^{i(\eta x - ct)}.$$

$Case\ 2{:}\ if\ \ a_0 = \dfrac{-3}{2}\dfrac{\ln(\rho)^2\alpha^2 - \beta}{\gamma}\ , a_1 = \dfrac{-6\lambda\ \alpha^2\ln(\rho)^2}{\gamma}\ ,$

$a_2 = \dfrac{-6\ \alpha^2\ln(\rho)^2\lambda^2}{\gamma}\ , b_1 = 0, \quad b_2 = 0, \mu = \dfrac{1}{4}\dfrac{\alpha^2\ln(\rho)^2 - \beta}{\lambda\ \alpha^2\ln(\rho)^2}$

1) When $v^2 - 4\mu\lambda < 0\ with\ \ \gamma \neq 0, R = \left(1 - \frac{\alpha^2\ln(\rho)^2 - \beta}{\alpha^2\ln(\rho)^2}\right).$

$$\Theta_7 = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta}{\gamma} + \frac{-3\ \alpha^2\ln(\rho)^2}{\gamma}\left[\ \left(-\frac{1}{2} - \frac{R}{2}tan_\rho{}^2\left(\frac{\sqrt{-R}}{2}\right)\xi\right)\right]\right]e^{i(\eta x - ct)}.$$

$$\Theta_8 = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta}{\gamma} + \frac{-3\ \alpha^2\ln(\rho)^2}{\gamma}\left[\ \left(-\frac{1}{2} - \frac{R}{2}cot_\rho{}^2\left(\frac{\sqrt{-R}}{2}\right)\xi\right)\right]\right]e^{i(\eta x - ct)}.$$

$$\Theta_9 = \frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-3\,\alpha^2\ln(\rho)^2}{\gamma}\left(-1+\frac{\sqrt{-R}}{2}(tan_\rho\left(\frac{\sqrt{-R}}{4}\right)\xi - cot_\rho\left(\frac{\sqrt{-R}}{4}\right)\xi)\right) + \frac{-3\,\alpha^2\ln(\rho)^2}{2\gamma}\left(-1+\frac{\sqrt{-R}}{2}(tan_\rho\left(\frac{\sqrt{-R}}{4}\right)\xi - cot_\rho\left(\frac{\sqrt{-R}}{4}\right)\xi)\right)^2 e^{i(\eta x-ct)}.$$

2) $When\ (R) > 0\ with\ \gamma \neq 0$

$$\Theta_{10} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-3\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(-\frac{1}{2}+\frac{R}{2}tanh_\rho{}^2\left(\frac{\sqrt{R}}{2}\right)\xi\right)\right]\right]e^{i(\eta x-ct)}.$$

$$\Theta_{11} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-3\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(-\frac{1}{2}+\frac{R}{2}coth_\rho{}^2\left(\frac{\sqrt{R}}{2}\right)\xi\right)\right]\right]e^{i(\eta x-ct)}.$$

3) When $\mu\lambda > 0\ with\ \ v = 1$

$$\Theta_{12} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-6\lambda\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(\sqrt{\frac{\mu}{\lambda}}tan_\rho\left(\sqrt{\mu\lambda}\right)\xi\right) + \mu\left(tan_\rho\left(\sqrt{\mu\lambda}\right)\xi\right)^2\right]\right]e^{i(\eta x-ct)}.$$

4) When $\mu\lambda < 0\ with\ \ v \neq 0$.

$$\Theta_{13} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-6\lambda\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(-\sqrt{\frac{\mu}{\lambda}}itanh_\rho\left(\sqrt{\mu\lambda}i\right)\xi\right) + \mu\left(-itanh_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right)^2\right]\right]e^{i(\eta x-ct)}.$$

$$\Theta_{14} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-6\lambda\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(-\sqrt{-\frac{\mu}{\lambda}}coth_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right) + \mu\left(-icoth_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right)^2\right]\right]e^{i(\eta x-ct)}.$$

$$\Theta_{15} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{-6\lambda\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(-tanh_\rho(\mu\lambda)\xi\right) + \lambda\left(-tanh_\rho(\mu\lambda)\xi\right)^2\right]\right]e^{i(\eta x-ct)}.$$

$$\Theta_{16} = \left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta}{\gamma} + \frac{6\lambda\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(coth_\rho(\mu\lambda)\xi\right) - \lambda\left(-coth_\rho(\mu\lambda)\xi\right)^2\right]\right]e^{i(\eta x-ct)}.$$

$$\Theta_{17} = \Bigg[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta}{\gamma} + \frac{-6\lambda\,\alpha^2\ln(\rho)^2}{\gamma}\left[\left(-\frac{1}{2}tanh_\rho\left(\frac{\mu}{2}\lambda\right)\xi + cot_\rho(\frac{\mu}{2}\lambda)\xi)\right) + \lambda\left(-\frac{1}{2}tanh_\rho\left(\frac{\mu}{2}\lambda\right)\xi + cot_\rho(\frac{\mu}{2}\lambda)\xi)\right)^2\right]\Bigg]e^{i(\eta x - ct)}.$$

$Case\ 3: if\ \ a_0 = \frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta^2}{\gamma}\ , a_1 = 0\ \ , a_2 = \ 0, \qquad b_1 = \frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta^2}{\lambda\gamma},$

$b_2 = \frac{-3}{8}\frac{(\ln(\rho)\alpha^2 - \beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}, \mu = \frac{\ln(\rho)^2\alpha^2 - \beta^2}{\lambda\ln(\rho)\alpha^2}.$

1) When $1 - 4\mu\lambda < 0\ with\ \ \lambda \neq 0.$

$$\Theta_{18} = \Bigg[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta^2}{\gamma} + \frac{-3\ln(\rho)^2\alpha^2 - \beta^2}{\gamma}\left(-1 + \sqrt{-(1-4\mu\lambda)}tan_\rho\left(\frac{\sqrt{-(1-4\mu\lambda)}}{2}\right)\xi\right)^{-1} + \frac{-3}{2}\frac{(\ln(\rho)\alpha^2 - \beta)^2}{\gamma\ln(\rho)^2\alpha^2}\left(-1 + \sqrt{-(1-4\mu\lambda)}tan_\rho\left(\frac{\sqrt{-(1-4\mu\lambda)}}{2}\right)\xi\right)^{-2}\Bigg]e^{i(\eta x - ct)}.$$

$$\Theta_{19} = \Bigg[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2 - \beta^2}{\gamma} + \frac{-3\ln(\rho)^2\alpha^2 - \beta^2}{\gamma}\left(-1 + \frac{\sqrt{-(1-4\mu\lambda)}}{2}\left(tan_\rho\left(\frac{\sqrt{-(1-4\mu\lambda)}}{4}\right)\xi - cot_\rho\left(\frac{\sqrt{-(1-4\mu\lambda)}}{4}\right)\xi\right)\right)^{-1} + \frac{-3}{2}\frac{(\ln(\rho)\alpha^2 - \beta)^2}{\gamma\ln(\rho)^2\alpha^2}\left(-1 + \frac{\sqrt{-(1-4\mu\lambda)}}{4\lambda}\left(tan_\rho\left(\frac{\sqrt{-(1-4\mu\lambda)}}{4}\right)\xi - cot_\rho\left(\frac{\sqrt{-(1-4\mu\lambda)}}{4}\right)\xi\right)\right)^{-2}\Bigg]e^{i(\eta x - ct)}.$$

2) $When\ (1 - 4\mu\lambda) > 0\ with\ \lambda \neq 0.$

$$\Theta_{20}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3\ln(\rho)^2\alpha^2-\beta^2}{\gamma}\ \left(-1+\sqrt{1-4\mu\lambda}\,tanh_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{2}\right)\xi\right)^{-1}\right.$$
$$\left.+\frac{-3}{2}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\alpha^2}\left(-1+\sqrt{1-4\mu\lambda}\,tanh_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{2}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{21}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3\ln(\rho)^2\alpha^2-\beta^2}{\gamma}\ \left(-1+\sqrt{1-4\mu\lambda}coth_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{2}\right)\xi\right)\right.$$
$$\left.+\frac{-3}{2}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\alpha^2}\left(-1+\sqrt{1-4\mu\lambda}\,coth_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{2}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{22}=$$
$$\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\ \left(-\frac{1}{2\lambda}+\frac{\sqrt{1-4\mu\lambda}}{4\lambda}(tanh_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{4}\right)\xi-coth_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{4}\right)\xi\right)^{-1}+$$
$$\frac{-3}{2}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\alpha^2}\left(-1+\frac{\sqrt{1-4\mu\lambda}}{2}(tanh_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{4}\right)\xi-coth_\rho\left(\frac{\sqrt{1-4\mu\lambda}}{4}\right)\xi\right)^{-2}e^{i(\eta x-ct)}.$$

3) When $\mu\lambda>0\ with\ v=1$

$$\Theta_{23}=$$
$$\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\ \left(\sqrt{\frac{\mu}{\lambda}}tan_\rho\left(\sqrt{\mu\lambda}\right)\xi\right)^{-1}+\right.$$
$$\left.\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda\mu\alpha^2}\left(tan_\rho\left(\sqrt{\mu\lambda}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{24}=$$
$$\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}-\frac{3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\ \left(\sqrt{\frac{\mu}{\lambda}}cot_\rho\left(\sqrt{\mu\lambda}\right)\xi\right)^{-1}-\right.$$
$$\left.\frac{3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(\sqrt{\frac{\mu}{\lambda}}cot_\rho\left(\sqrt{\mu\lambda}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{25}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3\ln(\rho)^2\alpha^2+3\beta^2}{\lambda\gamma}\ \left(\sqrt{\frac{\mu}{\lambda}}(tan_\rho\left(\frac{\sqrt{\mu\lambda}}{2}\right)\xi-cot_\rho\left(\frac{\sqrt{\mu\lambda}}{2}\right)\xi\right)^{-1}+\right.$$
$$\left.\frac{-24\ln(\rho)^2\lambda^2}{\gamma}\left(\frac{1}{2}\sqrt{\frac{\mu}{\lambda}}(tan_\rho\left(\frac{\sqrt{\mu\lambda}}{2}\right)\xi-cot_\rho\left(\frac{\sqrt{\mu\lambda}}{2}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

4) When $\mu\lambda<0\ with\ v\neq 0$

$$\Theta_{26}=$$
$$\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\ \left(-\sqrt{-\frac{\mu}{\lambda}}tanh_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right)^{-1}+\right.$$
$$\left.\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-\sqrt{-\frac{\mu}{\lambda}}tanh_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{27}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\ \left(-\sqrt{-\frac{\mu}{\lambda}}coth_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right)^{-1}\right.$$
$$\left.+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-\sqrt{-\frac{\mu}{\lambda}}coth_\rho\left(\sqrt{-\mu\lambda}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{28}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left(-\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}(-tanh_\rho\left(\frac{\sqrt{-\mu\lambda}}{2}\right)\xi\pm coth_\rho\left(\frac{\sqrt{-\mu\lambda}}{2}\right)\xi\right)^{-1}+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-\frac{1}{2}\sqrt{-\frac{\mu}{\lambda}}(-tanh_\rho\left(\frac{\sqrt{-\mu\lambda}}{2}\right)\xi\pm coth_\rho\left(\frac{\sqrt{-\mu\lambda}}{2}\right)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{29}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left(tan_\rho(\mu\lambda)\xi\right)^{-1}+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(tan_\rho(\mu\lambda)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{28}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left(-cot_\rho(\mu\lambda)\xi\right)^{-1}+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-cot_\rho(\mu\lambda)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{29}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}-3\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left((tan_\rho\left(\frac{\mu}{2}\lambda\right)\xi-cot_\rho(\frac{\mu}{2}\lambda)\xi)\right)^{-1}+\frac{-3}{2}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left((tan_\rho\left(\frac{\mu}{2}\lambda\right)\xi-cot_\rho(\frac{\mu}{2}\lambda)\xi)\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{30}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left(-tanh_\rho(\mu\lambda)\xi\right)^{-1}+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-tanh_\rho(\mu\lambda)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{31}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left(-coth_\rho(\mu\lambda)\xi\right)^{-1}+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-coth_\rho(\mu\lambda)\xi\right)^{-2}\right]e^{i(\eta x-ct)}.$$

$$\Theta_{32}=\left[\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\gamma}+\frac{-3}{2}\frac{\ln(\rho)^2\alpha^2-\beta^2}{\lambda\gamma}\left(-\frac{1}{2}tanh_\rho\left(\frac{\mu}{2}\lambda\right)\xi+cot_\rho(\frac{\mu}{2}\lambda)\xi)\right)^{-1}+\frac{-3}{8}\frac{(\ln(\rho)\alpha^2-\beta)^2}{\gamma\ln(\rho)^2\lambda^2\alpha^2}\left(-\frac{1}{2}tanh_\rho\left(\frac{\mu}{2}\lambda\right)\xi+cot_\rho(\frac{\mu}{2}\lambda)\xi)\right)^{-2}\right]e^{i(\eta x-ct)}.$$

Table 1- Comparison between the solution of the new path popularized HB method and the solution of the hyperbolic tanh method [22].

The solution of new path popularized HB method	The solution of tanth method
If suppose $\beta = 1, \rho = 0, \alpha = 1, c = 0, \eta = 0$ $, \gamma = 1$ *and* $R = 3$ $\Theta_7 = \frac{3}{2}\ \left(1 + 3tan_\rho{}^2\left(\frac{\sqrt{3}}{2}i\right)\xi\right)]$. $\Theta_8 = \frac{3}{2}\ \left(1 + 3cot_\rho{}^2\left(\frac{\sqrt{3}}{2}i\right)\xi\right)$. $\Theta_{10} = \ \frac{3}{2}\ \left[\left(1 - 3tanh_\rho{}^2\left(\frac{\sqrt{3}}{2}\right)\xi\right)\right]$. $\Theta_{11} = \frac{3}{2}\ [\left(1 - 3coth_\rho{}^2\left(\frac{\sqrt{3}}{2}\right)\xi\right)$.	If $k = 1, c = 0, b = 3$ *and* $a = -1$ then the following equations became $u = \frac{3}{2}\ \left(1 + 3tan^2\left(\frac{\sqrt{3}}{2}i\right)\xi\right)$. *(67)* $u = \frac{3}{2}\ \left(1 + 3cot^2\left(\frac{\sqrt{3}}{2}i\right)\xi\right)$. *(68)* $u = \ \frac{3}{2}\ \left[\left(1 - 3tanh^2\left(\frac{\sqrt{3}}{2}\right)\xi\right)\right]$. (63) $u = \frac{3}{2}\ \left[\left(1 - 3coth^2\left(\frac{\sqrt{3}}{2}\right)\xi\right)\right]$. (64)

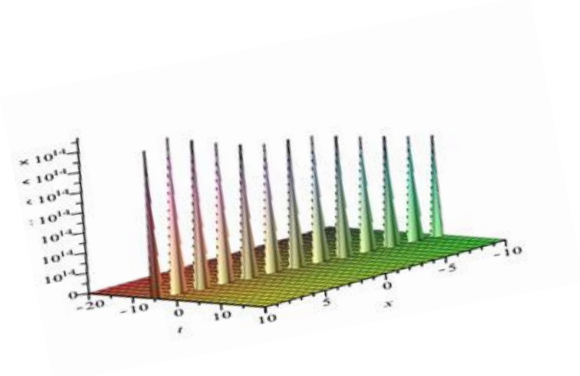

Figure1- The diagram 3D of the travelling wave if
$\omega = 2\,, \beta = -1\ \ , k = 0, \gamma = 1\,, t = -10, \dots, 10, x = -10, \dots, 10\ \ \ of\ G_2$

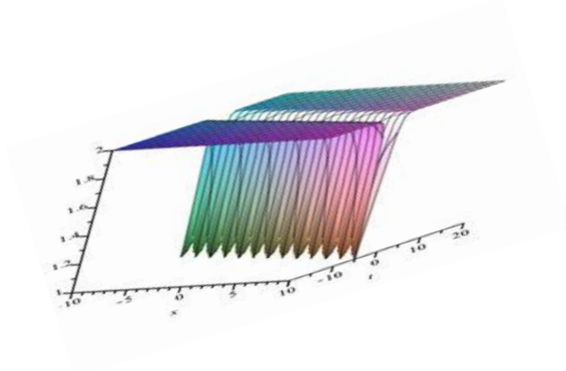

Figure2- The diagram 3D of the travelling wave if
$\omega = 2, \beta = -1\ \ , k = 0, \gamma = 1\,, t = -10, \dots, 10\,, x = -10, \dots, 10\ \ \ of\ G_4$

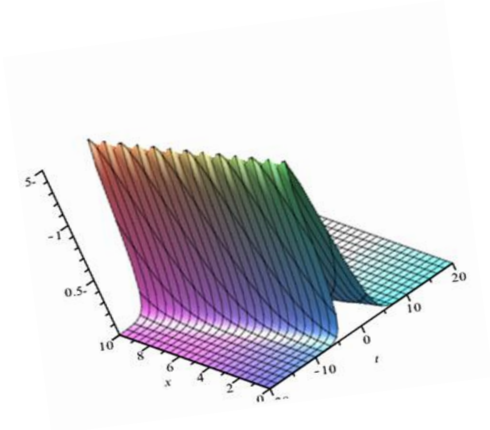

Figure 3-The diagram 3D of the travelling wave if
$\beta = 1\ ,\gamma = 1, k = 1\ , R = 3, t = -20, \dots, 20, x = -20, \dots, 20 \quad of\ G_1$

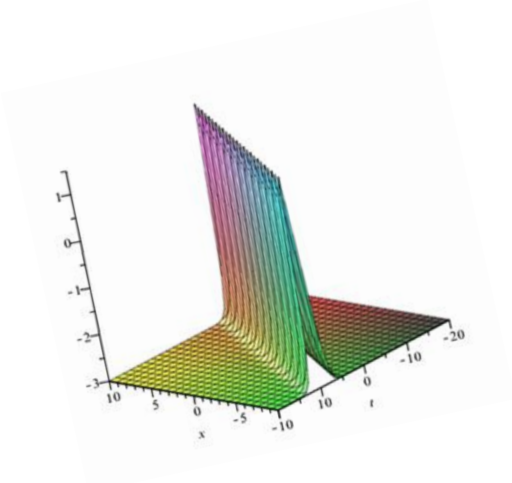

Figure 4-The diagram 3D of the travelling wave if
$\beta = 1\ ,\gamma = 1, k = 1\ , R = 3, t = -10, \dots, 10, x = -20, \dots, 20 \quad of\ G_7$

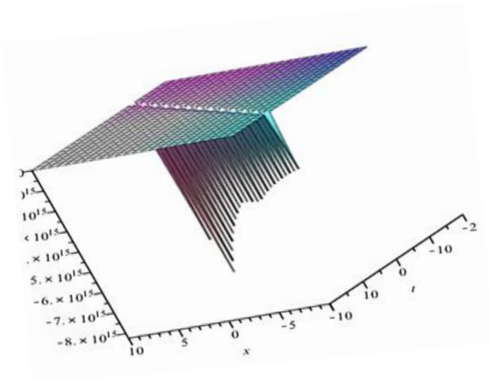

Figure 5- The diagram 3D of the travelling wave if
$\omega = 2, \rho = 0\ , \beta = 1\ ,\gamma = 1, C = 0, k = 1\ , \alpha = 1, t = -20, \dots, 20, x = -10, \dots, 10 \quad of\ G_{11}$

3. Conclusion

The complex valued solutions, periodic solutions and soliton solutions are obtained. In this paper, new path popularized HB method for the Klein-Gorden equation as a basic tool to find many exact travelling wave solutions. When compared with the methods in the literature, more solutions have been found with our method in Table 1. We noted that our solutions are in more general forms results and many known solutions to these equations are only special cases from our solutions. Also, by plotting 3D graphics of the some exact solutions, we noted that our results contain different of travelling wave solutions such as, periodic, solitary, soliton and kink waves.

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