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On the Dynamics of One Parameter Family of Functions $f_k(x) = kcsc(x)$

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Abstract

In this research, we study the dynamics of one parameter family of meromorphic functions $H = \{f_k(x) = kcsc(x): k \in \mathbb{R}\ and x \in \mathbb{R}\}$. Furthermore, we describe the nature of fixed points of the functions in H,and we explain the numbers of real fixed points depending on the critical point k. So, we develop some necessary conditions for the convergence of the sequence $\{f_k^n(x)\}$when $n \to \infty$.

Keywords: soft Picard iteration processes

ديناميكية عائلة الدوال ذات المعلمة الواحدة $f_k(x) = kcsc(x)$

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الخلاصه

في هذا البحث, قمنا بدراسة ديناميكية الدوال الميرومورفية ذات المعلمة الواحدة
$H = \{f_k(x) = kcsc(x): k \in \mathbb{R}\ and x \in \mathbb{R}\}$
. بالأضافة إلى ذلك قمنا بإعطاء وصف كامل للنقاط الثابتة للدوال في H, وبينا أن النقاط الحرجة k تؤثرعلى عدد النقاط الثابتة الحقيقية. كما وأعطينا بعض الشروط الضرورية الجديدة لتقارب المتتابعة $\{f_k^n(x)\}$ عندما $n \to \infty$.

1-Introduction

Fixed point theory works as an essential tool for different branches of mathematical analysis and its applications. One of these applications is the study of real or complex dynamic function. The real dynamics of functions has been explained by Devaney [1], [2], Fadil [3] and Sajid [4],while, Akbari and Rabii [5], Magrenan and Gutierrez [6] and Radwan [7] have suggested and analyzed the real dynamics of the cubic polynomials, generalized logistic maps and one parameter family of transcendental functions. Faris [8] has discussed the dynamics of one parameter families $H = \{h_k(z) = ke^z/(z-1): k \in \mathbb{R}\}$ and $H = \left\{g_k(z) = \frac{kcosh(z)}{z^2}: k > 0\right\}$ of critically and finite non-critically finite transcendental meromorphic functions respectively. For more details see [9], [10].

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In this paper we present the real dynamics of the one parameter family $H = \{f_k(x) = kcsc(x): k \in \mathbb{R}\ and x \in \mathbb{R})\}$. A distinction is made between points for which $f_k^n(x)$ remains bounded as $n \to \infty$ and points for which $f_k^n(x)$ diverges. We will prove the following result.

Theorem 1:

Let $f_k \in H$; then there are $k_1, k_2, k_3 \in \mathbb{R}$ such that $k_1 < k_2 < 0 < k_3$ such that:

1) At k_1, there exists a fixed point x_1 and $b_1 \in \mathbb{R}$ where $f_{k_1}(b_1) = x_1$ satisfies :as $n \to \infty$,
 i. $f_{k_1}^n(x) \to x_1$ when $x \in (b_1, x_1)$,
 ii. $f_{k_1}^n(x) \to \infty$ when $x \in (\pi, b_1) \cup (x_1, 2\pi)$,
 iii. $f_{k_1}^n(x) \to -\infty$ when $x \in (0, \pi)$.

2) At k_2, there exist two fixed points x_2 and $r_1 \in (x_1, 2\pi)$, and $b_2 \in \mathbb{R}$ where $f_{k_2}(b_2) = x_2$ satisfies : as $n \to \infty$,
 i. $f_{k_2}^n(x) \to x_2$ when $x \in (b_2, r_1)$,
 ii. $f_{k_2}^n(x) \to \infty$ when $x \in (\pi, b_2) \cup (r_1, 2\pi)$,
 iii. $f_{k_2}^n(x) \to -\infty$ when $x \in (0, \pi)$.

3) At k_3, there exist a fixed point x_3 and $b_3 \in \mathbb{R}$ where $f_{k_3}(b_3) = x_3$ satisfies : as $n \to \infty$,
 i. $f_{k_3}^n(x) \to x_3$ when $x \in (b_3, x_3)$,
 ii. $f_{k_3}^n(x) \to \infty$ when $x \in (0, b_3) \cup (x_3, \pi)$,
 iii. $f_{k_3}^n(x) \to -\infty$ when $x \in (\pi, 2\pi)$.

4) At $k \in (k_1, k_2)$, there exist two fixed points $a_1 \in (x_2, x_1)$, $r_2 \in (x_1, r_1)$ and $b_4 \in \mathbb{R}$ where $f_k(b_4) = r_2$ satisfies : as $n \to \infty$,
 i. $f_k^n(x) \to a_2$ when $x \in (b_4, r_2)$,
 ii. $f_k^n(x) \to \infty$ when $x \in (\pi, b_4) \cup (r_2, 2\pi)$,
 iii. $f_k^n(x) \to -\infty$ when $x \in (0, \pi)$.

5) At $k \in (k_2, 0)$, there exists two fixed points $r_3 \in (\pi, x_2)$, $r_4 \in (r_1, 2\pi)$ and $b_5 \in \mathbb{R}$ where $f_k(b_5) = r_4$ satisfy: as $n \to \infty$,
 i. $f_k^n(x) \to \infty$ when $x \in (\pi, b_5) \cup (r_4, 2\pi)$,
 ii. $f_k^n(x) \to -\infty$ when $x \in (0, \pi)$,
 iii. The orbit $\{f_k^n(x)\}$ is periodic or chaotic for $x \in (r_3, r_4)$.

6) At $k \in (0, k_3)$, there exist two fixed points $a_2 \in (0, x_3)$, $r_5 \in (x_3, \pi)$ and $b_6 \in \mathbb{R}$ where $f_k(b_6) = r_5$ satisfy: as $n \to \infty$,
 i. $f_k^n(x) \to a_2$ when $x \in (b_6, r_5)$,
 ii. $f_k^n(x) \to \infty$ when $x \in (0, b_6) \cup (r_5, \pi)$,
 iii. $f_k^n(x) \to -\infty$ when $x \in (\pi, 2\pi)$.

7) At $k \in (-\infty, k_1) \cup (k_3, \infty)$, $f_k^n \to \infty$ as $n \to \infty$, for all $x \in (0,2\pi)\backslash\{\pi\}$.

2- Preliminary Results

In this section, we describe the behavior of the fixed points of the one parameter family H of transcendental meromorphic functions. Let $\phi(x): \mathbb{R} \to \mathbb{R}$ be a mapping which is defined by $\phi(x) = x \sin x$.

Now for all $f_k \in H$, a fixed point of f_k must satisfy the equation $\phi(x) = k$. By solving this equation, we can find that f_k has two fixed points $x_1 \simeq 4.913$ and $x_3 \simeq 2.029$. So ϕ has two critical values $k_1 \simeq -4.814$ and $k_3 \simeq 1.82$. Since ϕ is even and continuous then we can reduce the domain of ϕ to $(0, \pi)$.

The following propositions describe the number of fixed points of f_k with respect to k.

Proposition 2: Let $f_k \in H$, then there are three cases for the number of fixed points for f_k with respect to k:

1- f_k has no fixed point if $k < k_1$ or $k > k_3$.

2- f_k has one fixed point at k_1 and at k_3.

3- f_k has two fixed points on $(k_1, 0)$ and on $(0, k_3)$.
Proof: -
let$\phi(x) = x\sin x$ and $\emptyset'(x) = \sin x + x\cos x$ then:
1- $\phi''(x) = 2\cos x - x\sin x$, $\phi''(x_3) < 0$, $\phi''(x_1) > 0$ where $x_1 \simeq 4.913$ and $x_3 \simeq 2.029$. Thus x_3 is a maximum point and x_1 is a minimum point for $\phi(x)$ in the $(0,2\pi)$.Then $\phi(x) = k$ has no solutions for $k < k_1 (k > k_3)$. So that $f_k$have no fixed points in this step.
2- When $k = k_1 (k = k_3)$, because of $k_1 = \phi(x_1)(k_3 = \phi(x_3))$ is the minimum(maximum) value of $\phi(x)$ in $(0,2\pi)$.Then f_k has only one fixed point at $x = x_1 (x = x_3)$.
3- When $k \in (k_1, 0)$, the point $x = x_1$ is a minimum value in $(0,2\pi)$. Since ϕ is strictly decreasing in (x_3, x_1) and it is strictly increasing in $(x_1, 2\pi)$, then the line $k = c$ intersects the plot of ϕ at exactly one point in all of the intervals (x_3, x_1) and $(x_1, 2\pi)$. Similarly, when $k \in (0, k_3)$ the point $x = x_3$ is a maximum value in $(0,2\pi)$. Hence ϕ is strictly increasing $(0, x_3)$ and it is strictly decreasing in $(x_3, 2\pi)$, so the line $k = c$ intersects the plot of ϕ at exactly one point in interval $(0, x_3)$ and $(x_3, 2\pi)$. Then f_k has two fixed points on $(k_1, 0)$ and on $(0, k_3)$.
The purpose of the following proposition is to study the nature of fixed points of the function f_k on $\mathbb{R}$. That is, we must study the equation $|f'_k(x)| = 1$, since $f'_k(x_1) = 1 = f'_k(x_3)$, then$x_1, x_3$ are indifferent fixed points of f_k. While the positive solution of the equation$f'_k(x) = tanx - x = -1$, is $x_2 \simeq 4.493$, hence $k_2 = \phi(x_2) \simeq -4.385$.
Proposition 3: Let $f_k \in H$,then if:
1- $k = k_2$, the two fixed points of f_k are:x_2 is indifferent, and $r_1 \in (x_2, 2\pi)$ is repelling ,
2- $k \in (k_1, k_2)$, the two fixed points of f_k are:$a_1 \in (x_1, x_2)$ is attracting , and $r_2 \in (x_1, r_1)$ is repelling ,
3- $k \in (k_2, 0)$,the two fixed points of f_k are: $r_3 \in (\pi, x_2)$ is repelling and $r_4 \in (r_1, 2\pi)$ is repelling,
4- $k \in (0, k_3)$,the two fixed points of f_k are:$a_2 \in (0, x_3)$ is attracting and $r_5 \in (x_3, \pi)$ is repelling.
Proof: let $f_k(x) = kcsc\, x$, hence $f'_k(x) = -kcsc\, x\cot x$ and the solutions of equation $k = \frac{x}{\csc x} = \phi(x)$ are the fixed points of f_k.
So $f'_k(x)$ at fixed point x is obtained by

$$|f'_k(x)| = \left|\frac{-x}{\csc x}\csc x\cot x\right| = |-x\cot x| = \frac{|x\cos x|}{|\sin x|}$$

Now, we define the function $\mu(x)$ as follows:
$\mu(x) = |x\cos x| - |\sin x|$, it is continuous and has 3 zeros when $x = x_1$, x_2 and x_3. From the graph we can show that $\mu(x)$ is decreasing in the intervals $(0, x_3)$ and (x_2, x_1), while it is increasing in the intervals (x_3, x_2) and $(x_1, 2\pi)$. So $\mu(x)$ has maximum point at π and it has minimum point at x=-1. From the above statements that $\mu(x) > 0$;
when $x \in (x_3, x_2) \cup (x_1, 2\pi)$, $\mu(x) = 0$; when $x = x_1$, x_2 ,x_3 and $\mu(x) < 0$ when $x \in (0, x_3) \cup (x_2, x_1)$ see Fig(1).
Thus
1- If $k = k_2$,the fixed point $r_1 \in (x_2, 2\pi)$ satisfies $|f'_k(r_1)| > 1$, then r_1 is repelling fixed point.
2- If $k \in (k_1, k_2)$, the fixed point $a_1 \in (x_2, x_1)$ satisfies $|f'_k(a_1)| < 1$, then a_1 is an attracting fixed point. While, if the fixed point $r_2 \in (x_1, r_1)$ satisfies$|f'_k(r_2)| > 1$; then r_2 is repelling fixed point.
3- If $k \in (k_2, 0)$, the fixed points $r_3 \in (\pi, x_2)$, and $r_4 \in (r_1, 2\pi)$ satisfy$|f'_k(r_i)| > 1$, i=3,4, then r_i are repelling fixed points for i=3,4.

4- If $k \in (0, k_3)$, the fixed points $a_2 \in (0, x_3)$ and $r_5 \in (x_3, \pi)$ satisfy $|f'_k(a_2)| < 1$ and $|f'_k(r_5)| > 1$ respectively. Then a_2 is an attracting fixed point and r_5 is a repelling fixed point.

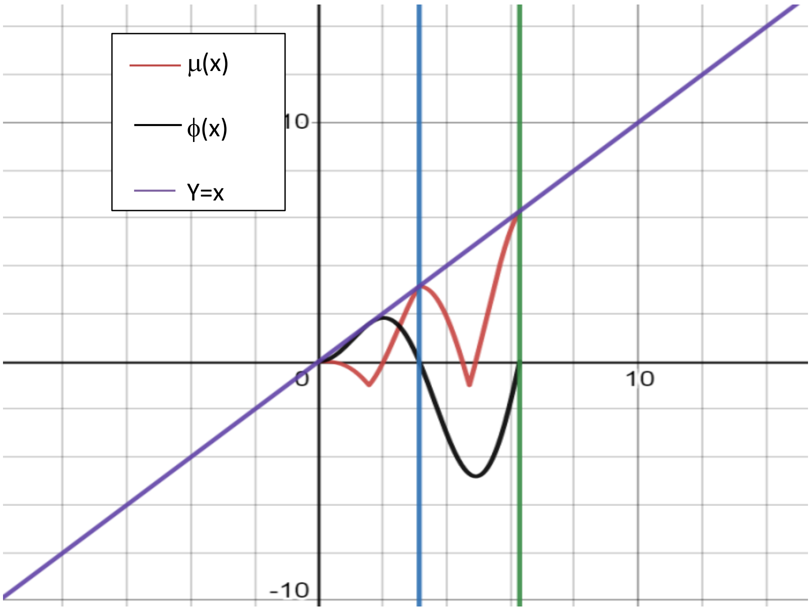

Figure 1- $\mu(x) = |x \cos x| - |\sin x|, \phi(x) = \frac{x}{\csc x}$

3-The Proof of the Main result

The proof of the main result is described as follows:

Proof of the main results:

Let $T_k(x) = f_k(x) - x$ then

1) when$\boldsymbol{k} = \boldsymbol{k_1}$, f_k has an indifferent fixed point x_1 by proposition (3).$T'_k(x_1) = 0$and $T''_k(x_1) > 0$, then T_k has minimum at x_1. Because of $T_k(x_1) = 0$, it follows that $T_k(x) > 0$ for each x in a neighborhood of x_1.Hence by continuity of T_k, for sufficiently small $m_1 > 0$, $T_k(x) > 0$ in $(x_1 - m_1, x_1) \cup (x_1, x_1 + m_1)$. From Fig.(2) we have$T_k(x) \neq 0$ in $(\pi, x_1) \cup (x_1, 2\pi)$, $T_k(x)$>0 for all $x \in (\pi, x_1) \cup (x_1, 2\pi)$ and $T_k(x) < 0$ for $x \in (0, \pi)$.

Next, we will study the dynamics of f_k as follows:

Case(1): For $x \in (b_1, x_1)$; $b_1 \in (\pi, x_1)$. $x_1$is a minimum point forT_k.$T_k(x) > 0$in $x \in (\pi, x_1) \cup (x_1, 2\pi)$, hence when $x \in (b_1, x_1)$, $T'_k(x) < 0$, so $f'_k(x) - 1 < 0$, then $f'_k(x) < 1$. Thus by the mean value theorem $|f_k(x) - f_k(x_1)| = f'_k(c)|x - x_1|$ such that $c \in (b_1, x_1)$. that is implies $|f_k(x) - f_k(x_1)| < |x - x_1|$ for all $x \in (b_1, x_1)$. Since x_1 is a fixed point of f_k, Thus $f_k^n(x) \to x_1$ as $n \to \infty$, for all $x \in (b_1, x_1)$.

Case (2): For $x \in (\pi, b_1) \cup (x_1, 2\pi)$, then $T_k(x) > 0$,hence $f_k(x) > x$, but $f_k(b_1) = x_1$, thus f_k maps the interval (π, b_1) into $(x_1, 2\pi)$, then it is enough to prove that $f_k^n(x) \to \infty$ as $n \to \infty$ when $x \in (x_1, 2\pi)$,

Since $f_k(x) > x$, then $\{f_k^n(x)\}$ is unbounded above and strictly increasing sequence in $x \in (x_1, 2\pi)$, so $f_k^n(x) \to \infty$ as $n \to \infty$, for all $x \in (x_1, 2\pi)$.

Case (3): When $x \in (0, \pi)$, $T_k(x) < 0$ and $f_k(x) < x$, therefore f_k is strictly decreasing in this interval then$\{f_k^n(x)\}$ is decreasing sequence and it is unbounded below. So for $x \in (0, \pi)$,$f_k^n(x) \to -\infty$ as $n \to \infty$.

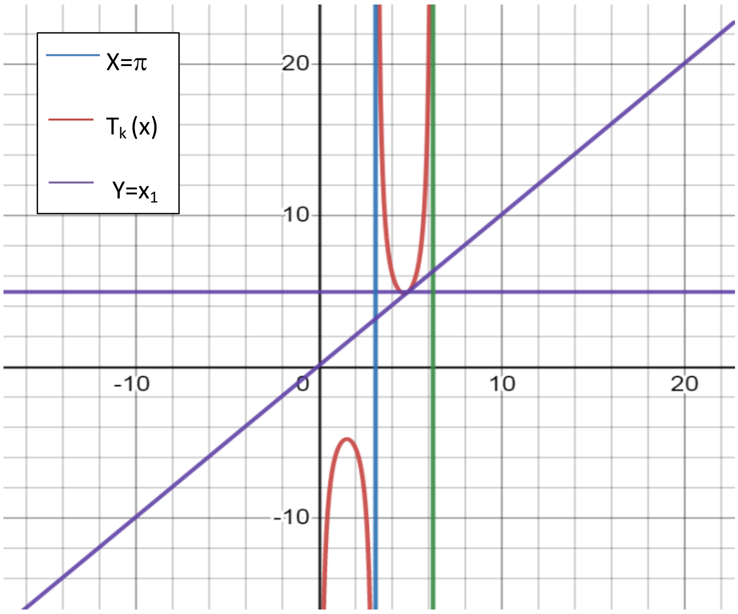

Figure 2- $T_k(x) = f_k(x) - x$, $k = k_1$

2)when $\boldsymbol{k = k_2}$, it is clear from Fig. (3) $T_k(x) > 0$, for all $x \in (\pi, x_2) \cup (r_1, 2\pi)$ and $T_k(x) < 0$ for all $x \in (x_2, r_1) \cup (0, \pi)$.
Now, we can describe the dynamic of f_k.
Case (1): For $x \in (b_2, r_1)$, we will show that $f_k^n(x) \to x_2$ since $T_k(x) < 0$ for $x \in (x_2, r_1)$ then$f_k(x) < x$. Since $f_k$is decreasing and by continuity forward iteration process we get
$x > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > x_2$.
Therefore, the sequence $\{f_k^n(x)\}$ is decreasing and bounded below by x_2. So $f_k^n(x) \to x_2$ as $n \to \infty$for $x \in (x_2, r_1)$.
Further since $f_k(b_2) = r_1$, and it is decreasing in (b_2, r_1), $f_k$maps the interval (b_2, r_1) into (x_2, r_1).It follows that byusing the previous arguments, $f_k^n(x) \to x_2$ as $n \to \infty$for $x \in (b_2, r_1)$.
Case (2): For $x \in (r_1, 2\pi)$, $f_k(x) > x$. Moreover f_k is strictly increasing in this interval, then

$$0 < x < f_k(x) < f_k^2(x) < \cdots < f_k^n(x) < \cdots$$

Thus, the sequence $\{f_k^n(x)\}$ is increasing and it is unbounded above. Hence $f_k^n(x) \to -\infty$ as $n \to \infty$for $x \in (r_1, 2\pi)$.
Now, for $x \in (\pi, b_2)$;$b_2 \in (\pi, x_2)$, we have $f_k(x) > x$.$f_k(b_2) = r_1$. Then f_k maps the interval(π, b_2)into $(r_1, 2\pi)$. Thus $f_k^n(x) \to -\infty$ as $n \to \infty$for $x \in (\pi, b_2) \cup (r_1, 2\pi)$.
Case (3): For $x \in (0, \pi)$ from Fig.(3), we have$T_k(x) < 0$ then$f_k(x) < x$ so that f_k is strictly decreasing in this interval, and hence

$$x > f_k(x) > f_k^2(x) > f_k^3(x) > \cdots > f_k^n(x) > \cdots$$

Thus $\{f_k^n(x)\}$ is decreasing sequence, which is unbounded below. Therefore, for $x \in (0, \pi)$ we have $f_k^n(x) \to -\infty$as $n \to \infty$.

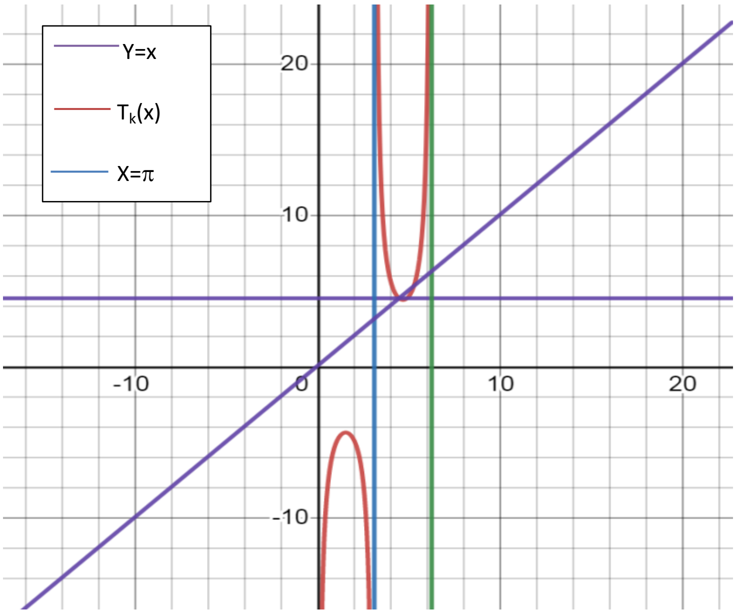

Figure 3- $T_k(x) = f_k(x) - x$, $k = k_2$

3)when $\boldsymbol{k = k_3}$, f_k has an indifferent fixed point x_3 by proposition (3).$T'_k(x_3) = 0$, and $T''_k(x_3) > 0$, then T_k has minimum at x_3. Because of $T'_k(x_3) = 0$, it follows that $T_k(x) > 0$ for each x in a neighborhood of x_3. Hence by continuity of T_k, for sufficiently small $m_1 > 0$, $T_k(x) > 0$ in $x \in (x_3 - m_1, x_3) \cup (x_3, x_3 + m_1)$. From Fig.(4) we have$T_k(x) \neq 0$ in $x \in (0, x_3) \cup (x_3, \pi)$, so $T_k(x)$>0 for all $x \in (0, x_3) \cup (x_3, \pi)$, and $T_k(x) < 0$ for $x \in (\pi, 2\pi)$.

Next, we will study the dynamics of f_k as follow:

Case(1): For $x \in (b_3, x_3)$; $b_3 \in (0, x_3)$,x_3 is a minimum point forT_k. $T_k(x) > 0$ in $(0, x_3) \cup (x_3, \pi)$, hence when $x \in (b_3, x_3)$, $T'_k(x) < 0$, so $T_k(x) < 0$, then $f'_k(x) < 1$. For $x > x_3$, $f'_k(x) > 1$. Thus by the mean value theorem $\left| f_k(x) - f_k\left(x_1 \right) \right| = f'_k(c)|x - x_1|$ such that $c \in (b_3, x_3)$. Since x_3 is a fixed point of f_k, that is implies $|f_k(x) - f_k(x_3)| < |x - x_3|$ for all $x \in (b_3, x_3)$. Thus $f_k^n(x) \to x_3$as $n \to \infty$, for all $x \in (b_3, x_3)$.

Case (2):for $x \in (0, b_3) \cup (x_3, \pi)$ then $T_k(x) > 0$ for all $x \in (x_3, \pi)$, hence $f_k(x) > x$, since f_k is strictly increasing in this interval, and

$$0 < x < f_k(x) < f_k^2(x) < \cdots < f_k^n(x) <..$$

Then $\{f_k^n(x)\}$ is increasing sequence which it is unbounded, above so $f_k^n(x) \to \infty$as $n \to \infty$, for all $x \in (x_3, \pi)$. Since $f_k(b_3) = x_3$ and $f_k$maps the interval $(0, b_3)$ into (x_3, π) hence we can use the same arguments to prove$f_k^n(x) \to \infty$ as $n \to \infty$when $x \in (0, b_3)$.

Case (3):when $x \in (\pi, 2\pi)$, $T_k(x) < 0$ then$f_k(x) < x$, therefore f_k is strictly decreasing in this interval and

$x_3 > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > \cdots$.

Then $\{f_k^n(x)\}$ is decreasing sequence and it is unbounded below. So for $x \in (\pi, 2\pi) f_k^n(x) \to -\infty$ as $n \to \infty$.

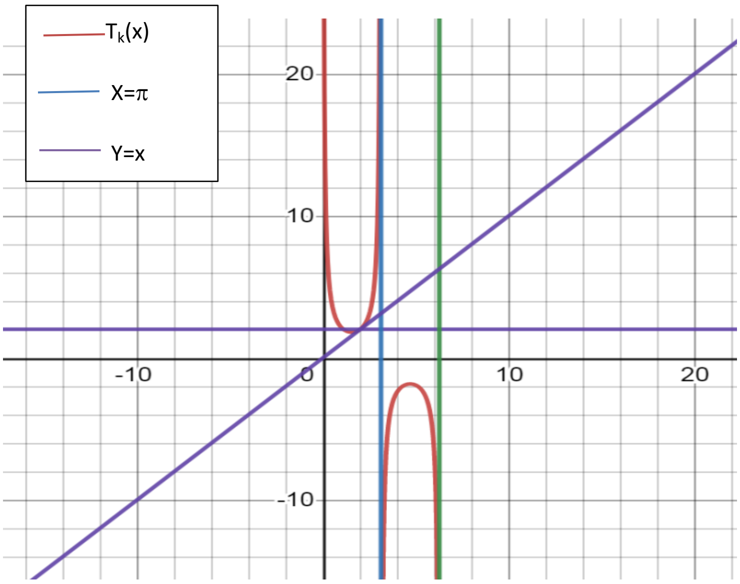

Figure 4- $T_k(x) = f_k(x) - x,\, k = k_3$

4)when $\boldsymbol{k} \in (\boldsymbol{k_1}, \boldsymbol{k_2})$, by proposition (3)$f_k$ has an attracting fixed point $a_1 \in (x_2, x_3)$ and repelling fixed point $r_2 \in (x_1, r_1)$. From Fig.(5)$T_k(x) \neq 0$ in $(\pi, r_2) \cup (r_2, 2\pi)$,$T_k(x)$>0 for all $x \in (\pi, a_1) \cup (r_2, 2\pi)$ and $T_k(x) < 0$ for $x \in (0, \pi) \cup (a_2, r_2)$.
To describe the dynamics of f_k, we have three cases:-
Case(1) when $x \in (b_4, r_2)$. $f_k(x) < x$ for all $x \in (a_1, r_2)$and it is decreasing, so
$x > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > a_1$.
Hence the sequence $\{f_k^n(x)\}$ is decreasing and bounded below by a_1, and there is no fixed point larger than a_1. Therefore $f_k^n(x) \to a_1$ as $n \to \infty$for all $x \in (b_4, r_2)$.
Case (2):-For $x \in (\pi, b_4) \cup (r_2, 2\pi)$, $f_k(x) > x$ for all $x \in (r_2, 2\pi)$. Since f_k is increasing and by continuing forward iteration process, it follows

$$0 < x < f_k(x) < f_k^2(x) < \cdots < f_k^n(x) < \cdots$$

Hence, the sequence $\{f_k^n(x)\}$ is increasing and there is no fixed point larger than r_2, the orbit must go to ∞as $n \to \infty$. Then $f_k^n(x) \to \infty$ as $n \to \infty$ for all $x \in (r_2, 2\pi)$. $f_k(b_4) = r_2$, f_k maps the interval (π, b_4)into $(r_2, 2\pi)$. Then by using the above arguments $f_k^n(x) \to \infty$ as $n \to \infty$ for all $x \in (\pi, b_4)$.
Case (3): for $x \in (0, \pi)$, $T_k(x) < 0$, hence$f_k(x) < x$, therefore f_k is strictly decreasing in this interval and
$x > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > \cdots$.
Then $\{f_k^n(x)\}$ is decreasing sequence and it is unbounded below. So for $x \in (0, \pi) f_k^n(x) \to -\infty$ as $n \to \infty$.

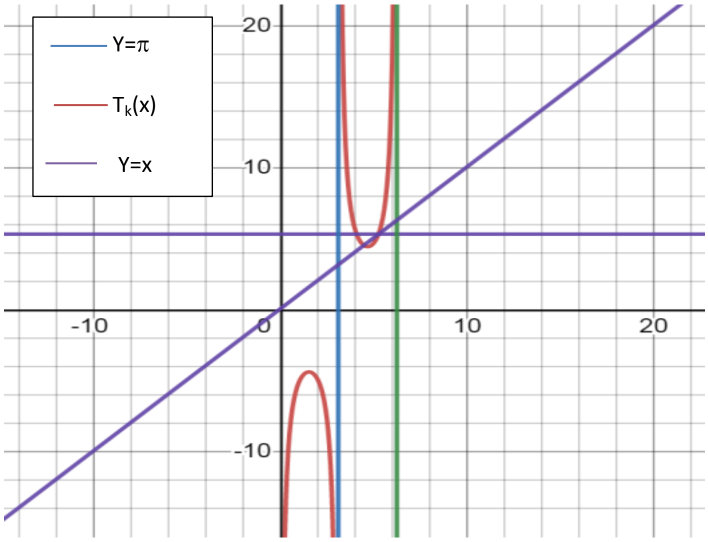

Figure 5- $T_k(x) = f_k(x) - x$, $k \in (k_1, k_2)$

5)when $\boldsymbol{k} \in (\boldsymbol{k_2}, \boldsymbol{0})$, f_k has two fixed points $r_3 \in (\pi, x_2)$and $r_4 \in (r_1, 2\pi)$ which are repelling by proposition (3). From Fig(6) we have $T_k(x) > 0$for all $x \in (\pi, b_5) \cup (r_4, 2\pi)$ and $T_k(x) < 0$ for $x \in (0, \pi) \cup (r_3, r_4)$.
To describe the dynamics of f_k, we have three cases:-
Case (1):- for $x \in (r_4, 2\pi)$,$f_k(x) > x$. Hence f_k is strictly increasing in $(r_4, 2\pi)$, then
$0 < r_4 < x < f_k(x) < f_k^2(x) < \cdots < f_k^n(x) < \cdots$.
So the sequence $\{f_k^n(x)\}$is increasing sequence which is unbounded above. So $f_k^n(x) \to \infty$as $n \to \infty$ for $x \in (r_4, 2\pi)$. Then $f_k^n(x) \to \infty$ as $n \to \infty$ for all $x \in (\pi, r_4)$. Because $f_k(b_6) = r_5$,andf_k maps the interval (π, r_4) into $(r_4, 2\pi)$. By using the above arguments $f_k^n(x) \to \infty$ as $n \to \infty$ for all $x \in (\pi, r_4)$.
Case (2):- for $x \in (r_3, r_4)$ the system of dynamics of f_k has no point attractors. Thus dynamical system will move indefinitely, and the orbit $\{f_k^n(x)\}$ will be periodic or chaotic in these intervals.
Case (3):-when $\in (0, \pi)$, $T_k(x) < 0$ then$f_k(x) < x$.Therefore f_k is strictly decreasing in this interval and

$$x_1 > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > \cdots$$

Then $\{f_k^n(x)\}$ is decreasing sequence and it is unbounded below. So for $x \in (0, \pi)$, $f_k^n(x) \to -\infty$ as $n \to \infty$.

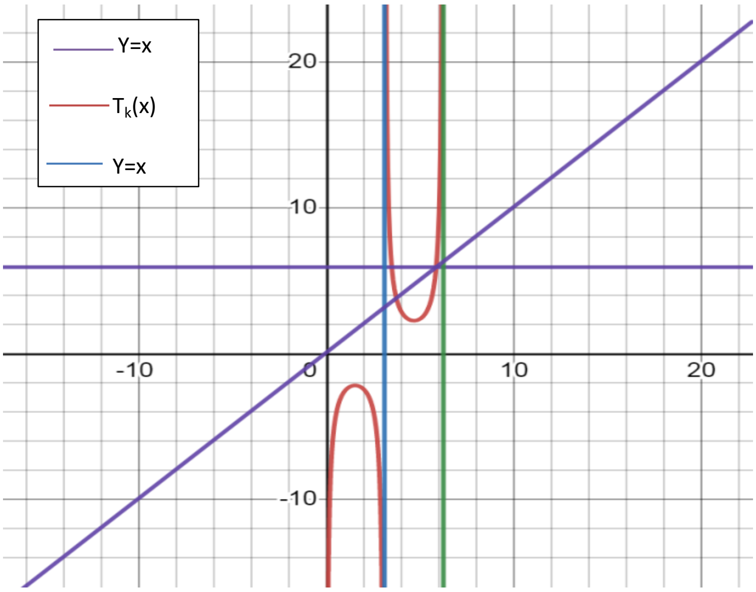

Figure 6- $T_k(x) = f_k(x) - x$, k ∈ (k_2 ,0)

6) when $k \in (0, k_3)$, so f_k has an attracting fixed point $a_2 \in (0, x_3)$ and repelling fixed point $r_5 \in (x_3, \pi)$ by proposition (3). From Fig. (7) we have $T_k(x) > 0$ for all $x \in (0, b_6) \cup (r_5, \pi)$ and $T_k(x) < 0$ for $x \in (\pi, 2\pi) \cup (a_2, r_5)$.
To describe the dynamics of f_k, we have three cases:-
Case(1) when $x \in (b_6, r_5)$, then $f_k(x) < x$ and it is decreasing, so

$$x > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > \cdots > a_2$$

Hence the sequence $\{f_k^n(x)\}$ is decreasing and bounded below by a_2, and there is no fixed point larger than a_2. Therefore $f_k^n(x) \to a_2$ as $n \to \infty$ for all $x \in (b_6, r_5)$.
Case (2):- for $x \in (0, b_6) \cup (r_5, \pi)$, $f_k(x) > x$ for all $x \in (r_5, \pi)$. Since f_k is increasing and by continuing forward iteration process, it follows that

$$0 < x < f_k(x) < f_k^2(x) < \cdots < f_k^n(x) < \cdots$$

Hence, the sequence $\{f_k^n(x)\}$ is increasing and there is no fixed point larger than r_5, the orbit must go to ∞as $n \to \infty$. Then $f_k^n(x) \to \infty$ as $n \to \infty$ for all $x \in (r_5, \pi)$.$f_k(b_6) = r_5$ and f_k maps the interval (r_5, π)into$(0, b_6)$. So by using the above arguments we can getting $f_k^n(x) \to \infty$ as $n \to \infty$ for all $x \in (0, b_6)$.
Case (3):for $x \in (\pi, 2\pi)$, $T_k(x) < 0$ then$f_k(x) < x$, therefore f_k is strictly decreasing in this interval and

$$x > f_k(x) > f_k^2(x) > \cdots > f_k^n(x) > \cdots.$$

Then $\{f_k^n(x)\}$ is decreasing sequence and it is unbounded below. So for all $x \in (\pi, 2\pi)$,$f_k^n(x) \to -\infty$ as $n \to \infty$.

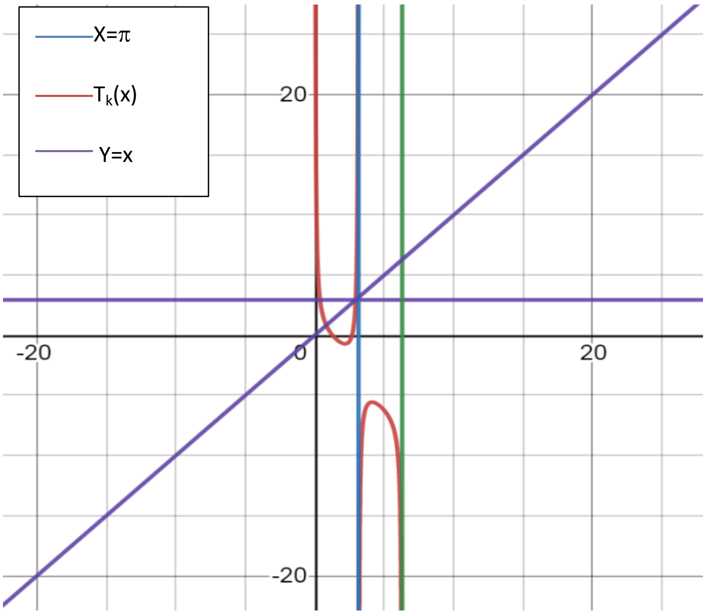

Figure 7- $T_k(x) = f_k(x) - x$, $\mathrm{k} \in (0, \mathrm{k}_3)$

7)when k $< k_1$ **(**$k > k_3$**)**, f_k has no fixed point by proposition (2). Since f_k is continues and differential for $x \in (0,2\pi)$ then T_k is continuous and differentiable. From Fig. (8) the sequence $\{f_k^n(x)\}$ is increasing and it is unbounded below for all $x \in (0,2\pi)$. Hence $f_k^n(x) \to \infty$ as $n \to -\infty$ for all $x \in (0,2\pi)$. When $k > k_3$ the proof is similar to the previous arguments.

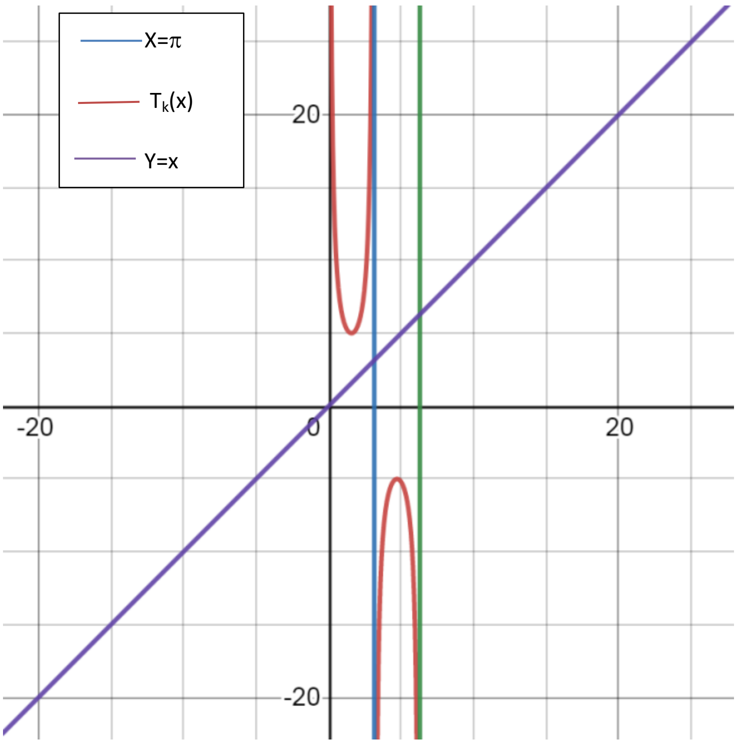

Figure 8- $T_k(x) = f_k(x) - x$, $\mathrm{k} > \mathrm{k}_3$)

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