DOI: 10.24996/ijs.2022.63.7.32

ISSN: 0067-2904

Almost Semi-extending Modules

Muna Abbas Ahmed*, Maysaa Riadh Abbas, Noor Riyadh Adeeb
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq

Received: 1/8/2021 Accepted: 3/9/2021 Published: 30/7/2022

Abstract
Fuchs introduced purely extending modules as a generalization of extending modules. Ahmed and Abbas gave another generalization for extending modules named semi-extending modules. In this paper, two generalizations of the extending modules are combined to give another generalization. This generalization is said to be almost semi-extending. In fact, the purely extending modules lies between the extending and almost semi-extending modules. We also show that an almost semi-extending module is a proper generalization of purely extending. In addition, various examples and important properties of this class of modules are given and considered. Another characterization of almost semi-extending modules is established. Moreover, the relationships with other related concepts are studied and discussed

Keywords: Extending modules, Purely extending modules, Almost semi-extending modules.

المقاسات شبه التوسعية تقريبا

منى عباس أحمد، ميساء رياض عباس، نور رياض أديب
قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق

الخلاصة
قدّم الباحث Fuchs مفهوم مقاسات التوسع النقي كأعمام لمفهوم المقاسات التوسعية، كما قدّم أحمد وعباس اعماماً أخر للمقاسات التوسعية سُمي بالمقاسات شبه التوسعية. في هذا البحث نقدم اعماماً أخر للمقاسات التوسعية له علاقة بكلا المفهومين, سنطلق عليه اسم المقاسات شبه التوسعية تقريباً. في الواقع ان مقاسات التوسع النقي تقع بين المقاسات التوسعية والمقاسات شبه التوسعية تقريباً. سنبرهن ان مقاسات التوسع النقي جزئية بشكل فعلي من المقاسات شبه التوسعية تقريباً. فضلا عن ذلك سندرس الخصائص الرئيسة والمهمة لهذا النوع من المقاسات، وإعطاء تشخيصاً اخراً له. اضافة الى ذلك سيتم دراسة مفصلة لعلاقة المقاسات شبه التوسعية تقريبا بالمقاسات الاخرى ذات العلاقة.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary left modules. The notation R and M are used to denote a ring and module, respectively . It is well known that a submodule N of M is called essential (briefly $N \leq_e M$), if whenever $N \cap L = (0)$, then $L = (0)$ for each submodule L of M [1], and a proper submodule P

*Email: munaaa_math@csw.uobaghdad.edu.iq

of M is called prime, if whenever rm ∈ P for r ∈ R and m ∈ M, then either m ∈ P or r ∈ $(P:_R M)$ [2].

A non-zero submodule N of M is called semi-essential (briefly $N \leq_{sem} M$), if N∩P ≠ (0) for each non-zero prime submodule P of M. Equivalently, a submodule N of an R-module M is called semi-essential if whenever N∩P = (0), implies that P = (0) for every prime submodule P of M [3]. A submodule N of M is called closed (briefly $N \leq_c M$), if N has no proper essential extensions in M that means if $N \leq_e K \leq M$, then N = K [1]. A submodule N of M is called St-closed in M (briefly $N \leq_{Stc} M$), if N has no proper semi-essential extensions in M, i.e if there exists a submodule K of M such that $N \leq_{sem} K \leq M$, then N = K [4]. An R-module M is called extending, (briefly CS-module), if every submodule of M is essential in a direct summand of M [5]. An R-module M is called semi-extending if every submodule of M is a semi-essential in a direct summand of M [6]. A submodule N of an R-module M is a pure submodule if IM∩N=IN for every finitely generated ideal I of R [7, P.30]. An R-module M is called purely extending if every submodule of M is essential in a pure submodule of M [8].

The aim of this paper is to extend the notion of an extending module to a generalization that includes the class of purely extending. An element of this class is called an almost semi-extending module. In fact, purely extending lies between extending and almost semi-extending module.

This work is organized as follows: In section 2, another characterization of this class of modules is obtained, see Theorem (2.5). Several important properties of almost semi-extending are investigated, namely the behavior of the submodules of an almost semi-extending module and the direct sum of almost semi-extending modules. We also discuss in Proposition (2.18) either the direct sum of fully idempotent submodules can be almost semi-extending module or not . Also, the relationships of almost semi-extending with semi-extending and purely extending modules are considered; see Remark (2.2)(2), Proposition (2.3), Theorem (2.4), Proposition (2.6), Proposition (2.9), Theorem (2.10) and Theorem (2.11).

In section 3 of this paper, the relationships of an almost semi-extending module with other known related concepts are studied such as semi-uniform, F-regular, St-semisimple, purely y-extending and CLS-modules; see Proposition (3.1), Theorem (3.2), Proposition (3.4), Corollary (3.5), Proposition (3.6), Proposition (3.7), Proposition (3.8), Proposition (3.9) and Theorem (3.10). Finally, the conclusions and discussions are given in the last section.

2. Almost semi-extending modules

This section devotes to studying the main properties of almost semi-extending modules. Also, the relationships of almost semi-extending with semi-extending and purely extending are given.

Definition (2.1): An R-module M is called almost semi-extending if every submodule of M is semi-essential in a pure submodule of M.

Remarks (2.2):

1. Every semisimple module is almost semi-extending. In fact, every submodule of semisimple module is a direct summand. However, every direct summand is pure submodule, so by definition of almost semi-extending the result is followed.

2. Every semi-extending is almost semi-extending.

Proof (2): Let M be semi-extending and let N is a submodule of M, so that N is a semi-essential in a direct summand of M. Since every direct summand is pure. Therefore, N is a semi-essential in a pure submodule.

3. To show that every extending is almost semi-extending. Let N ≤ M. By assumption N is essential in a direct summand of M. This implies that N is semi-essential in a direct summand of M [3], hence N is semi-essential in a pure submodule of M. Thus, M is almost semi-extending.

4. The converse of (3) is not true in general, for example the Z-module M=$Z_8 \oplus Z_2$ is a semi-extending module [6], and by (2), M is almost semi-extending. On the other hand, M is not extending module [6].

5. Every uniform is almost semi-extending module, where an R-module M is called uniform if every submodule in M is essential in M [1]. The converse is not true for example the Z-module Z_{36} is semi-extending [6], hence it is almost semi-extending Z-module while Z_{36} is not uniform, since Z_{36} contains a submodule $(\overline{18})$ of Z_{36} which is not essential.

The next proposition shows that the converse of Remark (2.2)(2) is not true in general. However, it is true under certain conditions. We recall that an R-module M is said to be divisible if M=rM for each non-zero divisor element r∈R [9, P.33], and an R-module M is called Noetherian if all submodules of M are finitely generated [1,P.7].

Proposition (2.3): Let M be a divisible module over principle ideal domain R. Then M is an almost semi-extending module if and only if M is semi-extending.

Proof: Suppose that M is an almost semi-extending R-module, and let N ≤ M, then N is semi-essential in a pure submodule of M, say L. But M is divisible over principle ideal domain, thus L is a direct summand of M [7, Corollary (2.9), P.62], and so that M is semi-extending. The converse is straightforward.

Theorem (2.4): Let M be a finitely generated module over a Noetherian ring R. Then, M is an almost semi-extending module if and only if M is semi-extending.

Proof: Assume that M is almost semi-extending, and let N be a submodule of M, then there exists a pure submodule H of M such that $N \leq_{sem} H$. Since M is finitely generated and R is a Noetherian ring, so that H is a direct summand of M [7, Proposition (2.10), P.62]. The converse is directly followed.

The following theorem gives another characterization of almost semi-extending

Theorem (2.5): An R-module M is almost semi-extending if and only if every St-closed submodule in M is a pure submodule of M.

Proof: Suppose that M is an almost semi-extending module, and K is an St-closed submodule in M. By the assumption, there exists a pure submodule B of M such that K is a semi-essential in B. But K is an St-closed submodule of M, therefore K=B [4, Remark (1.2)(4)]. Conversely, let A ≤ M, if A = (0), then it is clear that (0) is semi-essential in a pure submodule which is (0) itself. For the other case, there exists an St-closed submodule H of M such that $A \leq_{sem} H$ [4, Proposition (1.4)]. Since H is an St-closed of M, so by assumption H is a pure in M. Thus, M is an almost semi-extending module.

Proposition (2.6): Every purely extending module is almost semi-extending.

Proof: This follows by the direct implication between essential and semi-essential submodules.

The converse of Proposition (2.6) is not true in general, as the following example shows.

Example (2.7): Consider the Z-module M=$Z_8 \oplus Z_2$.We have to show that M is almost semi-extending. Note that, the St-closed submodules in M are $Z(\bar{0},\bar{1})$, $Z(\bar{4},\bar{1})$ and M itself, and all of them are direct summand, that is M=$Z(\bar{1},\bar{1}) \oplus Z(\bar{0},\bar{1})$, M=$Z(\bar{1},\bar{0}) \oplus Z(\bar{4},\bar{1})$, and M=M⊕(0). Since every direct summand is pure, therefore each of $Z(\bar{0},\bar{1})$, $Z(\bar{4},\bar{1})$ and M is pure in M. By Theorem (2.5), M is an almost semi-extending module. On the other hand, there exists a submodule N=Z(2,1) in M, which is generated by the element (2,1), it is closed and does not pure submodule of M [8, Example (2.4)(4)]. Thus, M is not purely extending.

Recall that an R-module M is called purely semisimple if for each pure submodule N of M, there exists a direct summand K of M such that $K \leq_e N$. [10, Definition (3.1.1), P.95]. Before the next proposition, we need the following.

Lemma (2.8): [10, Proposition (3.1.2)]

An R-module M is purely semisimple if and only if every pure submodule of M is a direct summand of M.

Proposition (2.9): Let M be a purely semisimple module, then M is semi-extending if and only if M is an almost semi-extending module.
Proof: ⇒ It follows by Remark (2.2)(2).
⇐) Let N $\leq_{Stc}$ M. Since M is an almost semi-extending module, then by Theorem (2,5), N is a pure. But M is purely semisimple, so by Lemma (2.8), N is a direct summand of M. that is M is semi-extending.

An R-module M is called fully prime if every proper submodule of M is a prime submodule [2].
Theorem (2.10): If M is a purely semisimple module, then we have the following:
1. M is extending if and only if M is purely extending.
2. If M is purely extending, M is almost semi-extending.
If M is fully prime and M is almost semi-extending, then M is extending**.**
Proof:

(1) It is obvious by the definition of purely extending which was mentioned in [8] For the other direction, we have assumed that M is purely extending, and N ≤ M. By assumption N is essential in a pure submodule say H. But M is purely semisimple, so by Lemma (2.8), N is a direct summand of M.

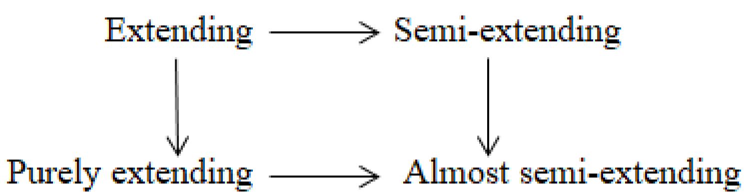
(2) It is directly obtained by Proposition (2.6).

(3) Let N be a closed submodule of M. We have two cases: if N = (0), then N is a direct summand. Otherwise, since M is fully prime, then N is an St-closed submodule in M [4, Remark (1.8)]. Since M is almost semi-extending, then N is a pure. But M is purely semisimple, so by Lemma (2.8), N is a direct summand of M, thus M is extending.
Theorem (2.11): If M is a purely semisimple module, then we have the following:
1. If M is extending, then M is semi-extending.
2. If M is semi-extending, then M is almost semi-extending.
3. If M is fully prime, then we have the following two cases:
i) If M is almost semi-extending, then M is extending.
ii) If is semi-extending, then M is extending
Proof:

(1) It follows by [6, Remark (1.2)(2)].

(2) It follows by Remark (2.2)(2).

(3)(i) It follows from Theorem (2.10).
(3)(ii) Since M is fully prime, so the result follows directly by [6, Theorem (2.3)].
Remark (2.12): According to the above argument about the three kinds of generalizations; semi-extending, purely extending and almost semi-extending, we conclude the following implications:



Proposition (2.13): Let M be almost semi-extending module. For any two submodules A and B of M, if A∩B is an St-closed submodule in M, then A∩B is pure in A and B.
Proof: Since M is almost semi-extending module, then by Theorem (2.5), A∩B is pure in M. Now we have A∩B ≤ A ≤ M, therefore A∩B is pure in A and A∩B is pure in B [11, Proposition (7.2)(2)].
Remark (2.14): The submodule of an almost semi-extending module is not necessary to be almost semi-extending. In fact, we can take a module M such that it is not almost semi-extending module, we also consider the injective hull of M, which is denoted by E(M). It is clear that E(M) is extending, hence it is almost semi-extending. On the other hand, it is known

that M $\leq_{sem}$E(M). But M itself is not almost semi-extending module. However, that is true under certain condition as the following proposition shows.

Recall that an R-module M is called chained if every submodule A and B of M either A ≤ B or B ≤ A [12].

Proposition (2.15): Let M be a chained almost semi-extending module. Then, every St-closed submodule of M is almost semi-extending.

Proof: Let N $\leq_{Stc}$ M, and assume that L $\leq_{Stc}$ N. Since M is a chained so by [4], L $\leq_{Stc}$ M. But, M is almost semi-extending this implies that L is pure submodule of M. Now, since L ≤ N, then Lis pure in N [11], that is N is almost semi-extending.

Proposition (2.16): Let $M_1$and M_2 be fully prime and almost semi-extending modules with $ann_R M_1+ann_R M_2$ = R, then $M_1 \oplus M_2$ is almost semi-extending.

Proof: If both of M_1 and M_2 are zero modules, then trivially $M_1 \oplus M_2$=(0), and there is no thing we do. So that (0) ≠ $M_1 \oplus M_2$. Suppose that (0)≠A≤ $M_1 \oplus M_2$. Since $ann_R M_1+ann_R M_2$ = R, then by similar proof of [13, Proposition (4.2), Ch1], A=C⊕D where C ≤ M_1 and D ≤ M_2. Since A ≠ (0), then either C ≠ (0) or D ≠ (0). We have two cases: the first case is when C ≠ (0) and D = (0), so that A is a submodule of M_1. But M_1 is almost semi-extending, therefore A$\leq_{sem}$H, where H is a pure submodule of M_1. Now, M_1 is a direct summand of $M_1 \oplus M_2$, hence it is pure, thus H is a pure in $M_1 \oplus M_2$ [11]. Second case: C ≠ (0) and D ≠ (0); Since both of M_1 and M_2 are almost semi-extending modules, then C $\leq_{sem}$ H and D $\leq_{sem}$ K, where H and K are pure in M_1 and M_2 respectively. On the other hand, M_1 and M_2 are fully prime, therefore C $\leq_e$ H and D $\leq_e$ K [3, Proposition (2.1)]. But H is pure in M_1 and K is pure in M_2 then H⊕K is pure in $M_1 \oplus M_2$ [7, Lemma (4.2), P.44]. Thus $M_1 \oplus M_2$ is an almost semi-extending module.

It is known that an R-module M is called multiplication if every submodule of M is of the form IM, for some ideal I of R [15, P.174]. In addition, we have $(N:M) = \{r \in \mathbb{R} |\ rM \subseteq N\}$, so M is multiplication if and only if $N = (N:M)M$ [16].

A submodule N of an R-module M is called idempotent if N= Hom_R(M,N)N, where: Hom_R(M,N)N= $\sum\{f(N); \text{such that } f: M \longrightarrow N \text{ is a homomorphism}\}$.
And if every submodule of M is idempotent, then M is called fully idempotent [17].

We also need the following lemma.

Lemma (2.17):

(1) Let M = N ⊕ L be a multiplication R-module with N and L are fully idempotent submodules, then M is fully idempotent [14, Theorem (2.11)].

(2) Every submodule of fully idempotent R-module is pure [14, Lemma (2.13)].

Now, we can introduce the following result.

Proposition (2.18): Let M = N ⊕ L be a multiplication R-module with fully idempotent submodules N and L of M, then M is almost semi-extending.

Proof: Let K $\leq_{Stc}$ M. Since N and L are fully idempotent submodules of M, so by Lemma (2.17)(1), M is fully idempotent module. On the other hand, Lemma (2.17) (2) implies that K is pure. By Theorem (2.5), M is almost semi-extending.

Theorem (2.19): Let M be an R-module. Assume that for each direct summand N of E(M); N∩M$\leq_{Stc}$ M, then the following statements are equivalent.

1. M is an almost semi-extending module.
2. Every St-closed in M is a pure submodule of M.
3. If N is a direct summand of E(M), then N∩M is pure in M.

Proof: (1)⇒(2) It follows by Theorem (2.5).

(2)⇒(3): Let N be a direct summand of E(M). By assumption N∩M $\leq_{Stc}$ M, and by (2), we have N∩M is a pure submodule of M.

(3)⇒(1): Let N be a submodule of M, and B be a relative complement of N, then N⊕B $\leq_e$ M [1, Proposition (1.3), P.17]. Since M $\leq_e$ E(M) therefore N ⊕ B $\leq_e$ E(M) [1, Proposition (1.1)

(a) P.16]. This implies that E(M) = E(N ⊕B) = E(N)⊕E(B). So that E(N) is a direct summand of E(M). By (3), E(N)∩M is pure submodule of M. Now, N$\leq_e$ E(N) and M $\leq_e$ M thus N = N∩M $\leq_e$ E(N)∩M [1, Proposition (1.1), P.16]. This implies that N $\leq_{sem}$ E(N)∩M which is pure in M. Therefore, N is semi-essential in a pure submodule of M that is M is an almost semi-extending module.

We need the following lemma which is appeared in [16], we give another proof.

Lemma (2.20): Let M be a finitely generated faithful and multiplication R-module. If I is a pure ideal of R, then IM is pure submodule of M.

Proof: Since I is pure ideal of R, so that $I \cap J = IJ$ for every ideal J of R. This implies that $(I \cap J)M = (IJ)M$. Since R is commutative, then $(I \cap J)M = J(IM)$. But M is finitely generated faithful and multiplication, then $(I \cap J)M = IM \cap JM$, hence $IM \cap JM = J(IM)$ That is IM is pure in M.

Proposition (2.21): let M be finitely generated, faithful and multiplication R-module. If R is almost semi-extending then M is almost semi-extending.

Proof: Let N $\leq_{Stc}$ M. Since M is multiplication, so $N = IM$ for some ideal I of R. Moreover, M is finitely generated, faithful and N $\leq_{Stc}$ M, so that I $\leq_{Stc}$ R [4]. But, R is almost semi-extending. Therefore, I is pure ideal of R. By Lemma (2.20), $IM = N$ is pure submodule of M that is M is almost semi-extending.

3. Almost semi-extending modules and other related concepts.

The purpose of this section is to study the relationships of almost semi-extending module with other related concepts such as semi-uniform, regular, St-semisimple, purely y-extending and CLS-modules.

Recall that a module M is called semi-uniform if every non-zero submodule of M is semi-essential [3]. so we have the following.

Proposition (3.1): If M is a semi-uniform module then M is almost semi-extending.

Proof: Let N be a submodule of M. If N=(0), then N is semi-essential in itself which is a pure submodule of M. Otherwise, since M is semi-uniform, then N is a semi-essential submodule of M, but M is pure in itself so we are done.

The converse of Proposition (3.1) is not true, for example Z_{24} is almost semi-extending Z-module, but not semi-uniform, since the submodule $(\bar{8})$ of Z_{24} is not semi-essential submodule of Z_{24}.

Recall that an R-module M is called pure simple if (0) and M are the only pure submodules of M [18].

Theorem (3.2): Let M be a pure simple module. Then M is semi-uniform if and only if M is almost semi-extending.

Proof: ⇒) It comes from Proposition (3.1).

⇐) Assume that M is an almost semi-extending module, and N is a non-zero submodule of M. Since M is almost semi-extending, then N is semi-essential in a pure submodule say H. But M is pure simple so either H = (0) or M. If H =(0), then N = (0), which is a contradiction. If H = M, then N is semi-essential in M. Thus M is semi uniform module.

Note that both of Z and Z_{p^∞} are pure simple Z-module [18], and we can easily show that each of them satisfies theorem (3.2).

The next example shows that we cannot drop the condition pure simple from Theorem (3.2).

Example (3.3): Consider the Z-module Z_{12}. Note that Z_{12} is not pure simple module since there exists $(\bar{4}) \subsetneqq Z_{12}$ which is a pure submodule. We know that Z_{12} is an almost semi-extending Z-module, since it is semi-extending. On the other hand Z_{12} is not semi-uniform, since $(\bar{4})$ is not semi-essential submodule of Z_{12}.

Recall that a module M is called F-regular if every submodule of M is pure [11].

Proposition (3.4): Every F-regular module is almost semi-extending .

Proof: Let M be an F-regular module and N ≤ M. Since M is F-regular, then N is a pure submodule of M. On the other hand, N is semi-essential in itself, so we conclude that N is semi-essential in a pure submodule which is itself (i.e $N \leq_{sem} N$). Thus M is almost semi-extending.

Corollary (3.5): Every fully idempotent module is almost semi-extending.

Proof: The result follows directly by Lemma (2.17)(2) and Proposition (3.4).

Ahmed in [19] defined the St-semisimple module as a module M in which every submodule is St-closed.

Proposition (3.6): Every St-semisimple module is almost semi-extending.

Proof: Since M is St-semisimple, then every submodule of M is St-closed. This implies that every submodule of M is direct summand [4, Proposition (1.13)]. But every direct summand is pure, so by Theorem (2.5), M is almost semi-extending.

Recall that a singular submodule is defined by Z(M) ={x ∈ M, ann (x) $\leq_e$ R}. If Z(M) = M, then M is called a singular module, and when Z(M) = (0), then M is called nonsingular [1, P.31].

A submodule N of module M is called y-closed submodule if $\frac{M}{N}$ is a nonsingular module [1,P.42]. A module M is called purely y-extending if every y-closed submodule of M is pure [20].

We cannot find a direct implication between almost semi-extending and purely y-extending but in the terms of a nonsingular module, we have the following Proposition.

Proposition (3.7): Let M be a nonsingular module. If M is purely y-extending, then M is almost semi extending.

Proof: Let $N \leq_{Stc} M$. Since M is a nonsingular, so that N is y-closed [4]. But, M is purely y-extending, therefore N is pure, and the proof is finished.

Also, under the class of fully prime modules, purely y-extending is almost semi-extending, as the following proposition shows.

Proposition (3.8): In the class of fully prime module every almost semi-extending module is purely y-extending.

Proof: Suppose that M is a fully prime and almost semi-extending module. Let N be a y-closed submodule in M. If N=(0), then N is clearly pure in M. If N≠(0); since M is fully prime, then N is an St-closed submodule in M. By assumption, N is pure submodule of M, that is M is purely y-extending.

An R-module M is called CLS-module if every y-closed submodule of M is direct summand [5].

We do not know if there is a direct implication between this class of module and almost semi-extending. In fact, we investigate that under certain class of modules as we show in the following.

Proposition (3.9): Let M be a nonsingular module. If M is a CLS-module then M is almost semi-extending.

Proof: Let N be an St-closed submodule of M. Since M is nonsingular, then N is y-closed [4, Proposition (1.24)]. But, M is CLS-module, therefore N is direct summand of M, hence N is pure submodule of M, so that M is almost semi-extending.

Theorem (3.10): if M is a nonsingular R-module, then we have the following:

1. If M is a CLS-module, then M is a purely y-extending module.

2. If M is a purely y-extending module, then an almost semi-extending module.

3. I if M is fully prime and purely semisimple ,and M is an almost semi-extending module, then M is a CLS-module.

Proof:

(1) Let N be y-closed submodule of M. Since M is CLS-module, then N is a direct summand of M. This implies that N is a pure submodule of M, and we are done.

(2) It comes from Proposition (3.7).

(3) Let N be y-closed submodule of M. If N=(0), then clearly N is direct summand of M. Otherwise, since M is fully prime, then N is a St-closed [4, Proposition (1.23)]. But M is almost semi-extending, then N is pure. Beside that M is purely semisimple , so by Lemma (2.8), N is a direct summand of M, hence M is CLS-extending.

Theorem (3.11): If M be a nonsingular and fully prime module, then we have the following statements:

1. If M is a semi-extending module, then M is an almost semi-extending module.

2. M is an almost semi-extending module if and only if M is a purely y-extending module

3. If M is purely semisimple and a purely y-extending module, then it is a semi-extending module

4. Proof:

(1) It comes straightforward.

(2) It follows from Proposition (3.8). For the other direction**.** Since M is nonsingular, so by Proposition (3.7), we get M is an almost semi-extending.

(3) Let N be an St-closed submodule in M. Since M is a nonsingular, then N is y-closed [4, Proposition (1.24)]. But M is purely y-extending, therefore N is pure. On the other hand, M is purely semisimple, so by Lemma (2.8), N is a direct summand of M. Thus M is semi-extending.

4. Conclusions and discussions:

In the light of the above results we focus on some point, and we conclude the following:

1. The relationships between almost semi-extending and both of semi-extending with purely extending modules are studied in Remark (2.2)(2), Proposition (2.3), Theorem (2.4), Proposition (2.6), Proposition (2.9), Theorem (2.10) and Theorem (2.11).

2. Another characterization of almost semi-extending modules is given in Theorem (2.5). Some equivalent statements for almost semi-extending under certain condition are given in Theorem (2.19).

3. The behavior of submodules of the almost semi-extending modules is discussed in Remark (2.14) and Proposition (2.15).

4. The direct sum of two almost semi-extending modules is studied in Proposition (2.16). Moreover, we discuss how can the direct sum of fully idempotent submodules is to be almost semi-extending module. This is done in Proposition (2.18).

5. The relationships of almost semi-extending modules with other related concept are considered such as semi-uniform, F-regular, St-semisimple, purely y-extending and CLS-module, see: Proposition (3.1), Theorem (3.2), Proposition (3.4), Corollary (3.5), Proposition (3.6), Proposition (3.7), Proposition (3.8), Proposition (3.9), Theorem (3.10) and Theorem (3.11).

Furthermore, some other results are given to describe other important properties of almost semi-extending modules.

Acknowledgment: The authors of this paper would like to thank the referees for their valuable suggestions and helpful comments.

References

[1] K. Goodearl, “Ring Theory, Nonsingular Rings and Modules”, Marcel Dekker, New York, 1976.

[2] M. Behboodi, O.A.S. Karamzadeh, and H. Koohy, “Modules Whose Certain Submodules are Prime”, *Vietnam J. of Mathematics*, v.32, n.3, pp.303-317, 2004.

[3] M.A. Ahmed, and M.R. Abbas, “On Semi-essential Submodules”, *Ibn Al-Haitham J. for Pure and Applied Science*, v.28, n.1, pp.179-185, 2015.

[4] M.A. Ahmed, and M.R. Abbas, “St-closed Submodules”, *Journal of Al-Nahrain University*, v.18, n.3, pp.141-149, 2015.

[5] A. Tercan, “On CLS-Modules”, *Rocky Mountain J. Math*., v.25, pp.1557-1564, 1995.

[6] M.A. Ahmed, and M.R. Abbas, "Semi-extending Modules", *International J. of Advanced Scientific and Technical Research*, v.6, n.5, pp.36-46, 2015.
[7] B.H. Al-Bahraany, "Modules With the Pure Intersection Property", Ph.D. Thesis, College of Science, University of Baghdad, 2000.
[8] Z.T. AL- Zubaidy, "On Purely Extending Modules", M.Sc. Thesis, College of Science, University of Baghdad, 2005.
[9] D.W. Sharpe, and P. Vamos "Injective Modules", Cambridge At The University Press, 1972.
[10] F.D. Shyaa, "A Study of Modules Related With T-semisimple Modules", Ph.D. Thesis. College of Education for Pure Science Ibn Al-Haitham, University of Baghdad, 2018.
[11] D.J. Fieldhouse, "Pure Theories", *Math. Ann*., v.184, pp.1-18, 1969.
[12] B.L. Osofsky, "A Construction of Nonstandard Uniserial Modules Over Valuation Domain", *Bulletin Amer. Math. Soc*, v. 25, pp. 89-97, 1991.
[13] M.S. Abbas, "On Fully Stable Modules", Ph.D. Thesis, University of Baghdad, 1991.
[14] N.O. Ertas, "Fully Idempotent and Multiplication Modules", *Palestine Journal of Mathematics* v.3, pp. 432-437, 2014.
[15] A. Barnard, "Multiplication Modules", *J. Algebra*, v.71, pp. 174-178, 1981.
[16] M.A. Majid, and J.S. David, "Pure Submodules of Multiplication Modules", *Contributions to Algebra and Geometry*, vol. 45, no.1, pp. 61-74, 2004.
[17] D.K. Tutuncu, D.K., N.O. Ertas, R. Tribalc, and P.F. Smith, "On Fully Idempotent Modules", *Comm. Algebra*, v.39, pp. 2707-2722, 2011.
[18] D.J. Fieldhouse, "Pure Simple and Indecomposable Rings", *Can. Math. Bull*. v.13, pp. 71-78, 1970.
[19] M.A. Ahmed, "St-Polyform and Related Concepts", *Baghdad Science Journal*, v.15, n.3, pp. 335-343, 2018.
[20] B.H. Al-Bahraany, "On Purely y-extending Module", *Iraqi Journal of Science*, v.54, n.3, pp. 672-675, 2013.