DOI: 10.24996/ijs.2022.63.4.34

ISSN: 0067-2904

Fuzzy α- Topological Vector Space

Suhad K. Hameed
Department of Mathematics, College of Science, Diyala University, Diyala, Iraq

Received: 19/7/2021 Accepted: 9/10/2021 Published: 30/4/2022

Abstract
In this study, the concept of fuzzy α-topological vector space is introduced by using the concept fuzzy α-open set , some properties of fuzzy α-topological vector spaces are proved .We also show that the space is T_2-space iff every singleton set is fuzzy α- closed .Finally, the convex property and its relation with the interior points are discussed.

Keywords: Topological Vector Space, Fuzzy α-Topological Vector Space, Fuzzy T_2-space and Extremely points .

الفضائات التبولوجية المتجهة الضبابية نوع الفا

سهاد كريم حميد
قسم الرياضيات، كلية العلوم، جامعة ديالى، ديالى، العراق

الخلاصه
تناولت هذه الدراسة تعريف الفضاءات التبولوجيه المتجهه الضبابية نوع الفا و دراسة خصائصها , كما تم اثبات ان الفضاء يكون منفصلا ضبابيا اذا كانت كل مجموعة منفردة مغلقة ضبابيا , وتم مناقشة الخاصية الجبرية (التحدب) وعلاقتها بالنقاط الداخلية للمجموعات الموجودة في الفضاء المدروس.

Introduction and Preliminaries:

The aim of the study is to define the notation of fuzzy α- topological vector space in the term of fuzzy set. The fuzzy concept has used in many branches of mathematics since 1965.The introduction of fuzzy sets is done by Zadeh [1]. The theories of fuzzy topological vector spaces are introduced and developed by A.K.Katsaras [2],[3], and are generalized by many authors as in [4],[5]. We consider it in terms of fuzzy α-open set by the sense of Bin Shahna, [6]. A subset A of a topological space X is called fuzzy α-open(fα-open) if $A \subset (Int(Cl(Int(A)))$ [7] , while a complement for fα-open set is called fuzzy α-closed(fα-closed). The fuzzy α-closure of A that subset of X is represented by $Cl_\alpha(A)$ which is intersection for all fα-closed subsets on X containing A. Recall that the subset $U \subset X$ is called fα-open neighborhood for x if there is a fα-open set A with $x \in A \subset U$. The point x for a subset A is said to be fα-interior point on A which is denoted by $Int_\alpha(A)$,if there is fα-open subset U , $x \in U$, and $U \subseteq A$, . A function $f: X \to Y$ is called fα- irresolute continuous if an inverse image for each fα-open subset in Y is fα-open on X where X and Y is a fuzzy topological spaces[8]. Also, a mapping $f: X \to Y$ is called fα-irresolute continuous at point x on X if for all fα-open

*Email: suhadkhamid@gmail.com

subset *V* on *Y* contained *f(x)*,there is a fα-open subset *U* with $x \in U$ satisfy f*(U)* is subset of *V* [8]. A function *f* from a fuzzy topological space X to Y is called fα-open function if the image for each fα-open subset of *X* is fα-open set in *Y*. Moreover, the notion of fα-homeomorphism[8] that defines a mapping *f* from X to Y is called fα-homeomorphism if it's bijective , both *f* and f^{-1} are fα-irresolute. The separation axiom and the compactness property are discussed.

suhadkareem@uodiyala.edu.iq

1. Fuzzy α- Topological Vector Space (FαTVS).

Definition 1.1: A fuzzy α- topological vector space is a vector space V over a field F with a fuzzy topology τ such that the two functions :

(a) The vector addition map $S:X\times X\to X$

(b) The multiplication by scalar map $M:F\times X\to X$

Are fuzzy α- irresolute continuous . For each $x \in X$ the mapping $T_x :X\to X$ which is defined by $T_x(y) = y+ x$, and the mapping $M_\lambda : X\to X$ that is defined by $M_\lambda(x) = \lambda.x$ are called the translation, and multiplication, respectively .

We refer to the collection of fα -neighborhoods for $x\in X$, by N_x , and we denotes the set of fα -neighborhoods of zero vector space 0 of *X* by N_0 . A subset A of a topological spaces (X,τ) is called sp-open if A ⊂ cl (int (cl(A)))[8].

Lemma 1.2[9]: Let. *A* be a sp-open(α-open, fα-open) subset of a topological space *X* , and *B* be any open subset of *X* , then the set $A\cap B$ is sp-open (α-open, fα-open, respectively)set .

Lemma 1.3[9]: Let $.f:X\to Y$ be sp-irresolute mapping where *X* and *Y* are topological spaces, so that for sp-neighborhood *V* for *f*(x), there sp-neighborhood *U* of *x* satisfies $f(U)\subseteq V$.

Theorem 1.4 : Let. *X* be FαTVS , the following statements hold.

a. If $U \in N_x$, and *V* is a neighborhood for *x* in *X* , then $U \cap V \in N_x$.

b. If $U \in N_0$, then $\lambda U \in N_0$ for a non-zero element $\lambda \in R$.

c. If $U\in N_0$, then $x+ U \in N_x$

Proof: a) Suppose that *U* is fα-neighborhood of *x* ,and *V* is a neighborhood for *x*, then there exists a fα-open subset *A* and an open set *B* s.t $x \in A\subset U$ and $x\in B\subset V$.

Then $x \in A\cap B\subseteq U\cap V$ and by Lemma 1.2, $A\cap B$ is fα-open. So that $A\cap B$ is a fα-neighborhood for *x*. To prove (b) and(c), let *U* is fα- neighborhood for *zero* since $S:(x, y)$ is fα -irresolute , we define the function T_x: $X\to X$ by $T_x(y)= y+ x$. Therefore $T_x(y)=S_x(x, y)$, $T_x(y)$ is fα -irresolute also $T^{-1}_x (y)= S_x(x, -y)$ is also fα -irresolute(from the definition of addition) ,thus T_x is fα-homeomorphism ,by lemma 1.3 for a fα -neighborhood *U* for *zero*, there fα -neighborhood $U+ x$ *for a point x*.

Definition 1.5[10]: The subset *A* on the vector spaces *X* is called balanced if $\lambda A \subseteq A$ for $|\lambda|\leq 1$and absorbing if every *x* belongs to *X* , there is $\varepsilon > 0$ such that $\lambda x\in A$ for $|\lambda|\leq \varepsilon$. It is called absolutely convex if the subset both balanced and convex.

Theorem 1.6 : Let. *X* be an FαTVS ,then

(a) Every fα -neighborhood *U* of *0* is absorbing.

(b) For a fα -neighborhood *U* of *0* there exists a balanced $V \in N_0$ satisfy that $V \subseteq U$.

Proof: (a) Suppose *U* be a fα-neighborhood for *0* , then there exists afα-open subset $U_1 \in N_0(X)$ such that $U_1\subseteq U$, since the scalar multiplication map M_λ is fα-irresolute , so that there exists fα-neighborhood of 0 (V_1 , V_2) satisfies that $M_\lambda (V_1\times V_2) \subseteq U_1$ .Let a set V_1 contains an open interval (-ε, ε) , therefore $tx \in U_1 \; \forall \; t\in(-\varepsilon, \varepsilon)$ and for all $x \in V_2$. This leads to U_1 which is absorbing.

(b). The map of multiplication $M_\lambda :R\times X\to X$ is fα-irresolute then for every fα -neighborhood *U* of 0 in *X* there exists fα-neighborhood of 0 , $M_\lambda (V) \subseteq U$, then there exists ε> 0 , $V= V_1\times V_2$, $(-\varepsilon,\varepsilon) \subseteq V_1$, V_1 is a fα-neighborhood for 0 on *R* ,

V_2 is fα-neighborhood of 0 in X. Let $W=\bigcup_{|t|<\varepsilon} t\, V_2$, tV_2 be fα-neighborhood of 0 from theorem 1.4 , for $t\neq 0$ and $tV_2\subseteq U$ for $t<\varepsilon$. Now we need to prove that W is balanced. If $r<1$, so $rW=\bigcup_{|t|<\varepsilon}(rt)\, V_2$ and $|r\,t|<\varepsilon$ $|r|<\varepsilon$, it implies that $rW=\bigcup_{|s|<\varepsilon} sV_2\subseteq W$, where $s=r\,t$, thus W is balanced.

Theorem 1.7 : Let. X be a FαTVS. If $A\subseteq X$, so $Cl_{f\alpha}(A)=\cap(A+U)$. In particular $Cl_{f\alpha}(A)\subseteq A+U$, $\forall U$ *belongs to* N_0.

Proof: Let. $x\in Cl_{f\alpha}(A)$, U be a fα-neighborhood of 0 then by theorem 1.6(b) V is the balanced neighborhood for zero s.t $(V\subseteq U)$, so $x+V$ fα-neighborhood for x and $x\in Cl_{f\alpha}(A)$, then $(x+V)\cap A\neq \mathbf{0}$, that implies $x\in A\text{-}V$. Since V is balanced, $A\text{-}V$ equal to $A+V$, then $x\in A+V$ *subset of* $A+U$, hence $Cl_{f\alpha}(A)\subseteq\cap(A+U)$. Conversely if $x\notin Cl_{f\alpha}(A)$,so that there balanced neighborhood U for zero satisfy that $(x+U)\cap A=\mathbf{0}$, so that $x\notin A\text{-}U=A+U$.

Theorem1.8: Let. X be a FαTVS. Then ,

(a) Every $U\in N_0$, $\exists\ V\in N_0$ satisfy $V+V\subseteq U$.

(b) For every $U\in N_0$, there is a fα-closed balanced $V\in N_0$ satisfy that $V\subseteq U$.

Proof: (a) Assume that $U\in N_0(X)$, since the map $S:X\times X\to X$ is fα-irresolute , then there are fα-neighborhoods for 0 say V_1 and V_2 satisfy $S(V_1,V_2)\subseteq U$, that means $V_1+V_2\subseteq U$, set $V=V_1\cap V_2$ which implies $V+V\subseteq V_1+V_2\subseteq U$.

(b) Assume that U is fα-neighborhood of 0 on X , by part (a) there is fα-neighborhood V for zero with $V+V\subseteq U$. Now from Part (b) of Theorem 1.6, there exists a neighborhood W for 0 which is balanced s.t $W\subseteq V$.*Now* from theorem 1.7 the $Cl_{f\alpha}(W)\subseteq W+V$ and $Cl_{f\alpha}(W)\subseteq W+V\subseteq V+V\subseteq U$. This shows that U contains the fα-closed neighborhood.

Definition 1.9: A topological space (X,τ) is called fα- Hausdorff , if for each two distinct points x and y in X , there exist disjoint fα-open sets U,V such that $x\in U$, and $y\in V$.

In the following theorem we give some properties of fα-Hausdorff space.

Theorem 1.10 : Let . X be a FαTVS, then X is fα- Hausdorff if and only if for each $x\in X$ there exists U_0 satisfies $x\notin U_0$.

Proof: (a) $\Rightarrow$ (b) Let $x\in X$ be a non-zero vector . Then , there are disjoint fα-neighborhoods U of 0, and V of x , where U *belong to* the collection of neighborhoods of 0 , V *belong to* the collection of neighborhoods of x and $x\notin U$.

(b) $\Rightarrow$(a) let $x, y\in X$ such that $x\neq y$, U_0 is a fα-neighborhood for 0 with $x\text{-}y\notin U_0$. By part (a) of Theorem 1.8, there exists fα-neighborhood W of 0 such that $W+W\subseteq U_0$ W is balanced (from part (b) of Theorem 1.6) .Now suppose that the sets $V_1=x+W$ and the set $V_2=y+W$. Therefore, V_1 , V_2 is a fα-neighborhood of x, y, respectively . We need to show that $V_1\cap V_2=\varnothing$, assume the point $s\in V_1\cap V_2$ then

$-(s-x)\in W$, note that W is balanced and $s-y\in W$,and this leads to

$x-y=(s-y)+(-(s-x))\in W+W\subseteq U_0$ which is a contradiction. So that we have $V_1\cap V_2=\phi$. Thus, the space X is fα- Hausdorff.

Corollary 1.11: Let X be a FαTVS, then the following statements are equivalent

(a) Let. X be a fα- Hausdorff.

(b) The intersection of all neighborhoods of 0 is $\{0\}$.

(c) The intersection of all neighborhoods of x is $\{x\}$.

Theorem 1.12:Let X be FαTVS, *then* X is fα-Hausdorff iff one-point subset of X be fα-closed in X .

Proof: Let $x\in X$, $y\in X/\{x\}$, and $x\neq y$ that means $x\text{-}y\neq 0$,therefore there exists a fα-neighborhood U for zero satisfies $y-x\notin U$. Then there exists fα-closed balanced neighborhood W of 0 with $W\subseteq U$ (by theorem 1.8,b) . Which implies that $y-x\notin W$ then

$y - x \in X - W$. Therefore $y \in (X - W) + \{x\}$, also, $(X-W)+\{x\}$ is fα-open , W is fα-closed, $(X - W) + \{x\}$ contained on $(X - \{x\})$. Which shows $X/\{x\}$ is fα-open, thus {x} is fα-closed. Conversely , let $x \in X$,and assume the singleton $\{x\}$ is fα-closed. From Theorem 1.7 the set $\{x\}=Cl_{f\alpha}\{x\}=\cap\{U+\{x\}:$ U is fα-neighborhood for zero}= {W: W is fα-neighborhood of x} where $W = U+\{x\}$. Thus by Corollary 1.11, X is fα- Hausdorff.

Definition 1.13:A topological space (X ,τ) is called fα-compact if for any cover for X by fα-open subsets have a finites subcover.

Theorem 1.14 : Let C, K be subsets in a FαTVS X , *and* $C\cap K=\emptyset$ such that C is a fα-closed, and K fα-compact. Then there exists a fα-neighborhood U for 0 such that
$(K+U)\cap(C+U)=\phi$.

Proof: If $K=\phi$, then the proof is trivial . Otherwise, let $x \in K$, and $x=0$. Then, $X - C$ is an fα-open set of 0 . Since the addition mapping is fα-irresolute and fα-continuous. Therefore , fα- neighborhood U for zero satisfies $3U = U + U + U \subset X - C$. Define $\tilde{U}= U\cap(-U)$ which is fα-open , symmetric and $3\tilde{U}= \tilde{U}+ \tilde{U}+ \tilde{U} \subset X-C$. That leads to $\emptyset=\{x +x+x ,x\in \tilde{U}\}\cap C =\{x+x ,x \in \tilde{U}\}$ intersected$\{y- x , x\in \tilde{U}, y\in C \}$ and $\tilde{U}\cap\{C+ \tilde{U} \} \subset \{2x ,x \in \tilde{U}\}\cap \{y- x , x\in \tilde{U}, y\in C \}$.This is for one point. Now we have K is fα-compact, then by the above argument for every $x \in K$, we have a symmetric fα-neighborhood V_x s.t. $(x+2V_x)\cap(C+V_x)=\phi$. The sets $\{V_x : x \in K\}$ are a fα-open that covers K, and since K is fα-compact subset therefore for finitely number for points $x_i \in K, i=1,...,n,$ we have $K \subset \bigcup_{i=1}^{n}(x_i + V_{x_i})$. Define the fα-neighborhood V of 0 by $V=\bigcap_{i=1}^{n} Vx_i$ Therefore $(K+V)$ *intersected* $(C+V)\subset\bigcup_{i=1} n\ (x_i+ Vx_i+V)\cap (C+V) \subset\bigcup_{i=1} n\ (x_i+ 2\ Vx_i\ )\cap(C+ Vx_i) = 0$, and the proof is finished

Lemma 1.15: Let U be fα-open subset of a FαTVS X, and A be any subset such that $U \cap A = \phi$, then $U\cap Cl_{f\alpha}(A) = 0$.

Proof: Let. $x\in (U\cap Cl_{f\alpha}(A))$. Thus $x\in Cl_{f\alpha}(A)$, U is fα-neighborhood for x , since U is fα-open subset , then $X - U$ is fα-closed subset contain A, so that $Cl_{f\alpha}(A) \subseteq X-U$, and $x\notin Cl_{f\alpha}(A)$ which implies a contradiction, therefore $U\cap Cl_{f\alpha}(A) =\emptyset$.

Corollary 1.16: Suppose that C, and K are disjoint sets in a FαTVS X with C fα-closed, K fα-compact. Therefore, there fα-neighborhood U for zero satisfies $Cl_{f\alpha}(K+U)\cap(C+U)=0$.

Proof: In theorem 1.14 we have for any disjoint fα-closed C set and fα-compact set K , so that there is a fα-neighborhood U for 0 such that $(K+ U) \cap(C+ U) =0$. The set $C+ U = \{y+ U : y\in C\}$ is a fα-open set , then by lemma 1.15 $Cl_{f\alpha}(K+U) \cap(C+ U) =0$.

Definition 1.17: Let X be a vector space with field K, an algebra dual for X is the collection of linear functional which is defined in X and represented by X^* .

Theorem 1.18: Let X be a FαTVS and $0 \neq f \in X^*$, then $f(F)$ is fα-open in K whenever F is fα-open in X .

Proof: Let. F be a non-empty subset of X , and $0 \neq x_0 \in X$ with $f(x_0)=1$. Then, for any point $a \in F$, we have to prove that $f(a) \in Int_{f\alpha}(f(F))$. F is fα-open neighborhood for a then, by Theorem 1.4 we have $F - a$ is fα-neighborhood of 0 . By Theorem 1.6 $F - a$ is absorbing , then there exists $\epsilon > 0$ such that $\lambda x_0 \in F -a$ for $\lambda\in R$ with $|\lambda| \leq \in$. Thus, for any $\beta \in R$ with $|\beta - f(a)| \leq \in$ we have the $(\beta-f(a))x_0 \in F-a$, thus $f((\beta- f(a)x_0)\in f(F-a)$, $(\beta- f(a))\ f(x_0) \in f(F-a)$ $(\beta- f(a))\ (1)\in f(F-a)= f(F)-f(a)$ which lead to $\beta \in f(F)$, $f(a) \in [\beta-\in , \beta+\in]$, f(a) is interior point so that f(a) $\in Int(f(F))\subseteq Int_{f\alpha}(f(F))$, which shows that $f(F)= Int_{f\alpha}(f(F))$.

Note that an extremely point of a convex set in a vector space X is a point which is not an interior point of a segment.

Lemma 1.19: [10] Let X be a vector space ,and $0\neq A\subseteq X$. For $x\in A$ the following statements are equivalents.

1) x is extremely point on A

2) if $a ,b\in A$ such that, $x=\frac{1}{2}(a+b)$, then, x *equal to* a *equal to* b.

3) let. $a,b \in A$, with $a\neq b$, let $\lambda\in(0,1)$, $x=\lambda a+(1-\lambda)b$. Then we have either $\lambda=0$, or $\lambda=1$

Theorem 1.20: Let X be FαTVS, *and* $F\subset X$ be convex . Therefore, [fα-interior $(F]\cap(\delta F)=0$

Proof: If $Int_{f\alpha}(F)=\emptyset$, the proof is trivial. Suppose that the $Int_{f\alpha}(F)\neq\emptyset$ and let $x\in Int_{f\alpha}(F)$. Therefore, there is a fα-neighborhood U for 0 s.t, $x+U\subset F$. Define the mapping $\varphi:\Re\to X$ where $\varphi(\mu)=\mu x$ continuous at $\mu=1$, for this the fα-neighborhood $x+U$, there is an $s>0$ s.t, $\mu x\in x+U$ whenever $|\mu-1|\leq s$. In particular, we have $(1+s)x\in x+U\subset F$ and $(1-s)x\in x+U\subset F$. Now consider $x=\lambda(1+s)x+(1-\lambda)(1-s)x$ and take $\lambda=\frac{1}{2}$. So that $x=\frac{1}{2}(1+s)\,x+(1-\frac{1}{2})(1-s)\,x$, which leads to the point x which is not extremely on F.

Conclusion: Throughout this research we have been proved some results, namely the fuzzy-α topological vector spaces satisfies ,the property of any neighbourhood for 0 containing a neighborhood for 0 which is balanced ,FαTVS satisfies the separation axiom (fuzzy-α hausdorff) and in a fuzzy-α topological vector space the fuzyy-α-interior of a convex set didn't intersect with the boundary of the set .

Reference

[1] L. A. Zadeh, "Fuzzy sets", *Information and Computation*, vol. 8, pp. 338–353, 1965.

[2] A.K.Katsaras and D.B.Liu , "Fuzzy vector spaces and fuzzy topological vector spaces", *J.Math. Anal.Appl*. vol. 58, pp. 135-146, 1977.

[3] A.K.Katsaras, "Fuzzy vector spaces and fuzzy topological space II", *Fuzzy Sets and Systems*, vol. 12, no. 2, pp. 143-154, 1984.

[4] Jin-Xuan Fang, Cong-Hua Yan, "Induced I(L)-fuzzy topological vector spaces", *Fuzzy Sets and Systems*, vol. 121, pp. 293-299, 2001.

[5] M.Alimohammady, M Roohi, "Fuzzy minimal structure and fuzzy minimal vector spaces", *Chaos,Solitons & Fractals*, vol. 27, no. 3, pp. 599-605, 2006.

[6] Bin Shahna , A.S. "On fuzzy strong semicontinuity and fuzzy pre-continuity". *Fuzzy Sets and Systems*, vol. 44, pp. 303-308, 1991.

[7] S.S. Thakur and R.K. Saraf, "α-compact fuzzy topological space" , *Mathematica Bohemica* , vol. 120, no. 3, pp. 299-303, 1995.

[8] V.Seenivasan , R.Renuka, "intuitionistic fuzzy α-irresolute function" , *Italian journal of pure and applied mathematics*, n.33, pp. 71-80, 2014.

[9] D. Andrijevie , "Semi-pre Open Sets" , *Math Vesnik*, vol. 38, pp. 24 -32, 1986.

[10] Horvath, J. "*Topological Vector Spaces and Distributions", Addison and Wesley publishing comp., London*, 1966.