DOI: 10.24996/ijs.2022.63.7.31

ISSN: 0067-2904

The generalized Cayley graph of complete graph K_n and complete multipartite graphs $K_{n,n}$ and $K_{n,n,n}$

Ayat Abdulaali Neamah[1]*, Abdulrahman H. Majeed[2], Ahmad Erfanian[3]

[1] *Department of Mathematics, Al-Nahrain University, Baghdad, Iraq*

[2] *Department of Mathematics, University of Baghdad,Baghdad, Iraq*

[3] *Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran*

Received: 17/7/2021 Accepted: 20/9/2021 Published: 30/7/2022

Abstract

Suppose that G is a finite group and S is a non-empty subset of G such that $e \notin S$ and $S^{-1} \subseteq S$. Suppose that $Cay(G,S)$ is the Cayley graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. In this paper, we introduce the generalized Cayley graph denoted by $Cay_m(G,S)$ that is a graph with vertex set consists of all column matrices X_m which all components are in G and two vertices X_m and Y_m are adjacent if and only if $X_m[(Y_m)^{-1}]^t \in M(S)$, where ${Y_m}^{-1}$ is a column matrix that each entry is the inverse of similar entry of Y_m and $M(S)$ is $m \times m$ matrix with all entries in S , $[Y^{-1}]^t$ is the transpose of Y^{-1} and $m \geq 1$. In this paper, we clarify some basic properties of the new graph and assign the structure of $Cay_m(G,S)$ when $Cay(G,S)$ is complete graph K_n, complete bipartite graph $K_{n,n}$ and complete 3-partite graph $K_{n,n,n}$ for every $m \geq 2$.

Keywords: Cayley graph; complete graph; bipartite graph; 3-partite graph; generalized Cayley graph; column matrix.

2020 Mathematics Subject Classification: Primary 05C25; 05C38; Secondary 05C07

تعميم الرسم البياني كيلي للرسم البياني الكامل K_n و الرسوم البيانية متعددة الأجزاء الكاملة $K_{n,n}$ و $K_{n,n,n}$

آيات عبدالعالي نعمه[1]*, عبدالرحمن حميد مجيد[2], احمد عرفانيان[3]

[1]قسم الرياضيات, كلية العلوم, جامعة النهرين, بغداد, العراق

[2]قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق

[3]قسم الرياضيات الصرفة, كلية الرياضيات, جامعة فردوسي, مشهد, ايران

الخلاصة

نفرض G مجموعة محدودة و S مجموعة جزئية غير فارغة من G بحيث $e \notin S$ و $S^{-1} \subseteq S$. إذا كان $Cay(G,S)$ رسم كيلي الذي رؤوسه تكون جميع عناصر G و رأسان x و y متجاورتان إذا وفقط إذا

*Email: erfanian@math.um.ac.ir

كان $xy^{-1} \in S$. نعرف التعميم لرسم البياني كيلي $Cay_m(G,S)$ وهو رسم بياني مجموعة رؤوسه تتكون من جميع مصفوفات الأعمدة X_m بحيث كل عناصر المصفوفة تؤخذ من G ورأسين X_m و Y_m متجاورتان إذا وفقط إذا كان $X_m[(Y_m)^{-1}]^t \in M(S)$، في هذا البحث، قمنا بتوضيح بعض الخصائص الأساسية للرسم البياني الجديد وقمنا بتعيين شكل الرسم البياني $Cay_m(G,S)$ عندما يكون $Cay(G,S)$ رسم البياني K_n و رسم بياني كامل ثنائي الأجزاء $K_{n,n}$ و كامل ثلاثي الأجزاء الرسم البياني $K_{n,n,n}$ لكل $m \geq 2$.

1. Introduction

A Cayley graph is a graph that encodes a group in a graph. It was defined by Arthur Cayley in 1878 [2]. He used the set of generators for the new geometric representation of a group. This translates groups into geometrical objects which can be fully considered from the geometric view. Particularly, it prepares a rich source of many symmetric graphs which are known as transitive graphs and it plays a serious work in many graph-theoretical problems as well as group theoretical problems, like Hamiltonian path and cycles, representation of interconnection networks, a width of groups and expanders that surely occur in computer science and etc. In this article, we purpose and present a new kind of generalization of the Cayley graph.

Formerly, some kinds of generalizations of the Cayley graph have been defined and studied by some authors. For example, Marušič in [5] defined a generalization of the Cayley graph with respect to an automorphism of group G. Later, Zho in [7] nominated Cayley graph on a semigroup. Recently, Erfanian [1] gave a new definition of a generalized Cayley graph namely $Cay_m(G,S)$ by using column $m \times 1$ matrices which is a new generalization of usual $Cay(G,S)$.

We mention that for any group G and any non-empty subset S of G such that $e \notin S$ and $S^{-1} \subseteq S$, the Cayley graph $Cay(G,S)$ is an undirected simple graph whose vertex set consists of all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. It is clear that, there are many kinds of Cayley graphs for each group that it was built by changing of S. We know that the Cayley graph $Cay(G,S)$ is connected if and only if S is a generating set of G. Also, it is regular and vertex transitive, see [3] for more details. Now, we define $Cay_m(G,S)$ as follows:

Definition 1.1 *For each $m \geq 1$, the generalized Cayley graph $Cay_m(G,S)$ of $Cay(G,S)$ is an undirected simple graph with vertex set consists of all $m \times 1$ matrices $[x_1 \quad x_2 \quad \cdots \quad x_m]^t$, where $x_i \in G, 1 \leq i \leq m$ and two vertices $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ and $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]^t$ are adjacent if and only if*

$$X(Y^{-1})^t = \begin{bmatrix} x_1{y_1}^{-1} & x_1{y_2}^{-1} & \cdots & x_1{y_m}^{-1} \\ x_2{y_1}^{-1} & x_2{y_2}^{-1} & \cdots & x_2{y_m}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_m{y_1}^{-1} & x_m{y_2}^{-1} & \cdots & x_m{y_m}^{-1} \end{bmatrix} \in M_{m\times m}(S), \text{ where}$$

$$M_{m\times m}(S) = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{bmatrix} \quad s.t. \quad x_{ij} \in S \ , \ 1 \leq i,j \leq m \right\}.$$

It is clear that, if $m = 1$, then $Cay_m(G,S)$ and $Cay(G,S)$ are coincide.

In all parts of this paper, We always suppose that $e \notin S$, $S^{-1} \subseteq S$ and S is generating set of G . Hence, $Cay(G,S)$ is connected graph. Note that if $e \in S$ then graph will have a loop and the condition , $S^{-1} \subseteq S$ deduces that the graph is undirected.

Before to continue stating some results about the generalized Cayley graph, some graph theory concepts are reminded. We observe that a graph Γ can be displayed by a pair of sets V and E as the vertex set and edge set of Γ, respectively. The graph Γ is said to be a simple graph if it has no loop (a vertex adjacent to itself) and multiple edges (having more than one edge between two vertices). Assume that Γ is an undirected simple graph. A vertex x in Γ is called an isolated vertex if there is no edge between x and any other vertices. An empty graph is a graph such that all vertices are isolated vertices. In other words, the edge set is empty. A path P_n of length $n-1$ from vertex x to vertex y in a graph Γ is a sequence of n distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph Γ, then Γ is connected otherwise, it is disconnected. A cycle C_n is a connected graph with n vertices where every vertex has exactly two neighbours. The smallest cycle is C_3 which is triangular. A graph with no triangular as a subgraph is called triangular free. The degree of a vertex x in Γ denoted by $deg(x)$ is the number of adjacent vertices of x. The length of the smallest cycle contained in a graph Γ is called a girth and it is denoted by $gr(\Gamma)$.

The distance between a and b in a graph Γ is the length of the shortest path between a and b. The diameter of a connected graph Γ is the length of the longest shortest path between two distinct vertices of Γ. A complete graph with n vertices denoted by K_n is a graph in which every pair of n distinct vertices is connected. A subset X of the vertex set V is called an independent set if there is no edge between any of two vertices in X. The size of the largest independent set is denoted by $\alpha(\Gamma)$ and is called an independence number. A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets A and B such that every edge has one ends in A and another ends in B. A complete bipartite graph is a bipartite graph such that every vertex a $\in$ A is adjacent to every vertex b $\in$ B. If $|A| = r$ and $|B| = s$, the complete bipartite graph on sets A and B will be denoted by $K_{r,s}$. The complete bipartite graph $K_{1,n}$ is called a star graph. A $k-$partite graph is a graph in which the set of graph vertices is decomposed into k disjoint sets such that no two graph vertices within the same set are adjacent. A complete $k-$partite graph is a $k-$partite graph such that every pair of graph vertices in the k sets are adjacent. If the size of k sets in a $k-$partite graphs are $p, q, \dots, r$, then we may denote it by $K_{p,q,\dots,r}$. A subset D of a vertex set of Γ is called dominating set, if for every vertex x outside of D, there exists at least one vertex y in D such that x adjacent to y. The minimum size of a dominating set is called dominating number, and will be denoted by $\gamma(\Gamma)$. The chromatic number of a graph Γ, written $\chi(\Gamma)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colours. The complement of a graph Γ is denoted by $\bar{\Gamma}$ and has the same vertex set as Γ, where vertices x and y are adjacent in $\bar{\Gamma}$ if and only if they are not adjacent in Γ.

Throughout the paper, we assume that G is always a finite group and all graphs are undirected and simple.

All notations and terminologies about the groups and graphs are standard here and we refer to [3].

In the following lemma, we find a necessary and sufficient condition for adjacency of two arbitrary vertices in $Cay_m(G,S)$.

Lemma 1.2 *[6] Let* $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ *and* $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]^t$ *be two arbitrary vertices of* $Cay_m(G,S)$ *where* x_i *and* y_j *are in* G *for all* $i,j \in \{1,2,\dots,m\}$. *Then* X *and* Y *are adjacent if and only if* x_i *is adjacent to* y_j *in* $Cay(G,S)$ *for all* $i,j \in \{1,2,\dots,m\}$.

In the next lemma, we use some right cosets of S to consider a formula for the degree of any vertex in $Cay_m(G,S)$.

Lemma 1.3 *[6] If* $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ *is a vertex of* $Cay_m(G,S)$. *Then* $deg(X) = |\bigcap_{i=1}^{m} Sx_i|$

It is interesting to see that when $Cay_m(G,S)$ has at least one isolated vertex. The following lemma states a condition of getting isolated vertex in $Cay_m(G,S)$.

Lemma 1.4 *[6] Suppose that $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ is a vertex of $Cay_m(G,S)$. If $d(x_i, x_j) \neq 2$ in $Cay(G,S)$ for some $1 \leq i \neq j \leq m$ and $Cay(G,S)$ is triangle free. Then X is an isolated vertex in $Cay_m(G,S)$.*

In the next lemma, the structure of $Cay_2(G,S)$ is determined when $Cay(G,S) = P_2$.

Lemma 1.5 *[6]Let $Cay(G,S)$ be a Cayley graph. Then*

(i) $Cay(G,S)$ is an empty graph if and only if $Cay_2(G,S)$ is an empty graph.

(ii) If $Cay(G,S) = P_2$, then $Cay_2(G,S) = K_2 \cup \overline{K}_6$.

We know that if S is a generating set, then the Cayley graph is connected. But, it is not true for $Cay_m(G,S), m \geq 1$. In fact, the generalized Cayley graph is not necessarily connected even when S is a generating set. Moreover, the Cayley graph is always regular and the degree of each vertex is $|S|$, but the generalized Cayley graph is not regular. For example, If $Cay(G,S) = K_2$ and S is a generating set. Then $Cay_2(G,S) = K_2 \cup \overline{K_2}$ which has two isolated vertices. Therefore, the generalized Cayley graph, is not regular.

This paper is allocated to discuss the generalized Cayley graph when the usual Cayley graph is complete graph K_n, complete bipartite graph $K_{n,n}$ and complete 3-partite graph $K_{n,n,n}$.

2. Case $Cay(G,S) = K_n$

First, we recall some definitions and lemmas and then we determine the structure of the generalized Cayley graphs $Cay_2(G,S)$ and $Cay_3(G,S)$ when $Cay(G,S) = K_n$.

Definition 2.1 *Let G and H be two graphs. Then the union of G and H denoted by $G \cup H$ is a graph which $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.*

Definition 2.2 *Suppose that G and H be graphs, then the corona product of G and H denoted by $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of $i-th$ copy of H to the $i-th$ vertex of G, where $1 \leq i \leq |V(G)|$.*

The structure of the generalized Cayley graph when the common Cayley graph is K_2 is studied in the following lemma.

Lemma 2.3 *Let $Cay(G,S)$ be a Cayley graph, then*

(i) $Cay(G,S)$ is an empty graph if and only if $Cay_m(G,S)$ is an empty graph.

(ii) If $Cay(G,S) = K_2$, then $Cay_m(G,S) = K_2 \cup \overline{K}_{2(2^{m-1}-1)}$ for all $m \geq 2$.

Proof: (i) It follows from lemma 1.2 directly.

(ii) Let $Cay(G,S) = K_2$. Then $G = \{e, x\}$ and e is adjacent to x, but S is a subset of G and $e \notin S$. Therefore, $S = \{x\}$ and $|V(Cay_m(G,S)| = 2^m$. It is clear that there is just one edge between $[e \quad e \quad \cdots \quad e]^t$ and $[x \quad x \quad \cdots \quad x]^t$. Other vertices are isolated and the number of these vertices is $2^m - 2$. Hence, $Cay_m(G,S) = K_2 \cup \overline{K}_{2(2^{m-1}-1)}$. □

It is interesting to know when the generalized Cayley graph is connected and when it is not connected. In the next lemma, we find the condition for the generalized Cayley graph when the Cayley graph is a complete graph.

Lemma 2.4 *Let $Cay(G,S) = K_n$, where $n \geq 1$.*

(i) If $n > m$, then $Cay_m(G,S)$ is connected.

(ii) If $n \leq m$, then $Cay_m(G,S)$ is not connected.

Proof: (i) Suppose on the contrary that $Cay_m(G,S)$ is not connected and it has at least two components X, Y. So, there is no path between X, Y where $A_1, \cdots, A_r \in X$ and $B_1, \cdots, B_{n^m - r} \in Y$. Thus, A_i is not adjacent to B_j for all $i \in \{1, \cdots, r\}$ and for all $j \in \{1, \cdots, n^m - r\}$. If $A_i = [a_{i1} \quad \cdots \quad a_{im}]^t$ and $B_j = [b_{j1} \quad \cdots \quad b_{jm}]^t$ then there exist $s, t \in \{1,2, \cdots, m\}$ such that a_{is} is not adjacent to b_{jt} which is contradiction Because, our assumption $Cay(G,S) = K_n$ implies that all vertices in $Cay(G,S)$ are adjacent. So, $Cay_m(G,S)$ is connected.

(ii) If $n \leq m$, then it is obvious that $[x_1, x_2, \cdots, x_n, x_1, x_2, \cdots, x_{m-n}]^t$ is an isolated vertex. □

For example, If $Cay(G,S)=K_3$, then $Cay_2(G,S)$ is connected, but $Cay_3(G,S)$ is not connected. (See Figure 1).

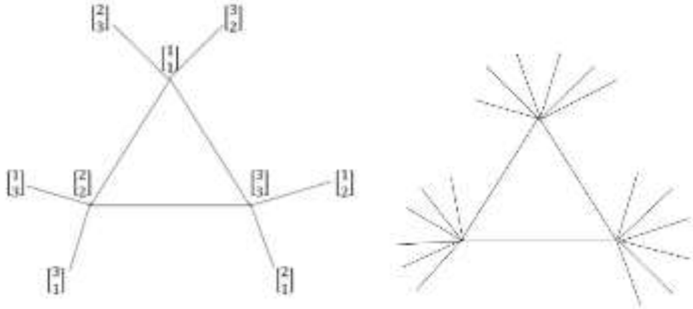

Figure 1- The graph $Cay_2(G,S)$ A component of $Cay_3(G,S)$ of K_3

Lemma 2.5 *Let $Cay(G,S)=K_n$. Then $Cay_m(G,S)$ has induced subgraphs K_n and $K_{r^m,m^m-m,\cdots,m^m-m}$ where $n=mq+r$.*

Proof: By lemma 1.2, it is clear that K_n is a subgraph of $Cay_m(G,S)$. Let $[a_{11} \quad a_{12} \quad \cdots \quad a_{1m}]^t$ be a vertex of $Cay_m(G,S)$ where $a_{11},a_{12},\cdots,a_{1m}$ are distinct and $A_{11}=\{[x_{111} \quad x_{112} \quad \cdots \quad x_{11m}]^t | x_{111},\cdots,x_{11m}\in\{a_{11},\cdots,a_{1m}\}\}$.
So, $|A_{11}|=m^m$. Define $B_{11}=A_{11}-\{[a_{11} \quad a_{11} \quad \cdots \quad a_{11}]^t,\cdots,[a_{1m} \quad a_{1m} \quad \cdots \quad a_{1m}]^t\}$. All vertices in B_{11} are not adjacent and so is an independent set. Now, Suppose $n=mq+r$ and $q=[\frac{n}{m}]$. By continuing this process, we get a vertex $[a_{q1} \quad a_{q2} \quad \cdots \quad a_{qm}]^t$ such that it is distinct with all previous vertices and
$A_{q1}=\left\{[x_{q11} \quad x_{q12} \quad \cdots \quad x_{q1m}]^t | x_{q11},\cdots,x_{q1m}\in\{a_{q1},\cdots,a_{qm}\}\right\}$. Define
$B_{q1}=A_{q1}-\{[a_{q1} \quad a_{q1} \quad \cdots \quad a_{q1}]^t,\cdots,[a_{qm} \quad a_{qm} \quad \cdots \quad a_{qm}]^t\}$. Likewise, the vertices in B_{q1} are not adjacent and they are adjacent to all vertices in $B_{11},B_{21},\cdots,B_{(q-1)1}$. For r vertices in $Cay(G,S)$, we have r^m vertices in $Cay_m(G,S)$ and these vertices are not adjacent to each other but they are adjacent to all vertices in $B_{11},\cdots,B_{q1}$. So, $Cay_m(G,S)$ has a subgraph $K_{r^m,m^m-m,\cdots,m^m-m}$, where $n=m.q+r$. □

In the following proposition, we find the biggest star graph in the generalized Cayley graph when the Cayley graph is a complete graph K_n.

Proposition 2.6 *Suppose that $Cay(G,S)=K_n$. Then $K_{1,(n-1)^m}$ is the biggest star graph of $Cay_m(G,S)$.*

Proof: If $A\in V(Cay_m(G,S))$ such that all entries are identity elements, then A is adjacent to maximum vertices in $Cay_m(G,S)$ and $deg(A)=(n-1)^m$. Hence, we have star graph $K_{1,(n-1)^m}$.
□

The following proposition is a direct consequence of Lemma 2.5.

Proposition 2.7 *If $Cay(G,S)=K_n$, then*

(i) $\chi(Cay_m(G,S))=n$.

(ii) If $n\le m$, then $\alpha(Cay_m(G,S))\ge m^m-m+\frac{n.m!}{(m-n+1)!}$.

(iii) If $n\le m$, then $\gamma(Cay_m(G,S))=m^m-m+\frac{n.m!}{(m-n+1)!}$

3. Case $Cay(G,S)=K_{n,n}$ and $K_{n,n,n}$

In this section, we are going to find the structure of $Cay_m(G,S)$ when $Cay(G,S)=K_{n,n}$ or $K_{n,n,n}$. First, we start the case $m=2$ and $m=3$ as the following:

Lemma 3.1 *If $Cay(G,S)=K_{n,n}$. Then $Cay_2(G,S)=K_{n^2,n^2}\cup\overline{K}_{2n^2}$.*

Proof: Suppose that $V(Cay(G,S)) = \{x_1, x_2, \cdots, x_n\} \cup \{y_1, y_2, \cdots, y_n\} = X \dot{\cup} Y$. Then $V(Cay_2(G,S)) = \{[\begin{smallmatrix} a \\ b \end{smallmatrix}] \mid a, b \in X \cup Y\}$ and $|V(Cay_2(G,S))| = 4n^2$. If $a, b \in X$, then $[\begin{smallmatrix} a \\ b \end{smallmatrix}]$ is adjacent to all vertices $[\begin{smallmatrix} c \\ d \end{smallmatrix}]$, where $c, d \in Y$. Similarly, if $a, b \in Y$, then $[\begin{smallmatrix} a \\ b \end{smallmatrix}]$ will adjacent to all vertices $[\begin{smallmatrix} c \\ d \end{smallmatrix}]$, for entry $c, d \in X$. Thus,the induced subgraph to set $\{[\begin{smallmatrix} a \\ b \end{smallmatrix}] \mid a, b \in X\} \cup \{[\begin{smallmatrix} a \\ b \end{smallmatrix}] \mid a, b \in Y\}$ is a complete bipartite graph K_{n^2,n^2}. The rest of the vertices are all isolated vertices which implies that $Cay_2(G,S) = K_{n^2,n^2} \cup \overline{K}_{2n^2}$ as required. □

Lemma 3.2 *If $Cay(G,S) = K_{n,n}$. Then $Cay_3(G,S) = K_{n^3,n^3} \cup \overline{K}_{6n^3}$.*

Proof: Assume that $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$. So, $V(Cay(G,S)) = X \dot{\cup} Y$, $V(Cay_3(G,S)) = \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in X \cup Y \right\}$ and $|V(Cay_3(G,S))| = 8n^3$. Now, if $a, b, c \in X$, then $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is adjacent to all vertices $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$, where $d, e, f \in Y$. Likewise, if $a, b, c \in Y$, then $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ will adjacent to all vertices $\begin{bmatrix} d \\ e \\ f \end{bmatrix}$, for entry $d, e, f \in X$. Thus, the induced subgraph to set $\left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in X \right\} \cup \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in Y \right\}$ is a complete bipartite graph K_{n^3,n^3}. The rest of vertices are all isolated vertices. Because if $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is adjacent to $\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ then w_i is adjacent to z_j for all $i, j \in \{1,2,3\}$ which is a contradiction. The number of these isolated vertices is $6n^3$. So, $Cay_3(G,S) = K_{n^3,n^3} \cup \overline{K}_{6n^3}$ as required. □

By using the method mentioned in the proof of Lemma 3.1 and Lemma 3.2, we are going to determine $Cay_m(G,S)$ for all $m \geq 2$, when $Cay(G,S) = K_{n,n}$.

Theorem 3.3 *Let $Cay(G,S) = K_{n,n}$. Then for every $m \geq 1$, $Cay_m(G,S) = K_{n^m,n^m} \cup \overline{K}_{2(2^{m-1}-1)n^m}$*

Proof: Let $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ and $V(Cay(G,S)) = X \dot{\cup} Y$. Then we have $V(Cay_m(G,S)) = \{[w_1 \quad w_2 \quad \cdots \quad w_m]^t \ : w_i \in V(Cay(G,S))\ ,\ 1 \leq i \leq m\}$. Now, put
$A = \{[v_1 \quad v_2 \quad \cdots \quad v_m]^t \ : v_i \in X\ ,\ 1 \leq i \leq m\}$ and $B = \{[w_1 \quad w_2 \quad \cdots \quad w_m]^t : w_i \in Y\ ,\ 1 \leq i \leq m\}$. Then every vertex in A is adjacent to every vertex in B and so the induced subgraph to the set $A \cup B$ is a complete bipartite K_{n^m,n^m}. Moreover, every vertex in $V(Cay_m(G,S)) \setminus (A \cup B)$ is an isolated vertex. Because, if $[w_1 \quad w_2 \quad \cdots \quad w_m]^t$ is one of such vertex, then there exists $1 \leq i \neq j \leq m$ such that $w_i \in X$ and $w_j \in Y$. Now, if $[w_1 \quad w_2 \quad \cdots \quad w_m]^t$ is adjacent to vertex $[z_1 \quad z_2 \quad \cdots \quad z_m]^t$, then we will have $w_i - z_1 - w_j$ which is not possible. If $z_1 \in X$, then w_i can not be adjacent to z_1 and if $z_1 \in Y$, then w_j can not be adjacent to z_1. Hence, all such vertices are isolated vertices. the number of such vertices is
$|V(Cay_m(G,S)| - |A| - |B| = (2n)^m - n^m - n^m = 2^m n^m - 2n^m = (2^m - 2)n^m = 2(2^{m-1} - 1)n^m$. Thus, $Cay_m(G,S) = K_{n^m,n^m} \cup \overline{K}_{2(2^{m-1}-1)n^m}$. □

From Theorem 3.3, we can state immediately the following proposition.

Proposition 3.4 *Let $Cay(G,S) = K_{n,n}$. Then*

(i) $\chi(Cay_m(G,S)) = 2$.
(ii) $\alpha(Cay_m(G,S)) = n^m(2^m - 1)$.
(iii) $\gamma(Cay_m(G,S)) = 2 + 2(2^{m-1} - 1)n^m$.

Now, we start with the case $Cay(G,S) = K_{n,n,n}$. Suppose that $Cay(G,S) = K_{n,n,n}$, where $n \geq 2$. In the following two lemmas, we determine $Cay_2(G,S)$ and $Cay_3(G,S)$ whenever $Cay(G,S) = K_{n,n,n}$.

Lemma 3.5 *Let $Cay(G,S) = K_{n,n,n}$. Then $Cay_2(G,S) = K_{n^2,n^2,n^2} \cup 3K_{(2n)^2,n^2}$.*

Proof: Suppose that $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_n\}$ and $Z = \{z_1, z_2, \cdots, z_n\}$ such that $V(Cay(G,S)) = X \cup Y \cup Z$. So, $V(Cay_2(G,S) = \{[w_1 \quad w_2]^t : w_1, w_2 \in V(Cay(G,S))\}$ and $|V(Cay_2(G,S)| = 9n^2$. Now, put $A_1 = \{[a_1 \quad a_2]^t : a_1, a_2 \in X\}$, $A_2 = \{[b_1 \quad b_2]^t : b_1, b_2 \in Y\}$, $A_3 = \{[c_1 \quad c_2]^t : c_1, c_2 \in Z\}$. Every vertex in A_1 is adjacent to every vertex in A_2 and A_3. So, the induced subgraph to the set $A_1 \cup A_2 \cup A_3$ is the complete 3-partite graph K_{n^2,n^2,n^2}. Now define
$D_{12} = \{[d_1 \quad d_2]^t : d_1, d_2 \in X \cup Y\} - (A_1 \cup A_2)$,
$D_{13} = \{[d_1 \quad d_2]^t : d_1, d_2 \in X \cup Z\} - (A_1 \cup A_3)$,
$D_{23} = \{[d_1 \quad d_2]^t : d_1, d_2 \in Y \cup Z\} - (A_2 \cup A_3)$ and
$E = V(Cay_2(G,S)) - (A_1 \cup A_2 \cup A_3 \cup D_{12} \cup D_{13} \cup D_{23})$. The induced subgraph to the set D_{12} ,D_{13} and D_{23} is complete bipartite $K_{(2n)^2,n^2}$. Moreover, the set E is an empty set. Since the number of the vertices in E is $|E| = (3n)^2 - 3(2n)^2 - 3n^2 = 0$. Therefore,the graph $Cay_2(G,S)$ is connected and $Cay_2(G,S) = K_{n^2,n^2,n^2} \cup 3K_{(2n)^2,n^2}$. □

Lemma 3.6 *Let $Cay(G,S) = K_{n,n,n}$. Then $Cay_3(G,S) = K_{(2n)^3,(2n)^3,(2n)^3} \cup K_{(2n)^3,n^3} \cup \overline{K}_{6n^3}$*

Proof: Suppose that $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_n\}$ and $Z = \{z_1, z_2, \cdots, z_n\}$ such that $V(Cay(G,S)) = X \cup Y \cup Z$. So,
$V(Cay_3(G,S) = \{[w_1 \quad w_2 \quad w_3]^t : w_1, w_2, w_3 \in V(Cay(G,S))\}$ and $|V(Cay_3(G,S)| = 27n^3$.
Now, put $A = \{[a_1 \quad a_2 \quad a_3]^t : a_1, a_2, a_3 \in X\}$, $B = \{[b_1 \quad b_2 \quad b_3]^t : b_1, b_2, b_3 \in Y\}$,
$C = \{[c_1 \quad c_2 \quad c_3]^t : c_1, c_2, c_3 \in Z\}$,
$D_1 = \{[d_1 \quad d_2 \quad d_3]^t : d_1, d_2, d_3 \in X \cup Y\} - (A \cup B)$,
$D_2 = \{[d_1 \quad d_2 \quad d_3]^t : d_1, d_2, d_3 \in X \cup Z\} - (A \cup C)$,
$D_3 = \{[d_1 \quad d_2 \quad d_3]^t : d_1, d_2, d_3 \in Y \cup Z\} - (B \cup C)$ and
$E = V(Cay_3(G,S)) - (A \cup B \cup C \cup D_1 \cup D_2 \cup D_3)$. Then every vertex in A is adjacent to every vertex in B and C and so the induced subgraph to the set $A \cup B \cup C$ is complete 3-partite K_{n^3,n^3,n^3} and the induced subgraph to the set D_1 ,D_2 and D_3 is complete bipartite $K_{(2n)^3,n^3}$. Moreover, every vertex in the set E is an isolated vertex. The number of such vertices is $|E| = (3n)^3 - 3(2n)^3 - 3((2n)^3 - 2n^3) = 6n^3$. Therefore, $Cay_3(G,S) = K_{(2n)^3,(2n)^3,(2n)^3} \cup 3K_{(2n)^3,n^3} \cup \overline{K}_{6n^3}$. □

The following theorem is the general case for $m \geq 2$.

Theorem 3.7 *Let $Cay(G,S) = K_{n,n,n}$. Then*
$Cay_m(G,S) = K_{n^m,n^m,n^m} \cup 3K_{(2n)^m,n^m} \cup K_{(3n)^m - 3n^m - 3((2n)^m - 2n^m)}$

Proof: Suppose that $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_n\}$ and $Z = \{z_1, z_2, \cdots, z_n\}$ such that $V\big(Cay(G,S)\big) = X \cup Y \cup Z$. So,
$V(Cay_3(G,S) = \big\{[w_1 \quad w_2 \quad \cdots \quad w_m]^t \quad : w_1, w_2, \cdots, w_m \in V\big(Cay(G,S)\big)\big\}$
and $|V(Cay_m(G,S) = (3n)^m$. Now, put $A = \{[a_1 \quad a_2 \quad \cdots \quad a_m]^t \quad : a_i \in X \ , \ 1 \leq i \leq m\}$,
$B = \{[b_1 \quad b_2 \quad \cdots \quad b_m]^t : b_i \in Y \ , \ 1 \leq i \leq m\}$,
$C = \{[c_1 \quad c_2 \quad \cdots \quad c_m]^t : c_i \in Z \ , \ 1 \leq i \leq m\}$,
$D_1 = \{[d_1 \quad d_2 \quad \cdots \quad d_m]^t \quad : d_i \in X \cup Y \ , \ 1 \leq i \leq m\} - (A \cup B)$,

$D_2 = \{[d_1 \quad d_2 \quad \cdots \quad d_m]^t \ : d_i \in X \cup Z \ , \ 1 \le i \le m\} - (A \cup C)$,
$D_3 = \{[d_1 \quad d_2 \quad \cdots \quad d_m]^t \ : d_i \in Y \cup Z \ , \ 1 \le i \le m\} - (B \cup C)$ and
$E = V(Cay_m(G,S)) - (A \cup B \cup C \cup D_1 \cup D_2 \cup D_3)$. Then every vertex in A is adjacent to every vertex in B and C and so the induced subgraph to the set $A \cup B \cup C$ is complete 3-partite K_{n^m,n^m,n^m} and the induced subgraph to the set D_1 ,D_2 and D_3 is complete bipartite $K_{(2n)^m,n^m}$. Moreover, every vertex in the set E is an isolated vertex. The number of such vertices is $|E| = (3n)^m - 3n^m - 3((2n)^m - 2n^m)$. Therefore, $Cay_m(G,S) = K_{n^m,n^m,n^m} \cup 3K_{(2n)^m,n^m} \cup \overline{K}_{|E|}$ and the proof of theorem is completed. □

Finally, we end the paper with the following problem.

Problem: Find $Cay_m(G,S)$ when $Cay(G,S)$ is complete r-partite graph for all $r \ge 1$.

4-Conclusions:

The aims of this paper is to introduce a generalization of the Cayley graph denoted by $Cay_m(G,S)$. Some basic properties of the new graph are given and investigated. Furthermore, the structure of $Cay_m(G,S)$ when $Cay(G,S)$ is complete graph K_n, complete bipartite graph $K_{n,n}$ and complete 3-partite graph $K_{n,n,n}$ for every $m \ge 2$ has been also assigned. Many important results have been also obtained and provided in this work.

Reference

[1] Al-Kaseabeh, S. and Erfanian, A., The structure of Cayley graphs of dihedral groups of valencies 1,2 and 3. *Journal of Proycciones* Vol. 40 No. 6. (2021).

[2] Cayley, P., Desiderata and suggestions: No. 2. the theory of groups: graphical representation. *American Journal of Mathematics 1*, Vol. 2, 174–176(1878).

[3] Godsil, C. and Royle, G. F., *Algebraic Graph Theory*, Vol. 207. Springer Science & Business Media, (2001).

[4] Hammack, R., Imrich, W., and Klavžar, S., *Handbook of Product Graphs*. CRC press, (2011).

[5] Marušič, D., Scapellato, R., and Salvi, N. Z., Generalized Cayley graphs. *Discrete Mathematics* Vol.102, No. 3, 279–285(1992).

[6] Mohammadi, S., Some generalizations of Cayley graph and Intersection graph, Ph.D Thesis, Ferdowsi University of Mashhad, Mashhad, Iran (March 2020).

[7] Zhu, Y. Generalized Cayley graphs of semigroups I. In *Semigroup Forum*, Vol. 84, 131–143(2012).