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The Dynamics of Sokol-Howell Prey-Predator Model Involving Strong Allee Effect

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Abstract

In this paper, a Sokol-Howell prey-predator model involving strong Allee effect is proposed and analyzed. The existence, uniqueness, and boundedness are studied. All the five possible equilibria have been are obtained and their local stability conditions are established. Using Sotomayor's theorem, the conditions of local saddle-node and transcritical and pitchfork bifurcation are derived and drawn. Numerical simulations are performed to clarify the analytical results.

Keywords: Prey-Predator model, Sokol-Howell functional response, Strong Allee Effect, stability, bifurcation.

ديناميكية نموذج سوكول–هاويل للفريسة–المفترس المتضمن تأثير آلي من النوع القوي

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الخلاصة

في هذه الورقة، تم اقتراح وتحليل نموذج سوكول–هاول للفريسة والمفترس الذي يتضمن تأثير آلي من النوع القوي. تمت دراسة وجود الحل ووحدانيته إضافة لإيجاد قيد الحل. تم ايجاد جميع نقاط التوازن الخمسة المحتملة، وتم وضع شروط الاستقرار المحلية الخاصة بهذه النقط. باستخدام نظرية سوتومايور تم ايجاد شروط التشعب وتقديم رسوم لكل من تشعب عقدة السرج المحلية والتشعب الحرج وتشعب المذراة. كما تم إجراء عمليات المحاكاة العددية لتوضيح النتائج التحليلية.

1. Introduction

The study of prey-predator dynamics is important in both theoretical ecology and applied mathematics. It is used to descript the relationships between species and their surrounding environment in addition to the connections between different species. Therefore, the prey-predator models have been receiving great interest in population dynamics. Consequently, varieties of mathematical models have been studied by many researchers. These models were written in the framework of the traditional work given by Lotka [1] and Volterra [2]. In fact, the traditional Lotka-Volterra model serves as the basis for many models used today to analyze population dynamics. Many of the well-known proposed classical models were developed by many researchers taking into consideration various environmental factors that affect the existence and stability of this system, such as prey refuge [3-5], disease,[6, 7], delay [8], harvesting [9] and many other factors [10-13].

Note that, the Lotka-Volterra prey-predator model in its general form can be expressed by [14]:

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$$\frac{dN}{dt} = f(N)N - a_1 g(N)P,$$
$$\frac{dP}{dt} = -dP + a_2 g(N)P,$$

where $N(t)$ and $P(t)$ denote the prey and predator individuals. While $f(N)$ is the prey's growth rate per capita, d is the predator's death rate in the absence of prey and a_1 and a_2 are the prey and predator interaction rates, respectively. The function $g(N)$ is the functional response of the predators, which corresponds to the saturation of their appetites and reproductive capacity. In theoretical ecology, several known functional responses in the prey-predator system exist, which include Holling type-I, type-II, type-III, type-IV, Beddington–DeAngelis type, and ratio-dependent type, etc. Some authors investigated and introduced several open questions for the structured prey-predator models using different types of functional responses, [12, 13, 15, 16]. In this paper, however, Sokol-Howell type of functional response [17] that is expressed as $g(N) = \frac{\alpha_1 N}{\alpha_2 + N^2}$ is used, which is simply a modified version of the Holling type III that is represented by $g(N) = \frac{\alpha_1 N}{\alpha_2 + N + \alpha_3 N^2}$. Although type III functional response and their modification represented by Sokol-Howell type are similar to type II at high levels of prey density, where the saturation occurs, at low prey density levels, the graphical relationship of the number of prey consumed by predators and the density of the prey population is an increasing function. The deacceleration property of Sokol-Howell type of functional response at the high levels of prey density can be used to describing the group defense property of some types of prey, such as buffalo against their predators such as lions or tigers.

Allee [18] was the first to describe the Allee effect in 1931. He reviewed the evidence regarding the impacts of population density on demographic and life-history attributes, demonstrating that the growth rate is not always positive for small densities, and it may not be decreasing as predicted by the logistic model. In other words, the Allee effect might reduce the intrinsic growth at low population densities, making the system unstable. There are a lot of factors that may cause the Allee effect; for example, difficulties in finding mates, predator avoidance of defense, social dysfunction in small population sizes, genetic drift, food exploitation, and several other causes. These effects can be observed in various species, including vertebrates, invertebrates, and plants. The effect usually saturates or fades with increasing population size [19]. There are two types of Allee effect: (i) The strong Allee effect has a negative per capita growth rate at a low population level and implies the existence of a threshold level of the population, so that the species become extinct below this level, causing an unstable equilibrium at some small, non-zero population size. The population must exceed this threshold to grow and avoid extinction. (ii) The weak Allee effect that has a decreasing per capita growth rate but remains positive at a low population level, which causes an unstable equilibrium [20].

The goal of this work is to investigate the Sokol-Howell prey-predator system, which has a strong Allee effect on the prey species. The existence and stability of equilibria are examined theoretically and numerically. Also, a rich investigation of the effects of varying the parameter values is given.

2. Mathematical Model Formulation

A mathematical model that simulates the dynamics of the prey-predator system involving the strong Allee effect in prey species and harvesting is formulated mathematically. It is supposed that the prey grows logistically in the absence of the predator, while the predator consumes the prey according to the Sokol-Howell type of functional response. Therefore, the dynamics of such a system can be defined as follows:

$$\begin{aligned} \frac{dN}{dT} &= rN\left(1 - \frac{N}{k}\right)\left(\frac{N}{M} - 1\right) - \frac{amNP}{b + \gamma m^2 N^2} - q_1 E_1 N, \\ \frac{dP}{dT} &= e\frac{amNP}{b + \gamma m^2 N^2} - dP - q_2 E_2 P, \end{aligned} \qquad (1)$$

where

$N(t) \geq 0$: the density of prey at time T,
$P(t) \geq 0$: the density of predator at time T
r: the prey-intrinsic growth rate,
k: the environment-carrying capacity,
$M \ll k$: the Allee threshold of the prey population in the absence of predators,
a: the maximum attack (predation) rate,

$(1-m)\in(0,1)$: the prey-refuge rate, such that mN is the available prey for predation,
b: the half-saturation constant,
$q_i\,,(i=1,2)$: the catchability constant,
$e\in(0,1)$: the food-conversion efficiency,
d: the predator-natural death rate,
γ: the inverse measure of inhibitory effect,
$E_i\,,(i=1,2)$: the harvesting effort.

All the above parameters are positive constants.

Note that, using the scaling parameters $rT=t$, $x=\frac{N}{k}$ and $y=\frac{aP}{r\gamma k^2}$ in the system (1) reduces the number of parameters from 13 to 7 and the system (1) takes the following dimensionless form:

$$\frac{dx}{dt}=x\left[(1-x)(w_1x-1)-\frac{my}{w_2+m^2x^2}-w_3\right]=xf(x,y),$$
$$\frac{dy}{dt}=y\left[\frac{w_4mx}{w_2+m^2x^2}-(w_5+w_6)\right]=yg(x,y), \tag{2}$$

where the dimensionless parameters are given by:

$$w_1=\frac{k}{M},\ w_2=\frac{b}{\gamma k^2},\ w_3=\frac{q_1E_1}{r},\ w_4=\frac{ea}{r\gamma k},\ w_5=\frac{d}{r},\ w_6=\frac{q_2E_2}{r}.$$

Note that, since the right-hand side of the interaction functions of the system (2) are continuous and have continuous partial derivatives, then system (2) has a unique solution that belongs to the positive quadrant $\mathbb{R}_+^2$.

3. Positivity and Boundedness

The positivity of the solution ensures that any solution of the system (2) with positive initial conditions remains at all times positive. While the boundedness confirms that the solution of the system (2) cannot increase without limit and hence it converges to an attractor.

Theorem 1: *All solutions of the system (2) that start in $\mathbb{R}_+^2$ remain positive forever.*

Proof: Let $(x(t),y(t))$ be any solution of the system (2). Then it is clear that:

$$x(t)=x(0)e^{\int_0^t f(x(s),y(s))ds}\ge 0.$$

and

$$y(t)=y(0)e^{\int_0^t g(x(s),y(s))ds}\ge 0.\blacksquare$$

Theorem 2: *All the solutions of the system (2) that start in $\mathbb{R}_+^2$ are uniformly bounded.*

Proof: Consider the function $G(x)=x[1+(1-x)(w_1x-1)]$. Hence it is easy to show that $G(x)$ reaches its maximum value at $x=\tilde{x}=\frac{2(1+w_1)}{3w_1}$, and then the maximum value of $G(x)$ is given by $G_{max}=\frac{4(1+w_1)^3}{27w_1^2}>0$. Let

$$W=Ax(t)+By(t),$$

where $A=\frac{1}{G_{max}}$, and $B=\frac{A}{w_4}$. Then

$$\dot{W}=A\dot{x}+B\dot{y}=1-Ax-Aw_3x-B(w_5+w_6)y=1-A(1+w_3)x-B(w_5+w_6)y$$
$$=1-(1+w_3)(Ax+B\sigma y)<1-(1+w_3)\tau W,$$

where $\sigma=\frac{w_5+w_6}{1+w_3}$ and $\tau=min\{1,\sigma\}$.

That is

$$\dot{W}+(1+w_3)\tau W\le 1.$$

So, we obtain that

$$0\le W\big(x(t),y(t)\big)\le\frac{1}{(1+w_3)\tau}+e^{-(1+w_3)\tau t}W(x(0),y(0)).$$

Therefore, as $t\to\infty$, yields

$$0\le W\le\frac{1}{(1+w_3)\tau}.$$

Thus all the solutions of the system (2) enter into the region:

$$\Omega=\left\{(x,y):0\le W\le\frac{1}{(1+w_3)\tau}+\varepsilon,\text{for any }\varepsilon>0\right\}.\blacksquare$$

4. Qualitative Analysis of Equilibria

In this section, the existence of the equilibria of the system (2) and the qualitative study of their stability are carried out. The computation shows that system (2) has five equilibria. The trivial equilibrium point $E_0(0,0)$ always exists. The axial equilibrium points, say $E_i(x_i^*,0)$ for $i=1,2$, where x_i^* is given in equation (3), exist under the condition (4).

$$x_i^* = \frac{1+w_1 \mp \sqrt{(w_1-1)^2-4w_1w_3}}{2w_1}. \tag{3}$$

$$(w_1-1)^2 > 4w_1w_3. \tag{4}$$

While $E(0,y)$ cannot exist due to the fact that the predators cannot coexist without the availability of their food given by the prey. The coexistence equilibrium points, say $E_i(x_i^*,y^*)$ for $i=3,4$, where x_i^* and y^*, are given by equation (5) and they exist together provided that the conditions (6a)-(6b) are satisfied simultaneously.

$$x_3^* = \frac{w_4+\sqrt{w_4^2-4w_2(w_5+w_6)^2}}{2m(w_5+w_6)}, x_4^* = \frac{w_4-\sqrt{w_4^2-4w_2(w_5+w_6)^2}}{2m(w_5+w_6)}. \tag{5}$$

$$y^* = \frac{w_2+m^2x_i^{*2}}{m}[(1-x_i^*)(w_1x_i^*-1)-w_3]$$

$$w_4^2 \geq 4w_2(w_5+w_6)^2. \tag{6a}$$

$$(1-x_i^*)(w_1x_i^*-1) > w_3. \tag{6b}$$

Note that, in the case of satisfying condition (6a) but not condition (6b), then we may or may not have a unique coexistence equilibrium point. However, when equality occurs in the condition (6a), both the coexistence equilibrium points are coincided with each other and the system (2) has a unique coexistence point, so that $E_3 = E_4 = E(x^*,y^*) = E\left(\frac{w_4}{2m(w_5+w_6)}, \frac{4w_2(w_5+w_6)^2+w_4{}^2}{4m(w_5+w_6)^2}[(w_1+1)x^* - w_1x^{*2} - (1+w_3)]\right)$.

The Jacobian matrix of the system (2) about an arbitrary point (x,y) is determined by:

$$J(x,y) = \begin{bmatrix} x\frac{\partial f}{\partial x}+f & x\frac{\partial f}{\partial y} \\ y\frac{\partial g}{\partial x} & y\frac{\partial g}{\partial y}+g \end{bmatrix}, \tag{7}$$

where

$$\frac{\partial f}{\partial x} = 1+w_1-2w_1x+\frac{2m^3xy}{(w_2+m^2x^2)^2}, \frac{\partial f}{\partial y} = \frac{-m}{w_2+m^2x^2}, \frac{\partial g}{\partial x} = \frac{mw_4(w_2-m^2x^2)}{(w_2+m^2x^2)^2}, \text{ and } \frac{\partial g}{\partial y} = 0.$$

Recall that, if all the eigenvalues of the Jacobian matrix at an equilibrium point have negative real parts, then this point is locally asymptotically stable. Accordingly, the following theorems present the local stability conditions for each of the above equilibria.

Theorem 3: *The trivial equilibrium point E_0 is always a locally asymptotically stable equilibrium point.*

Proof: Depending on the general Jacobian matrix that is given by (7), the Jacobian matrix at $E_0(0,0)$ is given by:

$$J(0,0) = \begin{bmatrix} -(1+w_3) & 0 \\ 0 & -(w_5+w_6) \end{bmatrix}. \tag{8}$$

Since the eigenvalues are $\lambda_1 = -(1+w_3) < 0$ and $\lambda_2 = -(w_5+w_6) < 0$, the trivial equilibrium point is local asymptotical stable.

Theorem 4: *The axial equilibrium point $E_1(x_1^*,0)$ of the system (2) is a locally asymptotically stable equilibrium point if condition (9) holds, while it is a saddle-node when this condition is reflected.*

$$\frac{x_1^*}{w_2+m^2x_1^{*2}} < \frac{w_5+w_6}{mw_4}. \tag{9}$$

Proof: At $E_1(x_1^*,0)$, the Jacobian matrix can be written as

$$J(x_1^*,0) = \begin{bmatrix} x_1^*[1+w_1-2w_1x_1^*] & \frac{-mx_1^*}{w_2+m^2x_1^{*2}} \\ 0 & \frac{w_4mx_1^*}{w_2+m^2x_1^{*2}} - (w_5+w_6) \end{bmatrix}, \tag{10}$$

where $x_1^* = \frac{1+w_1+\sqrt{\mu}}{2w_1}$ and $\mu = (w_1-1)^2 - 4w_1w_3$. Therefore, the eigenvalues of $J(x_1^*,0)$ are given by

$$\lambda_1 = x_1^*[1+w_1-2w_1x_1^*] = x_1^*\left[1+w_1-2w_1\frac{1+w_1+\sqrt{\mu}}{2w_1}\right] = -x_1^*\sqrt{\mu} < 0. \tag{11a}$$

$$\lambda_2 = \frac{w_4 m x_1^*}{w_2+m^2 x_1^{*2}} - (w_5 + w_6). \tag{11b}$$

Hence, the two eigenvalues are negative according to the given condition and the equilibrium points are locally asymptotically stable. However, λ_2 will be positive if condition (9) is reflected, and hence the axial equilibrium point $E_1(x_1^*, 0)$ becomes a saddle node.

Theorem 5: *The equilibrium point $E_2(x_2^*, 0)$ of the system (2) is an unstable node if the following condition holds, while it is a saddle-node when this condition is reflected.*

$$\frac{x_2^*}{w_2+m^2 x_2^{*2}} > \frac{w_5+w_6}{m w_4}. \tag{12}$$

Proof: At $E_2(x_2^*, 0)$, the Jacobian matrix is given by

$$J(x_2^*, 0) = \begin{bmatrix} x_2^*[1+w_1 - 2w_1 x_2^*] & \frac{-m x_2^*}{w_2+m^2 x_2^{*2}} \\ 0 & \frac{w_4 m x_2^*}{w_2+m^2 x_2^{*2}} - (w_5 + w_6) \end{bmatrix}, \tag{13}$$

where $x_2^* = \frac{1+w_1-\sqrt{\mu}}{2w_1}$. Clearly, the eigenvalues of $J(x_2^*, 0)$ are:

$$\lambda_1 = x_2^*[1+w_1 - 2w_1 x_2^*] = x_2^*\left[1+w_1 - 2w_1 \frac{1+w_1-\sqrt{\mu}}{2w_1}\right] = x_2^*\sqrt{\mu} > 0. \tag{14a}$$

$$\lambda_2 = \frac{w_4 m x_2^*}{w_2+m^2 x_2^{*2}} - (w_5 + w_6). \tag{14b}$$

Hence, the two eigenvalues are positive according to the condition (12) and the equilibrium point is an unstable node. However, λ_2 will be negative if condition (12) is reflected, and hence the axial equilibrium point $E_2(x_2^*, 0)$ becomes a saddle node.

Theorem 6: *The coexistence equilibrium point $E_4(x_4^*, y^*)$ of the system (2) is locally asymptotically stable, while $E_3(x_3^*, y^*)$ is a saddle point provided that the following conditions hold.*

$$2w_1 x_i^* > 1+w_1 + \frac{2m^3 x_i^* y^*}{(w_2+m^2 x_i^{*2})^2}. \tag{15a}$$

$$w_2 > m^2 x_i^{*2}. \tag{15b}$$

Proof: Depending on the general Jacobian matrix given by equation (7), the Jacobian matrix at $E_i(x_i^*, y^*)$ for $i = 3,4$ is given by:

$$J(x_i^*, y_i^*) = \begin{bmatrix} x_i^*\left[1+w_1 - 2w_1 x_i^* + \frac{2m^3 x_i^* y^*}{(w_2+m^2 x_i^{*2})^2}\right] & \frac{-m x_i^*}{w_2+m^2 x_i^{*2}} \\ y^* \frac{m w_4 (w_2 - m^2 x_i^{*2})}{(w_2+m^2 x_i^{*2})^2} & 0 \end{bmatrix}, \tag{16}$$

where x_i^* and y^* are given in equation (5).

Therefore, the eigenvalues of $J(x_i^*, y^*)$ are the roots of the characteristic equation given by:

$$\lambda^2 + \alpha_1 \lambda + \alpha_2 = 0, \tag{17}$$

where $\alpha_1 = -x_i^*\left[1+w_1 - 2w_1 x_i^* + \frac{2m^3 x_i^* y^*}{(w_2+m^2 x_i^{*2})^2}\right]$ and $\alpha_2 = \frac{m^2 x_i^* y^* w_4 (w_2 - m^2 x_i^{*2})}{(w_2+m^2 x_i^{*2})^3}$. Recall that, according to the Routh-Hurwitz criterion [21], when the coefficients α_1 and α_2 are positive, then the equilibrium point is locally asymptotically stable, while if α_1 is positive and α_2 is negative, then the equilibrium point is a saddle point. Straightforward computation shows that $\alpha_1 > 0$ and $\alpha_2 > 0$ under the conditions (15a)-(15b) at the point $E_4(x_4^*, y^*)$, however $\alpha_1 > 0$ and $\alpha_2 < 0$ under the conditions (15a)-(15b) at the point $E_3(x_3^*, y^*)$. Hence the coexistence equilibrium point $E_4(x_4^*, y^*)$ is locally asymptotically stable and $E_3(x_3^*, y^*)$ is a saddle point. ■

5. Bifurcation Analysis

The occurrence of the local bifurcation of the system (2) is discussed in this section using Sotomayor's theorem [22],[23], which gives the necessary and sufficient conditions for three types ((i) saddle-node, (ii) transcritical, and (iii) pitchfork) of local bifurcations to occur. Since the existence of non-hyperbolic equilibria is a necessary but not sufficient condition for the occurrence of the bifurcation near the equilibrium point [22], the candidate bifurcation parameter is chosen to ensure that the studied point will be non-hyperbolic for a specific parameter value. In other words, it is chosen to ensure that one of the Jacobian's eigenvalues at the bifurcation point is zero.

System (2) can be rewritten in the following vector forms to simplify the notations:

$$\frac{dX}{dt} = F(X), \text{ with } X = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } F = \begin{bmatrix} xf \\ yg \end{bmatrix}.$$

Then the second derivate of F with respect to X can be expressed as:

$$D^2F(X)(V,V)=\begin{bmatrix} v_1\left(\frac{2mv_2(m^2x^2-w_2)}{(m^2x^2+w_2)^2}+2v_1\left(1+w_1-3w_1x+\frac{m^3xy(3w_2-m^2x^2)}{(m^2x^2+w_2)^3}\right)\right) \\ \frac{2mv_1w_4\left(v_2(w_2^2-m^4x^4)+m^2v_1xy(m^2x^2-3w_2)\right)}{(m^2x^2+w_2)^3} \end{bmatrix} \quad (18a)$$

where $V=(v_1,v_2)^T$ is a general vector. Furthermore, we have

$$D^3F(X)(V,V,V)=\begin{bmatrix} 6v_1^2\left(-v_1w_1+\frac{m^3v_2x(3w_2-m^2x^2)}{(m^2x^2+w_2)^3}\right)+\frac{m^3v_1y(w_2^2-6m^2w_2x^2+m^4x^4)}{(m^2x^2+w_2)^4} \\ \frac{-6m^3v_1^2w_4(v_2x(3w_2-m^2x^2)(m^2x^2+w_2)+v_1(w_2^2-6m^2w_2x^2+m^4x^4)y)}{(m^2x^2+w_2)^4} \end{bmatrix} \quad (18b)$$

It is clear from the Jacobian matrix at the trivial equilibrium point E_0 given by equation (8) that there is no possibility to obtain a zero eigenvalue. Hence the trivial equilibrium point is not a non-hyperbolic point, which implies that no bifurcation may occur.

Theorem 7: *If the parameter w_4 passes through the value $w_4^*=\frac{w_5+w_6}{B}$, where $B=\frac{2mw_1\beta}{m^2\beta^2+4w_1^2w_2}$, with $\beta=(1+\alpha+w_1)$, and $\alpha=\sqrt{(w_1-1)^2-4w_1w_3}$, then the system (2) at the axial equilibrium point E_1 has*

i. No saddle-node bifurcation.

ii. Transcritical bifurcation provided that

$$m^2\beta^2\neq 4w_1^2w_2. \quad (19)$$

iii. A pitchfork bifurcation otherwise.

Proof: At E_1, the Jacobian matrix of system (2) with $w_4=w_4^*$ becomes:

$$J_1=DF(E_1,w_4^*)=\begin{bmatrix}\alpha+w_1+\frac{\beta}{w_1}-\frac{3\beta^2}{4w_1}-w_3 & -\frac{2mw_1\beta}{m^2\beta^2+4w_1^2w_2} \\ 0 & 0\end{bmatrix}.$$

Clearly J_1 has a zero eigenvalue, and the corresponding eigenvector for this eigenvalue can be written as:

$$U_1=\begin{bmatrix}-\frac{8mw_1^2\beta}{(m^2\beta^2+4w_1^2w_2)(3\alpha^2-(w_1-1)^2+2\alpha(1+w_1)+4w_1w_3)} \\ 1\end{bmatrix}.$$

While the eigenvector corresponding to the zero eigenvalue of ${J_1}^T$ is determined as:

$$W_1=\begin{bmatrix}0\\1\end{bmatrix}.$$

Differentiating F with respect to w_4 gives:

$$F_{w_4}=\begin{bmatrix}0\\ \frac{mxy}{w_2+m^2x^2}\end{bmatrix}.$$

Therefore, straightforward computation shows that:

$$W_1^TF_{w_4}(E_1,w_4^*)=0.$$

Consequently, by Sotomayor's theorem, the system (2) has no saddle-node bifurcation near E_1 and $w_4=w_4^*$. Moreover, direct computation gives that:

$$W_1^T\left[DF_{w_4}(E_1,w_4^*)U_1\right]=\frac{2mw_1\beta}{m^2\beta^2+4w_1^2w_2}=B\neq 0.$$

Also, using the form of D^2F given by equation (18a) and the eigenvectors U_1 and W_1 gives that:

$$W_1^T[D^2F(E_1,w_4^*)(U_1,U_1)]=\frac{32mw_1^3(m^2\beta^2-4w_1^2w_2)(w_5+w_6)}{(m^2\beta^2+4w_1^2w_2)^2(3\alpha^2-(w_1-1)^2+2\alpha(1+w_1)+4w_1w_3)}.$$

Therefore, condition (19) guarantees that $W_1^T[D^2F(E_1,w_4^*)(U_1,U_1)]\neq 0$. Hence, by Sotomayor's theorem, a transcritical bifurcation takes place.

Moreover, if the condition (19) is not satisfied, then we have $W_1^T[D^2F(E_1,w_4^*)(U_1,U_1)]=0$. In addition, using the form of D^3F given by equation (18b) and the eigenvectors U_1 and W_1 gives that:

$$W_1^T[D^3F(E_1,w_4^*)(U_1,U_1,U_1)]=\frac{1536m^4w_1^6\beta(m^2\beta^2-12w_1^2w_2)(w_5+w_6)}{(m^2\beta^2+4w_1^2w_2)^4(3\alpha^2-(w_1-1)^2+2\alpha(1+w_1)+4w_1w_3)^2}\neq 0.$$

Hence, a pitchfork bifurcation takes place and the proof is complete. ■

Theorem 8: *Assume that condition (15a) holds, then if the parameter w_4 passes through the value $w_4^{**}=2\sqrt{w_2}(w_5+w_6)$, the system (2) at the coexistence equilibrium points E_i $(i=3,4)$ has* *i. A saddle-node bifurcation provided that*

$$m(1+w_1) \neq \frac{w_1 w_2 + m^2(1+w_3)}{\sqrt{w_2}}. \tag{20}$$

ii. Otherwise, system (2) undergoes a pitchfork bifurcation but not a transcritical bifurcation.

Proof: Obviously, as $w_4 = w_4^{**}$, then the system (2) has a unique coexistence equilibrium point so that $E_3 = E_4 = E(x^*, y^*)$. The Jacobian matrix of the system (2), at $E(x^*, y^*)$ with $w_4 = w_4^{**}$, can be written as:

$$J_2 = DF(E, w_4^{**}) = \begin{bmatrix} \frac{2m(1+w_1)\sqrt{w_2} - 3w_1 w_2}{m^2} - (1+w_3) & -\frac{1}{2\sqrt{w_2}} \\ 0 & 0 \end{bmatrix}.$$

Therefore, J_2 has two eigenvalues; one negative $\lambda_1 = \frac{2m(1+w_1)\sqrt{w_2} - 3w_1 w_2}{m^2} - (1+w_3)$, which is negative under the condition (15a), while the other one is a zero eigenvalue $\lambda_2 = 0$, and the corresponding eigenvector of $\lambda_2 = 0$ can be written as:

$$U_2 = \begin{bmatrix} -\frac{m^2}{2\sqrt{w_2}(m^2(1+w_3) - 2m\sqrt{w_2}(1+w_1) + 3w_1 w_2)} \\ 1 \end{bmatrix}.$$

Clearly $m^2(1+w_3) - 2m\sqrt{w_2}(1+w_1) + 3w_1 w_2 > 0$ under the condition (15a), while the eigenvector corresponding to the $\lambda_2 = 0$ of ${J_2}^T$ is written as:

$$W_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, a simple calculation shows that under the condition (20) we obtain

$${W_2}^T F_{w_4}(E, w_4^{**}) = \frac{m w_2(1+w_1) - \left(w_1 w_2^{3/2} + m^2\sqrt{w_2}(1+w_3)\right)}{m^3} \neq 0.$$

and

$${W_2}^T[D^2F(E, w_4^{**})(U_2, U_2)] = -\frac{m^3\left(m w_2(1+w_1) - \left(w_1 w_2^{3/2} + m^2\sqrt{w_2}(1+w_3)\right)(w_5+w_6)\right)}{2w_2^{3/2}\left(3w_1 w_2 + m^2(1+w_3) - 2m\sqrt{w_2}(1+w_1)\right)^2} \neq 0.$$

Hence, system (2) has a saddle-node bifurcation.

Moreover, when the condition (20) is not satisfied, the following is obtained:

$${W_2}^T\left[DF_{w_4}(E, w_4^{**})U_2\right] = \frac{1}{2\sqrt{w_2}} \neq 0.$$

While: ${W_2}^T[D^2F(E, w_4^{**})(U_2, U_2)] = 0.$

Therefore, by Sotomayor's theorem, there is no transcritical bifurcation at E.

Moreover, using the form of D^3F given by equation (18b) and the eigenvectors U_2 and W_2 gives that:

$${W_2}^T[D^3F(E, w_4^{**})(U_2, U_2, U_2)] = \frac{3m^6(m(1+w_1) - 2w_1\sqrt{w_2})(w_5 w_6)}{4w_2^{3/2}\left(3w_1 w_2 + m^2(1+w_3) - 2m\sqrt{w_2}(1+w_1)\right)^3} \neq 0.$$

Hence, a pitchfork bifurcation takes place and this completes the proof. ■

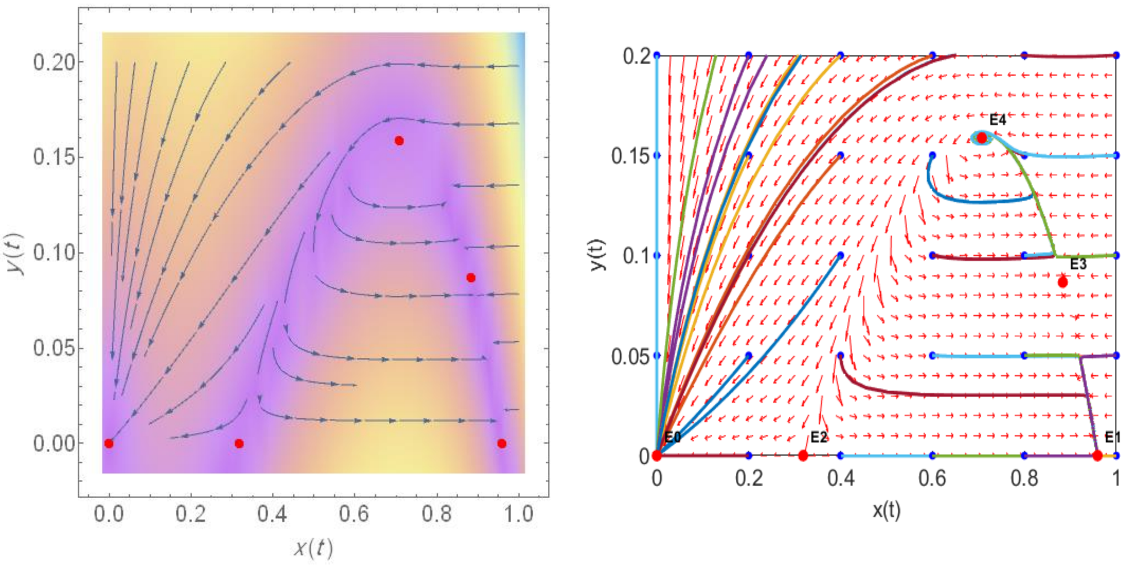
6. Numerical Analysis

In this section, the general dynamics of the system (2) are studied numerically for various sets of initial values with various sets of parameter values. The goals of this study are to show the effects of varying parameter values on the dynamical behavior of the system (2) as well as to approve the obtained theoretical results. The combination of parameter values, listed in Table 1, is used to carry out the numerical simulations.

Table 1- Parameters' values of system (2)

Parameters	Value
w_1	3.6
w_2	0.1
w_3	0.1
w_4	0.7
w_5	0.6
w_6	0.5
m	0.4

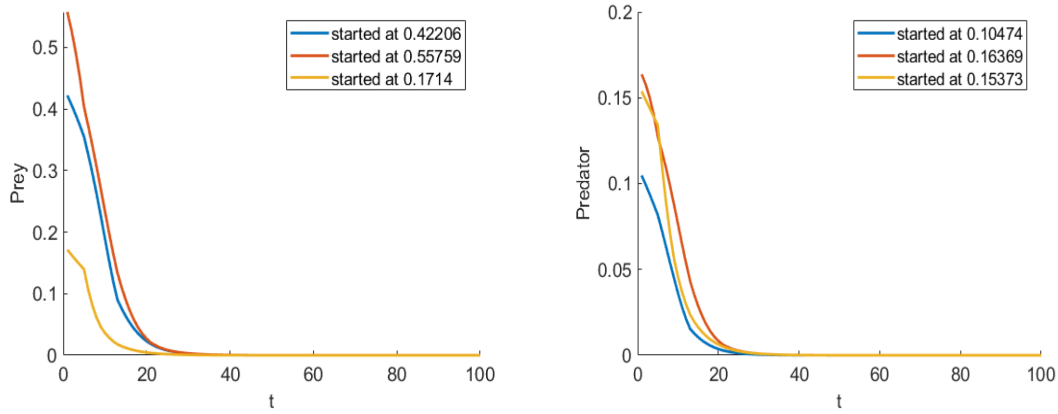
Now, using the parameters given in Table (1) with various sets of initial points, system (2) is solved numerically and then the trajectories that have been obtained are drawn in the form of phase portrait, as shown in Figure-1b, which coincides with line segments of the direction field of the system (1) given in Figure-1a, that is defined as the collection of small line segments passing through various points having a slope that will satisfy the given differential equation at that point. It is clear that Figure-1 shows the general dynamical behaviors of the system (2); in fact, the domain is divided into three disjoint regions (basins of attraction). The points in the first region approach asymptotically to the E_0, while the points in the second and third regions approach E_1 and E_4, respectively.



(a) **(b)**

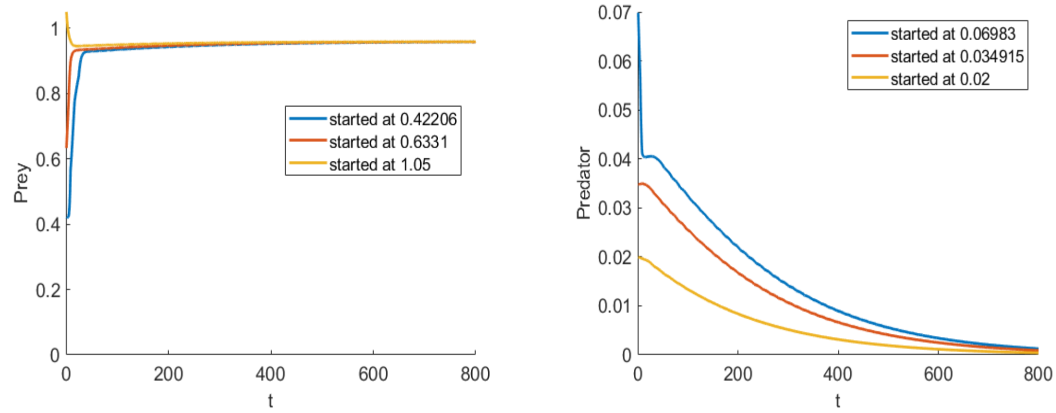
Figure 1- For the parameters given in Table (1) with different initial points. **(a)** The direction field of the system (2). **(b)** The phase portrait of the trajectories of the system (2) in which the system approaches asymptotically to three different points E_0, E_1, and E_4 depending on their initial points. (Red points are the equilibrium points and blue points are the initial points).

Three different initial points in each region are selected and then the time series for the obtained trajectories of system (2) are drawn in Figures- (2), (3), and (4). Clearly, the trajectories of system (2) approach to the E_0, E_1, and E_4 in Figures- (2), (3), and (4), respectively.



(a) **(b)**

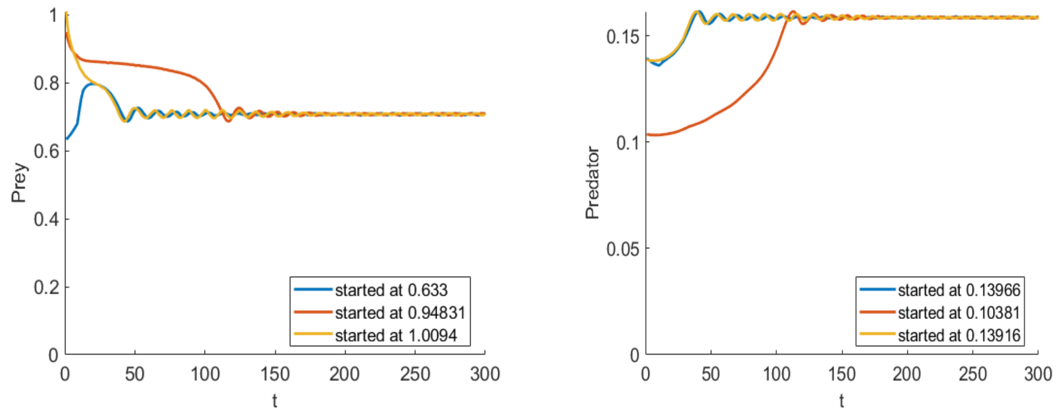
Figure 2- The system (2) approaches asymptotically to E_0 using parameters in Table (1). **(a)** Prey trajectories **(b)** Predator trajectories as a functions of time.



(a) **(b)**

Figure 3-The system (2) approaches asymptotically to $E_1(0.9592, 0)$ using parameters in Table (1). **(a)** Prey trajectories **(b)** Predator trajectories as a functions of time.

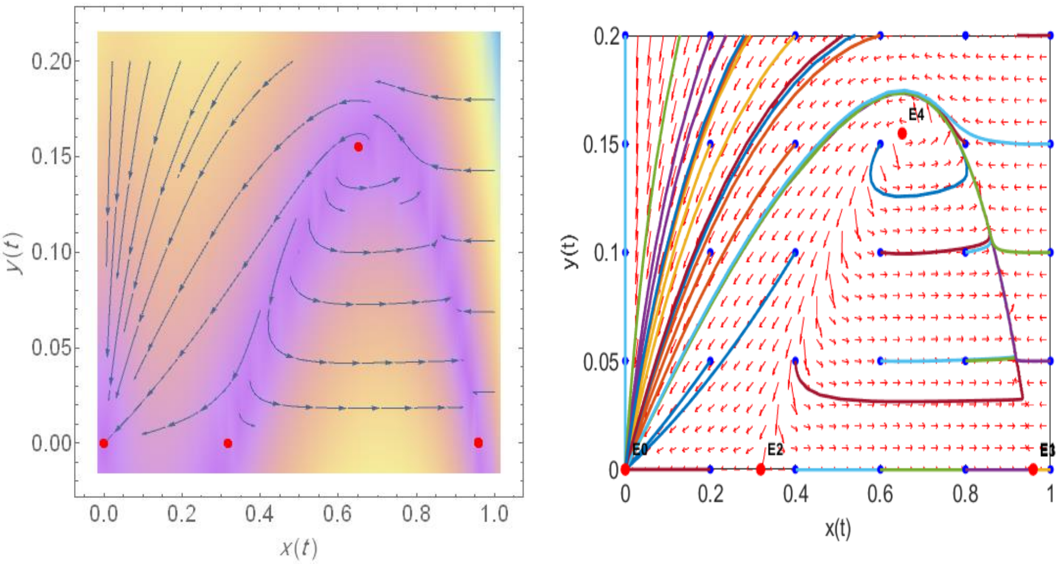
Obviously, for the parameters given in Table (1), the system (2) has five equilibrium points, as shown in Figure-1. Moreover, the local stability condition (9) for the axial equilibrium point $E_1(x_1^*, 0)$, and the conditions (15a) and (15b) for the coexistence point $E_4(x_4^*, y^*)$, are satisfied. Therefore, Figures- (2), (3) and (4) confirm the obtained theoretical finding and the system approaches asymptotically to E_0, E_1, and E_4, respectively, depending on the initial point's position.



(a) **(b)**

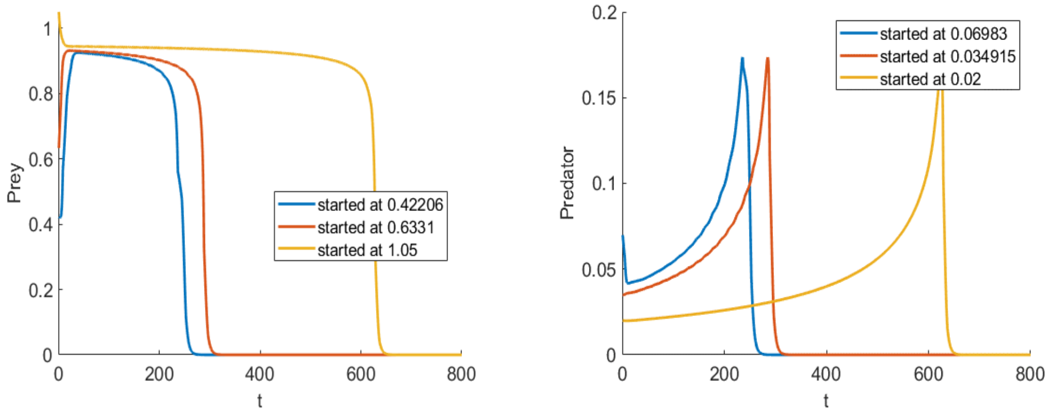
Figure 4- The system (2) approaches asymptotically to $E_4(0.7074, 0.1587)$ using parameters in Table (1). **(a)** Prey trajectories **(b)** Predator trajectories as a functions of time.

Now, the effect of the varying in the parameters set on the dynamical behavior of the system (2) is also investigated numerically. It is observed that as w_4 increases reaching the value $w_4 = 0.70875$, then $E_3 \equiv E_1$ and the equilibrium points E_1 and E_4 lose their stability so that all the trajectories of system (2) approach asymptotically to the origin E_0, as shown in Figure-5. Moreover for $w_4 > 0.70875$, keeping other parameters as given in Table-1, then E_3 dose not exist anymore and all the trajectories of system (2) approach E_0, as shown in Figure-6. These results are in agreement with what was proven in theorem (7).



(a) **(b)**

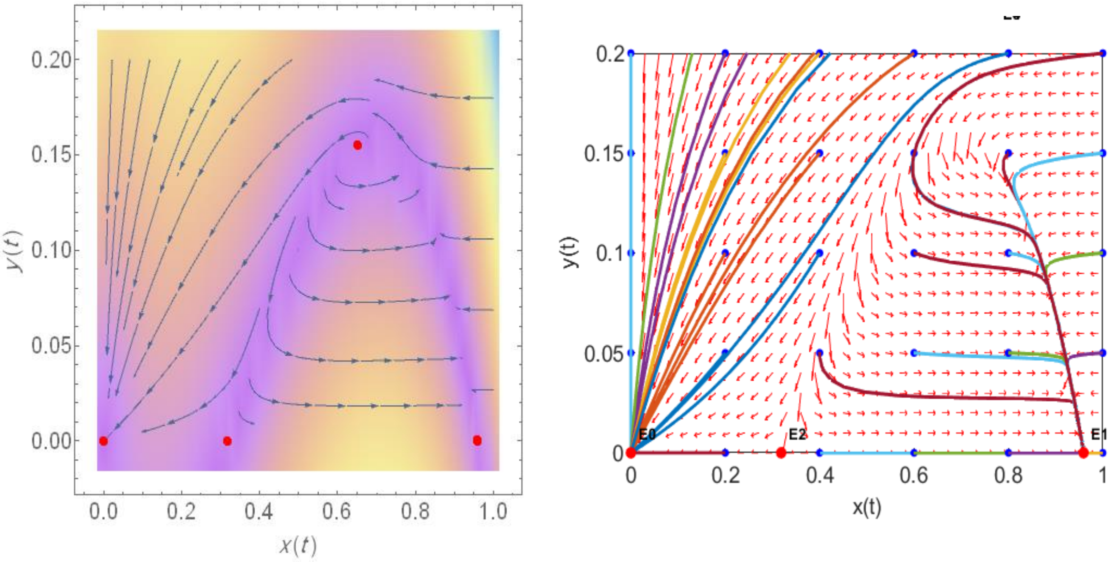
Figure 5- For $w_4 = 0.70875$ and the rest of the parameters as given in Table (1) with different initial points. (a) The direction field of the system (2). (b) Phase portrait of the trajectories of the system (2) in which the system approaches asymptotically to E_0 for all initial points, while E_2, E_3, and E_4 are unstable. (Red points are the equilibrium points and blue points are the initial points)



(a) **(b)**

Figure 6- The system (2) approaches asymptotically to E_0 using the parameters in Table (1) with $w_4 > 0.70875$ starting from different initial points. **(a)** Prey trajectories **(b)** Predator trajectories as a functions of time.

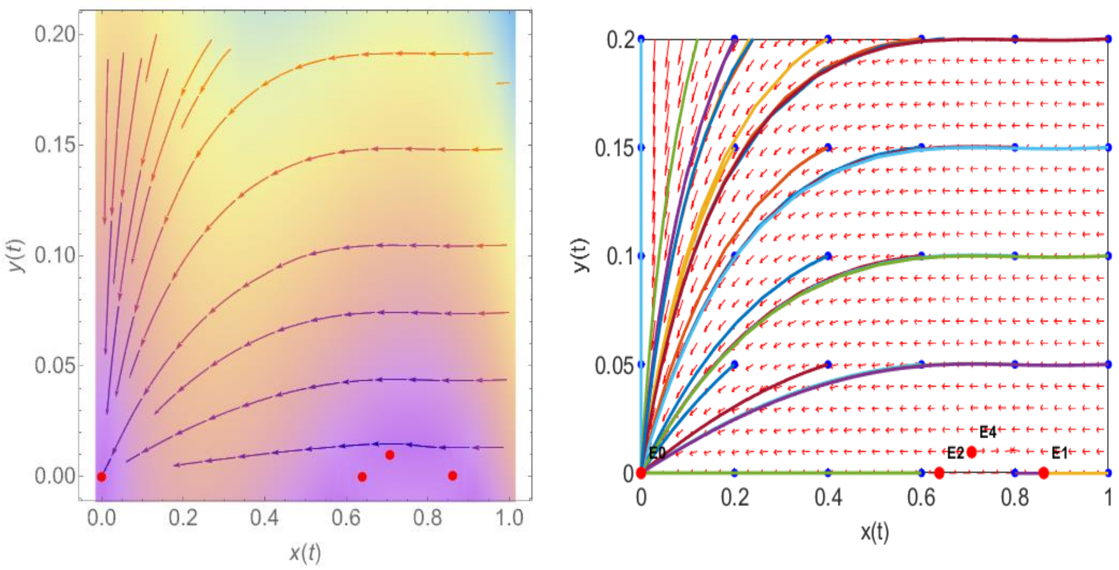
On the other hand, if we decrease w_4 reaching the value $w_4^{**} = 0.6957$, it is observed that $E_3 \equiv E_4$, and the coexistence equilibrium point loses its stability, as shown in Figure-7. Moreover, for $w_4 < 0.6957$, both E_3 and E_4 do not exist anymore. This result agrees with what was proven in theorem (8).



(a) **(b)**

Figure 7- For $w_4 = 0.6957$ and the rest of the parameters given in Table (1) with different initial points. **(a)** The direction field of the system (2). **(b)** Phase portrait of the trajectories of the system (2) in which the system approaches asymptotically to E_0 and E_1 depending on the initial points, while E_2 and E_4 are unstable. (Red points are the equilibrium points and blue points are the initial points)

While Figure-1 showed the dynamic behaviors of the system when $w_1 > 2.103269$, Figure-8 shows that when $w_1 < 2.103269$, the system has a unique equilibrium point E_0. Moreover, E_3 does not exist anymore.



(a) **(b)**

Figure 8- For $w_1 < 2.10329$ and the rest of the parameters given in Table-1 with different initial points. **(a)** The direction field of the system (2). **(b)** Phase portrait of the trajectories of the system (2) in which the system approaches asymptotically to E_0, while E_1, E_2, and E_4 are unstable and E_3 does not exist. (Red points are the equilibrium points and blue points are the initial points)

The dynamics of the system (2) are further investigated with varying other parameters and Table-2 summarizes the obtained results.

Table 2-The effects of varying the parameters' values on the dynamics of the system (2).(S=Stable, U=Unstable, and N=Not existing)

Parameter	Value	The dynamic behavior of the equilibria				
		E_0	E_1	E_2	E_3	E_4
w_1	$w_1 < 2.103269$	S	U	U	N	U
	$2.103269 < w_1$	S	S	U	U	S
w_2	$w_2 < 0.0649$	S	U	U	N	N
	$0.0649 < w_2 < 0.0969$	S	U	U	N	U
	$0.0969 < w_2 < 0.1012$	S	S	U	U	S
	$0.1012 < w_2$	S	S	U	N	N
w_3	$w_3 < 0.07$	S	S	U	U	U
	$0.07 < w_3 < 0.2541$	S	S	U	U	S
	$0.2541 < w_3$	S	U	U	N	S
w_4	$w_4 < 0.6957$	S	S	U	N	N
	$0.6957 < w_4 < 0.70875$	S	S	U	U	S
	$0.70875 < w_4$	S	U	U	N	U
w_5	$w_5 < 0.2674$	S	U	U	N	N
	$0.2674 < w_5 < 0.5865$	S	U	U	N	U
	$0.5865 < w_5 < 0.6068$	S	S	U	U	S
	$0.6068 < w_5$	S	S	U	N	N
w_6	$w_6 < 0.16734$	S	U	U	N	N
	$0.16734 < w_6 < 0.5681$	S	S	U	U	S
	$0.5681 < w_6$	S	S	U	N	N
m	$m < 0.295$	S	S	U	N	N
	$0.295 < m < 0.3683$	S	U	U	N	S
	$0.3683 < m < 0.8925$	S	S	U	U	S
	$0.8925 < m$	S	S	U	U	N

7. Discussion and Conclusions

This paper proposed and studied the dynamical behavior of an ecological system consisting of the Sokol-Howell prey-predator model with a Strong Allee effect on prey species. The proportional harvesting throughout the system is also included. The existence, uniqueness, and boundedness of its solution are discussed. All the possible equilibria of this system with their local stability conditions are obtained. It is observed that the system has at most five equilibria; the extinction equilibrium point that always exists and is locally asymptotically stable point, two axial equilibria, one is a saddle point while the other is locally asymptotically stable under some conditions, and two coexistent equilibria so that one of them is locally asymptotically stable under given conditions and the second is a saddle point. Finally, the local bifurcation that may occur near the equilibrium points is also investigated using Sotomayor's theorem. The proposed system is also investigated numerically in order to understand the global behavior of the system and detect the effects of the values of the parameter on the dynamical behavior of the system.

For the data given in Table-1, a different set of data may be used, it is observed that as the prey population crosses the Allee threshold value that is given by $w_1 = 2.103269$, the system becomes asymptotically stable at the coexistence equilibrium, otherwise the system approaches to the extinction equilibrium point asymptotically. When the prey's harvesting rate crosses a specific value, then the axial equilibrium points become unstable and the system approaches asymptotically to the extinction equilibrium point and the coexistence equilibrium point, depending on the initial point. However, when the predator's harvesting rate crosses a specific value, the system loses its persistence, due to the extinction of the predator, and the solution approaches asymptotically to the extinction equilibrium point and axial equilibrium point, depending on the initial point. The system has two bifurcation points regarding the conversion rate; at the first one, two coexistence equilibrium points were born, while at the second one the axial equilibrium point loses its stability, and the system approaches the extinction equilibrium point. Finally, increasing the availability of food (or decreasing the prey's refuge rate) leads first to create the coexistence equilibrium points, so that the system transfers its stability from the axial point to the coexistence point; however, increasing the availability of food further leads to an exchange in the stability of the system so that the coexistence equilibrium points lose their stability and the axial point becomes stable. According to the above conclusions and discussions, there is the possibility to control the system by choosing the ranges of its parameters' suitability.

Keeping the above in mind, the existence of a strong Allee effect in the prey population leads to specify the region of the persistence of the system through specifying the Allee threshold value. However, increasing the harvesting rate from the prey keeps the system persisting, while increasing the predator harvesting leads to losing the persistence of the system. Finally, decreasing slightly the refuge rate of the prey has a stabilizing effect on the system at first; however, further decreasing the refuge rate leads to destabilizing the system at the coexistence equilibrium point.

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