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Quasi-Hadamard products of New Subclass of Analytic Functions of β-Uniformly Univalent Function Defined by Salagean q-Differential Operator

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Abstract:
In this paper, we show many properties and results on the Quasi-Hadamard products of a new subclass of analytic functions of β-Uniformly univalent function that is defined by the Salagean q-differential operator.

Keywords: Uniformly functions, Analytic function, Salagean type q-difference, , quasi-Hadamard products, Negative coefficients.

شبه حاصل ضرب هادمرد لفئة جديدة من الدوال التحليلية الاحادية التكافؤ المنتظمة من النوع بيتا المعرفة بواسطة مؤثر سلاجيان التفاضلي من النوع q

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الخلاصة
في هذه الدراسة بينا العديد من النتائج على شبه حاصل ضرب هادمرد لفئة جديدة من الدوال التحليلية الاحادية التكافؤ المنتظمة من النوع بيتا المعرفة بواسطة مؤثر سلاجيان التفاضلي من النوع q

Introduction:

Let A be the class of analytic and univalent functions. A function in A has the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k \; z^k, \qquad z \in U = \{z: z \in \mathbb{C}: |z| < 1\} \;. \tag{1}$$

And let T be a subclass of . An element of T is defined by

$$f(z) = z - \sum_{k=2}^{\infty} a_k \; z^k \quad , \quad a_k \geq 0; \; z \in U \,. \tag{2}$$

Consider $S(\alpha)$ and $K(\alpha)$ are two subclasses of A which are starlike and convex functions of order $\alpha, 0 \leq \alpha < 1$ that satisfy

$$S(\alpha) = \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1) \tag{3}$$

and

$$K(\alpha) = \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (0 \leq \alpha < 1). \tag{4}$$

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It is easier to write S(0)=S and K(0)=K (see[1] and [2]) and from (3) and (4), we have

$$f(z) \in \mathrm{K}(\alpha) \Leftrightarrow zf'(z) \in \mathrm{S}(\alpha).$$

Let's $S^*(\alpha) = \mathrm{S}(\alpha) \cap T$ and $K^*(\alpha) = \mathrm{k}(\alpha) \cap T$ [3] defined the following subclass of S(K).

Circular arc γ contained in Y with center ζ is also in Y, the arc ϕ(γ)is convex (starlike) with respect to ϕ(ζ). The class of uniformly convex (starlike) functions is denoted by UCV and UST, respectively (see [4]).

Definition 1 [4],[5],[6]. A function ϕ(ζ)2 A is said to be in UCV, the class of uniformly convex functions, if it satisfies the following condition:

$$\mathrm{Re}\left\{1 + \frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right\} \geq \left|\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right| \qquad for\ z \in U. \tag{5}$$

Further, afunction ϕ(ζ)2 A is said to be in UST ,the class of uniformly starlike functions, if it satisfies the following condition:

$$\mathrm{Re}\left\{\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})}\right\} \geq \left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} - 1\right| \qquad for\ z \in U. \tag{6}$$

The class UCV was introduced by Goodman [4] and Ma and Minda [5]. While the class UST was introduced by Goodman [6] and Ronning [7]. One can see that

$$f(z) \in \mathrm{UCV} \Leftrightarrow zf'(z) \in \mathrm{UST}$$

In [6],[8]. Ronning generalized the classes UCV and UST by introducing a parameter α by the following definition.

Definition 2 [6]. A function ϕ(ζ)2 A is said to be in the class of uniformly starlike functions of order α , YΣT (α), if it satisfies the following condition:

$$\mathrm{Re}\left\{\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} - \alpha\right\} \geq \left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} - 1\right| \qquad (-1 \leq \alpha \leq 1; z \in U). \tag{7}$$

Replacing ϕ in (7) by $zf'(z)$, we have the condition:

$$\mathrm{Re}\left\{1 + \frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})} - \alpha\right\} \geq \left|\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right| \qquad (-1 \leq \alpha \leq 1; z \in U) \tag{8}$$

This requires for the function ϕ(ζ) to be in the class YXς (α)of uniformly convex functions of order α, you can see that

$$f(z) \in \mathrm{UCV}(\alpha) \Leftrightarrow zf'(z) \in \mathrm{UST}(\alpha)$$

Kanas and Wisniowska [9],[10]. introduced the classes of β −uniformly convex functions and β −uniformly starlike functions, β −YXς and β −YΣT ($0 \leq \beta < 1$) ,respectively, by the following definition.

Definition 3: [9],[10]. A function ϕ(ζ)2 A is said to be in the class of β −uniformly starlike functions, β −YΣT , if

$$\mathrm{Re}\left\{\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} - \alpha\right\} \geq \beta\left|\frac{\mathrm{z}f'(\mathrm{z})}{f(\mathrm{z})} - 1\right| \qquad (\beta \geq 0; z \in U), \tag{9}$$

and ϕ(ζ)2 A is said to be in the class of β −uniformly convex functions, β −YXς, if

$$\mathrm{Re}\left\{1 + \frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right\} \geq \beta\left|\frac{\mathrm{z}f''(\mathrm{z})}{f'(\mathrm{z})}\right| \qquad (\beta \geq 0; \mathrm{z} \in \mathrm{U}). \tag{10}$$

The relationship between β −YΣT and β − YXς is given by

$$f(z) \in \beta - UCV \Leftrightarrow zf'(z) \in \beta - UST.$$

Let $f(z) \in A$, then

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = D f(z) = zf'(z)$$
$$\vdots$$
$$D^n f(z) = D(D^{n-1} f(z))$$

$$z + \sum_{k=2}^{\infty} a_k \ k^n z^k \qquad z \in U, \tag{11}$$

$=D^n f(z)$

where $U = \{n \in N = N \cup \{0\}, N = 1,2,\dots\}$ and $D^n f(z)$ is introduced by salagean [11] , [12]

For ($0< q < 1$), the Jacksons q− derivative of a function $f(z) \in A$ is given by [13],[14],[15],[16],[17],[18],[19].

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{1-q(z)}, & for\ z \neq 0 \\ f'(0), & for\ z = 0\,. \end{cases} \tag{12}$$

And $D_q^2 f(z) = D_q\left(D_q f(z)\right)$. From (12), we have

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} a_k \ [k]_q z^{k-1}, \tag{13}$$

where, $[k]_q = \frac{(1-q^k)}{1-q}$ $(0 < q < 1)$. (14)

If $q \to 1^-, then\ [k]_q \to k$. For a function $h(z) = z^k$, we obtain

$D_q h(z) = D_q z^k = \frac{(1-q^k)}{1-q} z^{k-1} = [k]_q z^{k-1}$

and

$$\lim_{q\to 1^-} D_q h(z) = k z^{k-1} = h'(z),$$

Where h' is the ordinary derivative of h.

For $f(z) \in A$, Govindaraj and Sivasubramanian [20],[21] defined the Salagean q-defference operator as follows:

$$D_q^0 f(z) = f(z)$$
$$D_q^1 f(z) = zD_q f(z)$$
$$\vdots$$
$$D_q^n f(z) = zD_q\left({D_q}^{n-1} f(z)\right) \qquad n \in N$$

$$= z + \sum_{k=2}^{\infty} a_k \ [k]_q^n z^k \quad (n \in N, 0 < q < 1, z \in U). \tag{15}$$

For $\beta \geq 0, -1 \leq \alpha < 1, 0 < q < 1$ and $n \in N$, which is denoted by $A_{q,n,k}(\alpha,\beta)$, the subclass of A that satisfies

$$Re\left\{\frac{z\left(D_q\left(D_q^n f(z)\right)\right)'}{{D_q}^n f(z)} - \alpha\right\} > \beta\left|\frac{z\left(D_q\left(D_q^n f(z)\right)\right)'}{{D_q}^n f(z)} - 1\right|, \qquad z \in U \tag{16}$$

Is defined the class $A_q(k,n,\alpha,\beta)$ by

$$A_q(k,n,\alpha,\beta) = A_{q,n,k}(\alpha,\beta) \cap T.$$

Let $f_l(z)$ $(l = 1,2,\dots\dots..h)$ be given by

$$f_l(z) = z - \sum_{k=2}^{\infty} a_k \ z^k \qquad \left(a_{k,l} \ \geq 0\right). \tag{17}$$

The quasi-Hadamard product of these functions is defined by Kuang et al. [22] and Owa [23] as follows:

$$(f_1 * f_2 * ... * f_h)(z) = z - \sum_{k=2}^{\infty}(\prod_{l=1}^{h} a_{k,l})\, z^k. \tag{18}$$

In this paper, we obtain the quasi–Hadamard product results for $f(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha, \beta)$

2. Quasi-Hadamard products

Theorem 1. A function $f(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [k]_q^n \left\{(1+\beta)k\,[k]_q - (\alpha+\beta)\right\} a_k \le (1-\alpha). \tag{19}$$

Proof. If the equation (17) holds, then

$$\beta \left| \frac{z\left(D_q\left(D_q^n f(z)\right)\right)'}{D_q{}^n f(z)} - 1 \right| - \mathrm{Re}\left\{ \frac{z\left(D_q\left(D_q^n f(z)\right)\right)'}{D_q{}^n f(z)} - \alpha \right\} \le 1 - \alpha.$$

We have

$$\beta \left| \frac{z\left(D_q\left(D_q^n f(z)\right)\right)'}{D_q{}^n f(z)} - 1 \right| - \mathrm{Re}\left\{ \frac{z\left(D_q\left(D_q^n f(z)\right)\right)'}{D_q{}^n f(z)} - \alpha \right\}$$

$$\le \frac{(1+\beta)\sum_{k=2}^{\infty}[k]_q^n \left\{k\,[k]_q - 1\right\} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty}[k]_q^n\, a_k z^{k-1}}$$

$$\le \frac{(1+\beta)\sum_{k=2}^{\infty}[k]_q^n \left\{k\,[k]_q - 1\right\} a_k}{1 - \sum_{k=2}^{\infty}[k]_q^n\, a_k} \le 1 - \alpha.$$

Hence, $f(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha, \beta)$,

Conversely, let $f(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha, \beta)$, then

$$\mathrm{Re}\left\{ \frac{1 - \sum_{k=2}^{\infty}[k]_q^n\, [k]_q a_k k z^{k-1}}{1 - \sum_{k=2}^{\infty}[k]_q^n\, a_k z^{k-1}} - \alpha \right\} \ge \beta \left| \frac{\sum_{k=2}^{\infty}[k]_q^n \left\{k\,[k]_q - 1\right\} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty}[k]_q^n\, a_k z^{k-1}} \right|.$$

We get the desired inequality by letting $z \to 1^{-}$move along the real axis (19). The proof is completed.

Theorem 2: If $f_l(z) \in A_q(k, n, \alpha_l, \beta)$ for each $l = 1,2, \dots, h$, then $(f_1 * f_2 * ... * f_h)(z) \in A_q(\mathrm{k}, \mathrm{n}, \delta, \beta)$, where

$$\delta = 1 - \frac{\left\{2[2]_q (1+\beta) - \beta\right\} \prod_{l=1}^{h}(1-\alpha_l)}{\prod_{l=1}^{2}\left\{2[2]_q (1+\beta) - (\alpha_l + \beta)\right\}}. \tag{20}$$

This result is sharp for the functions.

$$f_l(z) = z - \frac{(1-\alpha_l)}{[k]_q^n \left\{k\,[k]_q (1+\beta) - (\alpha_l+\beta)\right\}} z^k \quad (k \ge 2, l = 1,2, \dots h). \tag{21}$$

Proof : For $h = 1$, we have that $\delta = \alpha_1$. For $h = 2$, Theorem 1 gives

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left\{(1+\beta)k\,[k]_q - (\alpha_l+\beta)\right\}}{(1-\alpha_l)}\, a_{k,l} \le 1 \quad (l = 1,2) \tag{22}$$

Note that from (22), we have

$$\sum_{k=2}^{\infty} [k]_q^n \sqrt{\prod_{l=1}^{2} \frac{\left\{(1+\beta)k\,[k]_q - (\alpha_l+\beta)\right\}}{(1-\alpha_l)} a_{k,l}} \le 1 \quad (l = 1,2) \tag{23}$$

When h=2, we have

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left\{k\,[k]_q (1+\beta) - (\delta+\beta)\right\}}{(1-\delta)} \quad a_{k,1} a_{k,2} \le 1 \tag{24}$$

Or, such that

$$\frac{\{k\,[k]_q\,(1+\beta)-(\delta+\beta)\}}{(1-\delta)}\sqrt{a_{k,1}a_{k,2}} \le \sqrt{\prod_{l=1}^{2}\frac{\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}}{(1-\alpha_l)}a_{k,l}} \qquad (k \ge 2)$$

Further, by using (23), we need the largest δ such that

$$\frac{\{k\,[k]_q\,(1+\beta)-(\delta+\beta)\}}{(1-\delta)} \le \prod_{l=1}^{2}\frac{\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}}{(1-\alpha_l)}\ a_{k,l} \quad (k \ge 2)$$

Which is equivalent to

$$\delta \le \left(\frac{\prod_{l=1}^{2}\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}-\{(1+\beta)k\,[k]_q-\beta\}\prod_{l=1}^{2}(1-\alpha_l)}{\prod_{l=1}^{2}(1-\alpha_l)}\right) \times\left(\frac{\prod_{l=1}^{2}(1-\alpha_l)}{\prod_{l=1}^{2}\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}}\right),$$

That is,

$$\delta \le \frac{\prod_{l=1}^{2}\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}-\{(1+\beta)k\,[k]_q-\beta\}\prod_{l=1}^{2}(1-\alpha_l)}{\prod_{l=1}^{2}\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}}$$

Or, equivalently that

$$\delta \le 1-\frac{\{(1+\beta)k\,[k]_q-\beta\}\prod_{l=1}^{2}(1-\alpha_l)}{\prod_{l=1}^{2}\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}}. \qquad (k \ge 2)$$

If we define the function $\Phi(k)$by

$$\Phi(k) = 1-\frac{\{(1+\beta)k\,[k]_q-\beta\}\prod_{l=1}^{2}(1-\alpha_l)}{\prod_{l=1}^{2}\{(1+\beta)k\,[k]_q-(\alpha_l+\beta)\}}. \qquad (k \ge 2)$$

Then, we see that$\Phi(k)$ for ≥ 2 . This implies that

$$\delta = \Phi(2) = 1-\frac{\{2\,[2]_q\,(1+\beta)-\beta\}\prod_{l=1}^{2}(1-\alpha_l)}{\prod_{l=1}^{2}\{2\,[2]_q\,(1+\beta)-(\alpha_l+\beta)\}};$$

When $h = 2$ the result is also correct for any positive integer$h$. Then, we have $(f_1 * f_2 * \dots * f_h * f_{h+1})(z) \in A_q(k, n, \nu, \beta)$, where

$$\nu = 1-\frac{\{2\,[2]_q\,(1+\beta)-\beta\}(1-\delta)(1-\alpha_{h+1})}{\{2\,[2]_q\,(1+\beta)-(\delta+\beta)\}\{2\,[2]_q\,(1+\beta)-(\alpha_{h+1}+\beta)\}\ )} \ . \tag{25}$$

Where δ is given by (20). It follows from (25) that

$$\nu = 1-\frac{\prod_{l=1}^{h+1}\{2\,[2]_q\,(1+\beta)-\beta\}(1-\alpha_l)}{\prod_{l=1}^{h+1}\{2\,[2]_q\,(1+\beta)-(\alpha_l+\beta)\}}. \tag{26}$$

So, the result is correct for $h+1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer h

Finally, we take $f_l\ (z)$ that is given by (21), then we see that

$$(f_1 * f_2 * \dots * f_h)(z) = z-\left\{\prod_{l=1}^{h}\left(\frac{1-\alpha_l}{[2]_q^n\{2\,[2]_q\,(1+\beta)-(\alpha_l+\beta)\}}\right)\right\}z^2 = z-\psi_2 z^2 \ .$$

Where

$$\psi_2 = \prod_{l=1}^{h}\left(\frac{1-\alpha_l}{[2]_q^n\{2\,[2]_q\,(1+\beta)-(\alpha_l+\beta)\}}\right).$$

Thus, we know that

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left\{(1+\beta)k\,[k]_q - (\delta+\beta)\right\}}{(1-\delta)} . \psi_k .$$

Then we have

$$\frac{[2]_q^n \left\{2\,[2]_q\,(1+\beta) - (\delta+\beta)\right\}}{(1-\delta)} . \prod_{l=1}^{h} \left(\frac{1-\alpha_l}{[2]_q^n \left\{2\,[2]_q\,(1+\beta) - (\alpha_l+\beta)\right\}} \right) = 1$$

Consequently, the result is sharp for the functions $f_l\,(z)$ that are given by (21).
Putting $h = 2$ and $\alpha_l = \alpha$ in Theorem 2, then we have the following corollary

Corollary 1: If $f_l\,(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha_l\,, \beta)(l = 1,2)$, then $(f_1 * f_2)\,(z)\; A_q(\mathrm{k}, \mathrm{n}, \delta, \beta)$, where

$$\delta = 1 - \frac{\left\{k\,[k]_q\,(1+\beta) - \beta\right\}(1-\alpha)^2}{\left\{k\,[k]_q\,(1+\beta) - (\alpha+\beta)\right\}^2} \qquad (k \geq 2).$$

The end result is sharp for the functions

$$f_l\,(z) = z - \frac{1-\alpha}{[k]_q^n \left\{k\,[k]_q\,(1+\beta) - (\alpha+\beta)\right\}} z^k \qquad (k \geq 2, l = 1,2).$$

Corollary 2 : Let $f_l\,(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha\,, \beta)$ for each $(l = 1,2\,...\,h)$, then $(f_1 * f_2 * ... * f_h)(z) \in A_q(\mathrm{k}, \mathrm{n}, \delta, \beta)$, where

$$\delta = 1 - \frac{\left\{k\,[k]_q\,(1+\beta) - \beta\right\}(1-\alpha)^h}{\left\{k\,[k]_q\,(1+\beta) - (\alpha+\beta)\right\}^h} \qquad (k \geq 2). \tag{28}$$

Theorem 3: Let $f_l\,(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha_l\,, \beta)$ for each $l = 1,2\,...\,h$ and suppose that

$$F(z) = z - \sum_{k=2}^{\infty} \left(\sum_{l=1}^{h} a_{k,l}^t \right) z^k \qquad (t > 1, k \geq 2). \tag{29}$$

Then, $F(z) \in A_q(\mathrm{k}, \mathrm{n}, \gamma_h\,, \beta)$, where

$$\gamma_h = 1 - \frac{h\,(1+\beta)(1-\alpha_l)^t\,[2]_q^n\,\left[2\,[2]_q + 1\right]}{[2]_q^n \left\{2\,[2]_q\,(1+\beta) - (\alpha_l+\beta)\right\}^t - h(1-\alpha_l)^t} \tag{30}$$

The result is sharp for the functions $f_l\,(z)\;(l = 1,2, \dots, h)$ that are given by (21).

Proof : Since $f_l\,(z) \in A_q(\mathrm{k}, \mathrm{n}, \alpha_l\,, \beta)$, by (19) , we obtain

$$\sum_{k=2}^{\infty} \frac{[k]_q^n \left\{(1+\beta)k\,[k]_q - (\alpha_l+\beta)\right\}}{(1-\alpha_l)} a_{k,l} \leq 1 \qquad (l = 1,2, \dots h)$$

By virtue of the Cauchy-Schwarz inequality, we get

$$\sum_{k=2}^{\infty} \left[\frac{[k]_q^n \left\{(1+\beta)k\,[k]_q - (\alpha_l+\beta)\right\}}{(1-\alpha_l)} \right]^t a_{k,l}^t$$

$$\leq \left(\sum_{k=2}^{\infty} \frac{[k]_q^n \left\{(1+\beta)k\,[k]_q - (\alpha_l+\beta)\right\}}{(1-\alpha_l)} a_{k,l} \right)^t \leq 1. \tag{31}$$

It follows from (31) that

$$\sum_{k=2}^{\infty} \left[\frac{1}{h} \left(\sum_{k=2}^{\infty} \frac{[k]_q^n \left\{(1+\beta)k\,[k]_q - (\alpha_l+\beta)\right\}}{(1-\alpha_l)} \right)^t a_{k,l}^t \right] \leq 1.$$

By setting

$$\alpha = \min_{1 \leq l \leq h} \{\alpha_l\}.$$

Therefore, to prove our result we need to find the largest γ_h such that

$$\sum_{k=2}^{\infty}\frac{[\mathrm{k}]_{\mathrm{q}}^{\mathrm{n}}\ \{(1+\beta)\mathrm{k}\,[\mathrm{k}]_{\mathrm{q}}\ -(\gamma_h\ +\beta)\}}{(1-\ \gamma_h\ )}\left(\sum_{l=1}^{h}a_{k,l}^{t}\right)\le 1.\qquad (t>1,k\ge 2)$$

That is that

$$\frac{[k]_q^n\ \{(1+\beta)k\,[k]_q\ -(\gamma_h\ +\beta)\}}{(1-\ \gamma_h\ )}\le\frac{1}{h}\left(\sum_{k=2}^{\infty}\frac{[k]_q^n\ \{(1+\beta)k\,[k]_q\ -(\alpha_l+\beta)\}}{(1-\alpha_l)}\right)^{t}.$$

Which leads to

$$\gamma_h\ \le 1-\frac{h(1+\beta)(1-\alpha_l)^t\big[[\mathrm{k}]_{\mathrm{q}}^{\mathrm{n}}\{k[\mathrm{k}]_{\mathrm{q}}\ +1\}\big]}{\big[[\mathrm{k}]_{\mathrm{q}}^{\mathrm{n}}\ \{(1+\beta)\mathrm{k}\,[\mathrm{k}]_{\mathrm{q}}\ -(\alpha_l+\beta)\}\big]^t-h(1-\alpha_l)^t}.$$

Now, let

$$\Psi(k)=1-\frac{h(1+\beta)(1-\alpha_l)^t\big[[k]_q^n\{k[k]_q\ +1\}\big]}{\big[[k]_q^n\ \{(1+\beta)k\,[k]_q\ -(\alpha_l+\beta)\}\big]^t-h(1-\alpha_l)^t}$$

Since $\Psi(\mathrm{k})$ is an increasing function of ($k\in N$), then we get

$$\gamma_h\ =\Psi(2)=1-\ \frac{h(1+\beta)(1-\alpha_l)^t\big[[2]_{\mathrm{q}}^{\mathrm{n}}\{2[2]_{\mathrm{q}}\ +1\}\big]}{\big[[2]_{\mathrm{q}}^{\mathrm{n}}\ \{(1+\beta)2\,[2]_{\mathrm{q}}\ -(\alpha_l+\beta)\}\big]^t-h(1-\alpha_l)^t},$$

So we can see that $0\le\ \gamma_h\ <1$. The result is sharp for the functions $f_l\ (z)\ (l=1,2,\dots,h)$ that are given by (21). The proof of Theorem 4 is thus completed.
If we put t=2 and $\alpha_l=\alpha\ (l=1,2,\dots,h)$ in Theorem 3, we obtain the following result.

Corollary 3 : Let $f_l\ (z)\in A_q(\mathrm{k},\mathrm{n},\alpha\ ,\beta)$ for each $(l=1,2\dots h)$ and suppose that

$$F(z)=z-\sum_{k=2}^{\infty}\left(\sum_{l=1}^{h}a_{k,l}^{2}\right)z^k\qquad (k\ge 2).\qquad (32)$$

Then, $F(z)\in A_q\big(\mathrm{k},\mathrm{n},\alpha,\ \gamma_h\ \ ,\beta\big)$ where

$$\gamma_h\ =1-\frac{h(1+\beta)(1-\alpha_l)^2\big[[2]_{\mathrm{q}}^{\mathrm{n}}\{2[2]_{\mathrm{q}}\ +1\}\big]}{\big[[2]_{\mathrm{q}}^{\mathrm{n}}\ \{(1+\beta)2\,[2]_{\mathrm{q}}\ -(\alpha_l+\beta)\}\big]^2-h(1-\alpha_l)^2},\qquad (33)$$

The result is sharp for the functions $f_l\ (z)\ (l=1,2,\dots,h)$ that are given by (21).

Theorem 4: Let $f_l\ (z)\in A_q\big(\mathrm{k},\mathrm{n},\alpha_l\ ,\beta\big)$ for each $(l=1,2,\dots,h)$ and suppose that the functions $g_s\ (z)$ are defined by

$$g_s\ (z)=z-\sum_{k=2}^{\infty}b_k\ z^k\qquad (\ b_k\ \ \ge 0,\mathrm{k}\ge 2),\qquad (34)$$

in the class $A_q\big(\mathrm{k},\mathrm{n},\alpha_s\ ,\beta\big)\ (s=1,2,\dots,t)$. Then
$(f_1\ *f_2\ *\dots*f_h*g_1\ *g_2\ *\dots*g_t\ )(z)\in\ A_q(k,n,\psi,\beta)$, where

$$\psi=1-\frac{\{\mathrm{k}\,[\mathrm{k}]_{\mathrm{q}}\,(1+\beta)-\beta\}\ \prod_{l=1}^{h}(1-\alpha_l)\ \prod_{s=1}^{t}(1-\alpha_s)}{\prod_{l=1}^{h}\{\mathrm{k}\,[\mathrm{k}]_{\mathrm{q}}\,(1+\beta)-(\alpha_l+\beta)\}\ \prod_{s=1}^{t}\{\mathrm{k}\,[\mathrm{k}]_{\mathrm{q}}\,(1+\beta)-(\alpha_s+\beta)\}}\ .\qquad (35)$$

The end result is sharp for $f_l\ (z)\ (l=1,2,\dots,h)$ that are provided by (21) and $g_s\ (z)$ that are given by

$$g_s\ (z)=z-\frac{1-\alpha_s}{[\mathrm{k}]_{\mathrm{q}}^{\mathrm{n}}\ \{(1+\beta)\mathrm{k}\,[\mathrm{k}]_{\mathrm{q}}\ -(\alpha_s+\beta)\}}z^k\qquad (k\ge 2,s=1,2,\dots t).\qquad (36)$$

Proof : From (20), we get that if $f(z)\in A_q(\mathrm{k},\mathrm{n},\delta\ ,\beta)$ and $g_s\ (z)\in A_q(k,n,\mu,\beta)$, then $(f*g)(z)\in A_q\big(\mathrm{k},\mathrm{n},\psi,\beta\big)$, where

$$\psi = 1 - \frac{\left\{k\,[k]_q\,(1+\beta) - \beta\right\}\,(1-\delta)(1-\mu)}{\left\{k\,[k]_q\,(1+\beta) - (\delta+\beta\right\}\left\{k\,[k]_q\,(1+\beta) - (\mu+\beta)\right\}}.$$

Since Theorem 2 leads to $(f_1 * f_2 * \ldots * f_h)(z) \in A_q(k, n, \delta, \beta)$, where δ is defined by (20) and $(g_1 * g_2 * \ldots * g_t)(z) \in A_q(k, n, \mu, \beta)$, with

$$\mu = 1 - \frac{\left\{k\,[k]_q\,(1+\beta) - \beta\right\}\prod_{s=1}^{t}(1-\alpha_s)}{\prod_{s=1}^{t}\left\{k\,[k]_q\,(1+\beta) - (\alpha_s+\beta)\right\}} \tag{37}$$

Then, we have $(f_1 * f_2 * \ldots * f_h * g_1 * g_2 * \ldots * g_t)(z) \in A_q(k, n, \psi, \beta)$, where ψ is given by (35) , this completes the proof of theorem 4.

Letting $\alpha_l = \alpha\ (l = 1,2, \ldots, h)$ and $\alpha_s = \alpha\ (l = 1,2, \ldots, t)$ in Theorem 4, we obtain the following corollary

Corollary 4: Let the functions $f_l(z) \in A_q(k, n, \alpha, \beta)$ $(l = 1,2, \ldots h)$ and the functions $g_s(z)$ that are defined by (36) in the class $A_q(k, n, \alpha, \beta)$. Then, we have $(f_1 * f_2 * \ldots * f_h * g_1 * g_2 * \ldots * g_t)(z) \in A_q(k, n, \psi, \beta)$, where ψ

$$\psi = 1 - \frac{\left\{k\,[k]_q\,(1+\beta) - \beta\right\}(1-\alpha)^{h+t}}{\left\{k\,[k]_q\,(1+\beta) - (\alpha+\beta)\right\}^{h+t}}. \tag{38}$$

Conclusions

In this work, properties and results on the Quasi-Hadamard products have been shown . This is done for a new subclass of analytic functions of β-Uniformly univalent function that is defined by the Salagean q-differential operator.

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