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Resultant Graphs of Block Designs

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Abstract
In this paper we have made different regular graphs by using block designs. In one of our applicable methods, first we have changed symmetric block designs into new block designs by using a method called a union method. Then we have made various regular graphs from each of them. For symmetric block designs with $\lambda = 1$ (which is named finite projective geometry), this method leads to infinite class of regular graphs. With some examples we will show that these graphs can be strongly regular or semi-strongly regular. We have also propounded this conjecture that if two semi-symmetric block designs are non-isomorphic, then the resultant block graphs of them are non-isomorphic, too.

Keywords: Block Designs, Regular Graph, Isomorphism of Graphs.

الرسوم البيانية الناتجة عن التصاميم الكتلية

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الخلاصة
قمنا في هذه المقالة بإنشاء رسوم بيانية منتظمة ومتنوعة بالاستعانة بالتصاميم الكتلية. في إحدى الطرق التي استخدمناها، قمنا أولاً بتحويل التصاميم الكتلية المتناظرة إلى تصاميم كتلية جديدة باستخدام طريقة تسمى طريقة الاجتماع، ثم قمنا بإنشاء العديد من الرسوم البيانية المنتظمة من كل منها. تؤدي هذه الطريقة إلى إنشاء مجموعة غير منتهية من الرسوم البيانية المنتظمة للتصاميم الكتلية المتناظرة مع $\lambda = 1$ (والتي تسمى الهندسة الإسقاطية المتناهية). سنوضح بالأمثلة أن هذه الرسوم البيانية يمكن أن تكون منتظمة بشكل فائق أو شبه فائق. سنقدم أيضاً تخميناً قائماً على أنه إذا كان لدينا تصميمان كتليان شبه متناظران غير متماثلي الشكل، فإن الرسوم البيانية الكتلية الناتجة عنهما ستكون غير متماثلة الشكل أيضاً.

1. Introduction
The graphs can be defined and constructed with different ways. In some of these ways it is got help used by combinatorial designs like Latin squares, Hadamard matrices, block designs, etc. We know that the resultant block graph of a pseudo-symmetric block design is a strongly regular graph. Now, this question naturally arises that what kind of graphs can be made by using other block designs? In this paper, we construct non-symmetric block designs with the help of symmetric block designs and then with this new block designs create some graphs. We will also consider some of the features of created graphs. Especially we examine the regularity of these graphs .

2. Definitions and Preliminaries

2.1. Block Designs
Definition 1.[1] A Partially Balanced Incomplete Block Design (PBIBD) is the partially ordered (X.B) such that X is a set of v elements and B is a family of subsets of X (which each of these subsets

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is called a block.) such that each two distinct members of X like y and z are appeared in exactly λ blocks (λϵN). In the case that each block has k elements, we call the above design a block design or 2-design and if the mention of parameters is necessary, we call it 2-(v.k.λ). We represent the number of blocks of B by b and the number of blocks containing an arbitrary element x of X by r. (It is easy to see that r is independent of x.)

Theorem 1 [1] The parameters v, b, r, k and λ satisfy the following relations:

$$\lambda(v-1)=r(k-1).bk=rv \qquad (1)$$

Definition 2 If necessary we show a 2-design with all its parameters as $2-(v.b.r.k.\lambda)$. If $v=b$, then the design is called *symmetric* design.

Example 1. Suppose that $X=\{1.2.\cdots.7\}$ and B is the following family of subsets of X.

$B_1=\{1.2.4\}.B_2=\{2.3.5\}.B_3=\{3.4.6\}.B_4=\{4.5.7\}.B_5=\{5.6.1\}.B_6=\{6.7.2\}.B_7=\{7.1.3\}.$

In this case $(X.B)$ forms a block design with parameters $v=7.k=3$ and $\lambda=1$. Since $b=v=7$, by definition this design is symmetric.

A block design is usually shown with its arrays such that each column represents a block. For example the block design of the previous example is displayed as below:

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

Hereafter, we will also follow this approach.

Example 2 In the below, there is an instance of $2-(9.3.1)$ −design:

1	2	3	4	5	7	7	8	9	6	6	1
2	3	4	5	8	9	8	9	6	1	2	3
4	5	6	7	1	2	3	4	5	7	8	9

In this design $v=9$ and $b=12$. Hence according to definition this design is asymmetric.

The symmetric block designs have some important properties. One of these features is the basis of an important definition.

Theorem 2 [1] In a symmetric block design $(v.k.\lambda)$ −design each two distinct blocks have exactly λ common members.

Similar to before, the above theorem has been proved in standard books like.

Definition 3.[1] A block design $(v.k.\lambda)$ −design is named *pseudo-symmetric*, if the intersection of each two distinct blocks is exactly one of two distinct values *x* and *y*.

An important category of strongly regular graphs is constructed using pseudo-symmetric block designs.

Definition 4 [1] Assume that *D* is a pseudo-symmetric block design $(v.k.\lambda)$ −design and the intersection of every two distinct blocks is one of two distinct values *x* and *y* such that $x>y$. Now, we construct a graph *G* as follows:

Set the vertices of G with the blocks of mentioned block design and two vertices are adjacent if and only if the intersection of them has exactly *x* members.

We call *G* the *block graph* of *D*.

In the following, we will express a theorem without its proof. The proof has come in [1].

Theorem 3 If *D* is a pseudo-symmetric block design $(v.b.r.k.\lambda)$ −design, then the block graph of *D* will be strongly regular.

Example 3 Consider a $2-(10.4.2)$ −design with the following blocks:

1	1	1	1	1	1	2	2	2	2	3	3	3	4	4
2	2	3	4	5	6	3	4	5	6	4	5	7	5	8
3	7	6	5	8	7	5	6	7	8	7	6	9	6	9
4	8	10	9	10	9	9	10	10	9	8	8	10	7	10

Observe that every two distinct blocks have one or two common elements. So the block graph of this design is a strongly regular graph.

$$(n=15.k=8.\lambda=4.\mu=4)$$

2.2 The graphs generated by other block designs

If *D* is not a pseudo-symmetric block design $(v.b.r.k.\lambda)$ −design, then for two distinct blocks the number of common elements can be various. Before proceeding to the discussion, we look at two notable examples.

1	2	3	4	5	6	7	8	9	10	11	12	13
2	3	4	5	6	7	8	9	10	11	12	13	1
4	5	6	7	8	9	10	11	12	13	1	2	3
10	11	12	13	1	2	3	4	5	6	7	8	9

1	2	3	4	5	6	7	8	9	10	11	12	13
3	4	5	6	7	5	9	10	11	12	13	1	2
6	7	8	9	10	11	12	13	1	2	3	4	5
7	8	9	10	11	12	13	1	2	3	4	5	6

In this design $v = 13$ and $b = 26$. So by definition this design is not symmetric. Furthermore the number of common members between two distinct blocks is one of the numbers zero or 1 or 2. Hence this design is not pseudo-symmetric, too. Now, consider the graph G_1 with vertices$B_1. \cdots . B_{26}$. Two distinct vertices are adjacent if and only if the intersection of them is empty. G_1 is a 3-regular graph which every two distinct vertices don't have any common neighbor. But the number of common neighbors of two non-adjacent distinct vertices of G_1 is not a constant number. Hence G_1 is a semi-strongly regular graph with $n = 26. k = 3$ and $\lambda = 0$.

By using this design, consider graph $G_2$with the same vertices. Two distinct vertices are adjacent if and only if the intersection of them has zero or two elements. The resulting graph is a 9-regular graph that every two distinct vertices have no common neighbor. But the number of common neighbors of two non-adjacent distinct vertices of G_2 is not a constant number. Hence G_2 is a semi-strongly regular graph with $n = 26. k = 9$ and $\lambda = 0$.

Reuse this design and consider graph $G_3$with the same vertices. In G_3 two distinct vertices are adjacent if and only if the intersection of them has exactly two elements. The resulting graph is a 6-regular graph that every two distinct vertices have no common neighbor. But similar to previous, the number of common neighbors of two non-adjacent distinct vertices of G_3 is not a constant number. Hence G_3 is a semi-strongly regular graph with $n = 26. k = 6$ and $\lambda = 0$.

In recent example we saw that sometimes it is possible to make different graphs with only one block design.

Next example has been extracted from reference [2].

Example 5 There exists a $2 - (21.7.12)$ block design with 120 blocks. Assume that *G* is a graph which its vertices are the blocks of this design. Two distinct vertices are adjacent if and only if the intersection of them has exactly three elements. This graph is strongly regular with parameters $n = 120. k = 77. \lambda = 52$ and $\mu = 44$.

3. Fundamental Theorems

In this section we will offer a new method for constructing regular graphs by using block designs and show that for some specific parameters, these graphs are strongly regular or semi-strongly regular.

$$\{B_i \cup B_j \big| 1 \leq i < j \leq v\}$$

$$k' = 2k - \lambda. \qquad \lambda' = \lambda v - \frac{\lambda(\lambda+1)}{2} + (k-\lambda)^2$$

Suppose that D is a symmetric block design $2 - (v.k.\lambda)$ −design. In 1946 Elisabeth Morgan proved that the block set $\{B_i \cup B_j \big| 1 \leq i < j \leq v\}$, forms a block design $(v.k'.\lambda')$ −design [2], subject to

$$k' = 2k - \lambda. \ \lambda' = \lambda v - \frac{\lambda(\lambda+1)}{2} + (k-\lambda)^2 \qquad (2)$$

(Here $B_1. \cdots . B_v$ are the blocks of D.)

In 1992 Mahmoudian and Shirdareh established that if $v \geq 2k$, then resulting block design is simple [3]. (It means the design has no repeated block.) The other parameters of this new block design can simply be obtained, too. In the following, we will name this method *union method* by conforming [3].

In the following, we will attend to condition that $\lambda = 1$. Let D be a symmetric block design $2 - (v.k.1)$ −design. We represent a block design which is obtained from D with union method by D'. Firstly we will ask this question that for each two distinct blocks in D' how many common members can be seen. In this context we will prove the following lemma.

Lemma 1 If D is a symmetric block design $2-(v.k.1)$ −design and D' is a block design obtained from D by union method, then the number of common members between two distinct blocks of D' is one of the numbers $1.2.3.4.k$ and $k+1$.
Proof- We consider two cases.
Case 1: The two blocks of D' are in the forms $A\cup B$ and $A\cup C$. Hence we have

$$|(A\cup B)\cap(A\cup C)| = |A\cup(B\cap C)| = |A| + |B\cap C| - |A\cap B\cap C| \\ = k+1-|A\cap B\cap C| \tag{3}$$

Since $|A\cap B\cap C| = 0$ or 1,

$$|(A\cup B)\cap(A\cup C)| = k$$

or

$$|(A\cup B)\cap(A\cup C)| = k+1.$$

Case 2: The two blocks of D' are in the forms $A\cup B$ and $C\cup D$. Hence we have

$$|(A\cup B)\cap(C\cup D)| = |(A\cap C)\cup(A\cap D)\cup(B\cap C)\cup(B\cap D)|$$

Each of sets $A\cap C$, $A\cap D$, $B\cap C$ and $B\cap D$ has exactly one member. Thus

$$|(A\cup B)\cap(C\cup D)| = |(A\cap C)\cup(A\cap D)\cup(B\cap C)\cup(B\cap D)|$$

is one of the values 1,2,3 and 4.
Lemma 2 Let D be a symmetric block design $2-(v.k.1)$ −design and D' be a block design obtained from D by union method. Let $k\geq 5$. If $A\cup B$ is a block of D',then the number of blocks of D' that have exactly k common members with $A\cup B$ is equal to $2(v-k)$ and the number of blocks of D' that have exactly $k+1$ common members with $A\cup B$ is equal to $2(k-2)$.
Proof- If $A\cup B$ is a block of D', then the blocks of D'that have k or $k+1$ common members with $A\cup B$ should be in the form $A\cup B$ or $C\cup B$ where *A*, *B* and *C* are distinct blocks of *D*. According to (3) if $|A\cap B\cap C| = 1$, then $A\cup B$ and $A\cup C$ have exactly k common members and if $|A\cap B\cap C| = 0$, then $A\cup B$ and $A\cup C$ have exactly $k+1$ common members. Furthermore (since D is a symmetric design,) $A\cup B$ has exactly one element that we call it *x*. The necessary and sufficient condition for $|A\cap B\cap C| = 0$ is that $x\notin C$. The number of blocks of *D* containing *x* is equal to *k*.

So the number of blocks that do not contain *x* is equal to $v-k$. Therefore if $A\cup B$ is a fix block of D', then the number of blocks like $A\cup C$ which have exactly $k+1$ common members with $A\cap B$ is equal to $v-k$ and there exists exactly the same number of blocks with form $C\cap B$ which have $k+1$ common members with $A\cap B$. Thus the number of blocks of D' which have k common members with $A\cup B$ is equal to $2(v-k)$. Similarly it is true for the number of blocks of D including *x*. Note that two of them are *A* and *B*. Hence there are $k-2$ blocks like $A\cup C$ and $k-2$ blocks like $A\cup C$ and finally $k-2$ blocks like $C\cup B$ such that have exactly $k+1$ common members with $A\cup B$. Thus the number of blocks of D' which have exactly $k+1$ common members with $A\cup B$ is equal to $2(k-2)$.
Theorems 4 and 5 are the main theorems of this section and the immediate result from the last lemma.
Theorem 4 Let D be a symmetric block design $2-(v.k.1)$ −design with $k\geq 5$ and D' be a block design generated by D from union method. We define graph G_1 with blocks of D' as its vertex set and two vertices are adjacent if and only if they have exactly $k+1$ common members. G_1 is a $2(k-2)$ −regular graph.
Theorem 5 Let D be a symmetric block design $2-(v.k.1)$ −design with $k\geq 5$ and D' be a block design generated by D from union method. We define graph G_2 with blocks of D' as its vertex set and two vertices are adjacent if and only if they have exactly $k+1$ common members. G_2 is a $2(v-k)$ −regular graph.
of the two recent theorems when *n* is the power of a prime number, important results can be obtained which will be expressed and proved in the following two theorems.
Theorem 6 If n is a power of a prime number, then there exists a $2(n-1)$ −regular graph with $\frac{(n^2+n+1)(n^2+n+2)}{2}$ vertices.
Proof- Notice that if n is a power of a prime number, then there exists a $2-(n^2+n+1.n+1.1)$ −design (finite projective geometry) [1].
Now, by theorem 4 the desirable result is obtained.

Theorem 7 If n is a power of a prime number, then there exists a $2n^2$ −regular graph with $\frac{(n^2+n+1)(n^2+n+2)}{2}$ vertices.

Proof- Similar to previous, if n is a power of a prime number, then there exists a $2-(n^2+n+1.n+1.1)$ −design (finite projective geometry) [1].

Now, the desirable result eventuates by theorem 5.

Example 6 Remark a symmetric block design $2-(21.5.1)$ −design with the following blocks:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	1	2	3
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	1	2	3	4
10	11	12	13	14	15	16	17	18	19	20	21	1	2	3	4	5	6	7	8	9
12	13	14	15	16	17	18	19	20	21	1	2	3	4	5	6	7	8	9	10	11

The block design D' which is obtained from D by union method is a $2-(21.9.36)$ −design with 210 blocks. Define graph G_1 with blocks of D' as its vertex set and two vertices are adjacent if and only if they have exactly 5 common members. G_1 is a 6 −regular graph. By using MAPLE software it is illustrated that this graph is a disconnected graph with 21 components where each component is the complement of Petersen graph. (So this graph is in fact a semi-strongly regular graph.)

Example 7 Like the previous example we define G_2 with blocks of D' as its vertex set and two vertices are adjacent if and only if the intersection of them is exactly 6. G_2 is a 32-regular graph. Again, by using MAPLE software it is turned out that this graph is a semi-strongly regular graph with $\lambda = 13$.

The two previous examples are the same as those that have described in two theorems 4 and 5. But by means of the same block design, the other graphs can be created, too.

Example 8. Define G_3 with vertex set blocks of D'. This time two vertices are adjacent if and only if their intersection is a single-member set. G_3 is a 3-regular graph. MAPLE software shows that this graph is a union of 21 distinct copies of Petersen graph. Therefore this graph is semi-strongly regular with $\lambda = 0$.

Example 9 Define G_4 with vertex set blocks of D' and also adjacency occurs if and only if their intersection is a three-member set. G_4 is a 96-regular graph. By using MAPLE software it is obvious that this graph is a semi-strongly regular graph with $\lambda = 36$.

Example 10 Define G_5 with vertex set blocks of D' and the adjacency occurs if and only if their intersection is a four-member set. G_5 is a 72-regular graph. By help of MAPLE software it is shown that this graph is a semi-strongly regular graph with $\lambda = 30$.

4. Isomorphism

For some parameters there is more than one non-isomorphic block design. Thus this question arises whether the resulting graphs of these designs are also non-isomorphic.

First of all, we will explain our meaning about block design isomorphism.

Definition 5 Two block designs D and D' are isomorphic if there exists a permutation σ on the elements of design such that for each block of D like B, $\sigma(B)$ is a block of D'. Obviously, if two block designs are isomorphic, then the graphs obtained from them will be isomorphic, too. But is the reverse true?

Example 11 In reference [4], it has been introduced two block designs $2-(13.3.1)$ −design where are not isomorphic. Each of them has 26 blocks. In each design the intersection of every two blocks is at most single-member. Hence the block graph of both designs is strongly regular. The calculation shows that in these graphs $n = 26. k = 10. \lambda = 3. \mu = 4$. Calculating with MAPLE also points that these two graphs are non-isomorphic.

Example 12 In reference [4], it has been mentioned eighty block designs $2-(15.3.1)$ −design where are not isomorphic. Each of them has 35 blocks. In each of these designs the intersection of every two blocks is at most 1. Therefore the block graph of both designs is strongly regular. The calculation shows that in these graphs $n = 35. k = 16. \lambda = 6. \mu = 8$. As the instances, with the help of MAPLE we have constructed some of these graphs and verified that they are not isomorphic.

Various calculations using block designs with the same parameters offers us the following conjectures:

Conjecture 1 If two semi-symmetric block designs are non-isomorphic, then the resultant block graphs of them are also non-isomorphic.

Conjecture 2 If two symmetric block designs are non-isomorphic, then the non-symmetric block designs generated by union method and therefore resultant regular graphs of them are also non-isomorphic.

Conclusions and other problems

The properties of regular graphs are one of the most important issues in graph theory. In the present article we have dealt with several methods of constructing these graphs using block designs. We have shown that infinite classes of regular graphs can be constructed using specific block designs. Two conjectures are also discussed about isomorphism of these graphs.

The following problems can also be raised.

Problem 1 Under what conditions the graphs obtained in theorems 4 and 5 are strongly or semi-strongly regular?

Problem 2 How can the properties of block designs be transferred to resultant graphs of them?

Problem 3 Which results can be earn from methods of this paper and also Cartesian product of graphs and line graphs?

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