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The Effect of Mutual Interaction and Harvesting on Food Chain Model

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Abstract
This paper treats the interactions among four population species. The system includes one mutuality prey, one harvested prey and two predators. The four species interaction can be described as a food chain, where the first prey helps the second harvested prey. The first and the second predator attack the first and the second prey, respectively, according to Lotka-Volterra type functional responses. The model is formulated using differential equations. One equilibrium point of the model is found and analysed to reveal a threshold that will allow the coexistence of all species. All other equilibrium points of the system are located, with their local and global stability being assessed. To back up the conclusions of the mathematical analysis, a numerical simulation examination of the model is carried out. The system's coexistence can be achieved as long as the harvesting rate of the prey population is lower than its intrinsic growth rate.

Keywords: Food chain model, Prey-predator model, Mutual interaction, Harvesting, Stability.

تأثير التفاعل المتبادل والحصاد على نموذج السلسلة الغذائية

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الخلاصة
يهدف هذا البحث الى دراسة التفاعلات بين أربعة أنواع من السكان. يشتمل النظام على فريسة متبادلة واحدة وفريسة محصودة واحدة واثنان من الحيوانات المفترسة. يمكن وصف تفاعل الأنواع الأربعة بأنه سلسلة غذائية. تم العثور على نقطة التوازن للنموذج وتحليلها لتكشف عن عتبة تسمح بالتعايش بين جميع الأنواع. تم حساب جميع نقاط التوازن للنظام وتقييم استقرارها المحلي والشامل. ولدعم استنتاجات التحليل الرياضي ، تم إجراء فحص محاكاة رقمي للنموذج. لوحظ بأنه يمكن تحقيق تعايش النظام طالما أن معدل الحصاد على الفريسة أقل من معدل النمو الداخلي.

1. Introduction

Ecosystems are the result of interactions between the environment and communities. In an ecosystem, a food chain plays a vital role in guaranteeing the stability of the populations [1]. The best method to understand the dynamics and behaviour of ecological interactions between prey and predator populations is to utilise a mathematical model. A simple model of prey-predator interactions was separately proposed by Lotka and Volterra, but the model is now known as the Lotka – Volterra model [2,3]. In

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[4], the first simple mathematical model of two prey and one predator has been investigated and analysed in terms of predicting their dynamics. Subsequently, some researchers have studied numerous properties, such as coexistence, persistence, stability and extinction [5- 8]. In [9], it has been used Holling type-I functional response in a system consisting of two prey-one predators. The authors [10,11] explored the difficulties in the dynamic behaviour of two prey-one predator systems following the Holling type II functional response with an influence impulsive. In [12], it was analysed the local and global stability of the prey-predator model including Holling type I functional response and the implications of group help. Further, in [13], it has been considered the interaction between two mutualistic prey and a predator population. In addition, the proportional harvesting function is taken into account in his model when these species interact. The stability of his model has been established for the positive equilibrium point.

In this paper, we consider the interaction among four populations: two prey and two predators. The first prey is assumed to help the second, whilst the latter is harvested. The first predator can attack the first prey, while the second predator (top predator) can only attack the first predator, according to the type I functional response.

The rest of this paper is orginsed as follows: In section two, the existence of the equilibrium points for the proposed model has been investigated. In section three, the stability of the possible equilibrium point has been analysed. Finally, in the last section, some numerical analyses have been provided to confirm our analytical result.

1. Assumptions of the Model

Suppose a food chain contains the following species: prey, a predator and a top predator, with the mathematics beings based on the following assumptions. $n_1(t)$ is the density of the first prey (the first species in the food chain), $n_2(t)$ is the density of the second harvested prey, which has a positive effect on the first prey, whilst $n_3(t)$ and $n_4(t)$ are the densities of the predator and top predator species, respectively.

Under the above assumptions, the model can be presented by the following system of differential equations:

$$\frac{dn_1}{dt} = rn_1\left(1-\frac{n_1}{k}\right) - \beta_1 n_1 n_3 + a n_1 n_2 = n_1 f_1(n_1,n_2,n_3,n_4),$$
$$\frac{dn_2}{dt} = sn_2\left(1-\frac{n_2}{l}\right) - qEn_2 = n_2 f_2(n_1,n_2,n_3,n_4),$$
$$\frac{dn_3}{dt} = \beta_2 n_1 n_3 - \beta_0 n_3 - \gamma_1 n_3 n_4 = n_3 f_3(n_1,n_2,n_3,n_4),$$
$$\frac{dn_4}{dt} = \gamma_2 n_3 n_4 - \alpha n_4 = n_4 f_4(n_1,n_2,n_3,n_4).$$

Here, model (1) has been analysed with the initial conditions $n_1(0) \geq 0$, $n_2(0) \geq 0$, $n_3(0) \geq 0$ and $n_4(0) \geq 0$. $p(n_1) = \beta_1 n_1$ and $q(n_3) = \gamma_1 n_3$ are the Lotka-Volterra type of functional responses. All parameters of the system (1) are assumed to be positive and described as:

The parameters k and l are the carrying capacities of the first and second prey, respectively, with intrinsic growth rates *r* and *s*; a is the positive effect on the first prey by the second prey; $E$and q are the effort and the catchability rate applied on the second prey, i.e., qE represents the harvesting rate of the second prey; β_1 and $\gamma_1$are the attack rate coefficient of the first prey and first predator species due to the first predator and top predator, respectively; β_0 and α represent the first and the second predator's natural death rate, respectively.

Apparently, the functions on the right-hand side of system (1) are continuously differentiable functions on $R_+^4 = \{(n_1,n_2,n_3,n_4), n_1 \geq 0, n_2 \geq 0, n_3 \geq 0, n_4 \geq 0\}$. Therefore, there exists a unique solution for system (1). And hence, they are Lipschitzian. Further, all solutions of the system (1) with any non-negative initial conditions are bounded, as shown in the following section.

Theorem (1) All solutions $n_1(t), n_2(t), n_3(t)$ and $n_4(t)$ of the system (1) with the initial conditions $(n_1,\ n_2,\ n_3,\ n_4)$ are uniformly bounded if the following conditions ($\beta_1 \geq \beta_2$) and ($\gamma_1 \geq \gamma_2$) hold.

Proof: - Let $\left(n_1(t), n_2(t), n_3(t), n_4(t)\right)$ be an arbitrary solution of the system (1) with a non-negative initial condition. Then for $H(t) = n_1(t) + n_2(t) + n_3(t) + n_4(t)$, we have

$$\frac{dH}{dt} = \frac{dn_1}{dt} + \frac{dn_2}{dt} + \frac{dn_3}{dt} + \frac{dn_4}{dt}$$

$$\frac{dH}{dt} = rn_1 - \frac{rn_1^2}{k} - (\beta_1 - \beta_2)n_1n_3 + an_1n_2 + Sn_2 - \frac{sn_2^2}{l} - qEn_2 - \beta_0 n_3 - (\gamma_1 - \gamma_2)\, n_3n_4 - \alpha n_4 .$$

Hence, according to the assumptions of the theorem, the following is obtained:

$$\frac{dH}{dt} \leq rn_1 - \frac{rn_1^2}{k} + an_1n_2 + sn_2 - \frac{sn_2^2}{l} - qEn_2 - \beta_0 n_3 - \alpha n_4.$$

$$\frac{dH}{dt} + \mu H \leq 2rn_1 - \frac{rn_1^2}{k} + an_1n_2 + 2sn_2 - \frac{sn_2^2}{l} .$$

Where μ = *min.* $\{r, s, qE, \beta_0, \alpha\}$. Then,

$$\frac{dH}{dt} + \mu H \leq 2rk + akl + 2sl = \xi.$$

Applying Gronwall's Inequality [6], the following is obtained:

$$0 \leq H\left(n_1(t), n_2(t), n_3(t), n_4(t)\right) \leq \frac{\xi}{\mu}\,(1 - e^{-\mu t}) + H(0)e^{-\mu t}$$

hence,

$$0 \leq \limsup_{t\to\infty} \mathrm{H(t)} \leq \frac{\xi}{\mu}.$$

Therefore, all the solutions of the system (4.1) that are initiated in R_+^4 are attracted to the region $\vartheta = \left\{(n_1, n_2, n_3, n_4) \in R_+^4 : H = n_1 + n_2 + n_3 + n_4 \leq \frac{\xi}{\mu}\right\}$ under the given conditions. Thus, these solutions are uniformly bounded, and the proof is complete. ■

2. Existence of equilibria

The harvested food chain prey-predator model with a mutual interaction given by the system (1) has eight non-negative equilibrium points, namely:

(1) The vanishing equilibrium point $I_1 = (0,0,0,0)$, always exists.

(2) The first prey equilibrium point $I_2 = (k, 0,0,0)$, always exists.

(3) The second prey equilibrium point $I_3 = \left(0, \frac{l}{s}(s - qE), 0,0\right)$, exists when

$$s > qE. \tag{2}$$

(4) The first two species equilibrium point $I_4 = (\bar{n}_1, \bar{n}_2, 0,0)$, where $\bar{n}_1 = \frac{k}{r}[r + a\bar{n}_2]$ and $\bar{n}_2 = \frac{l}{s}(s - qE)$, exists when

$$s > qE. \tag{3}$$

(5) The first and third species equilibrium point $I_5 = (\ddot{n}_1, 0, \ddot{n}_3, 0)$, where $\ddot{n}_1 = \frac{\beta_0}{\beta_2}$ and $\ddot{n}_3 = r\left(\frac{\beta_2 k - \beta_0}{\beta_1\beta_2 k}\right)$, exists when

$$\beta_2 k > \beta_0. \tag{4}$$

(6) The first three species equilibrium point $I_6 = (\hat{n}_1, \hat{n}_2, \hat{n}_3, 0)$, here $\hat{n}_1 = \frac{\beta_0}{\beta_2}$, $\hat{n}_2 = \frac{L}{s}(s - qE)$ and $\hat{n}_3 = \frac{rs\,(\beta_2 k - \beta_0) + alk\beta_2(s - qE)}{\beta_1\beta_2 ks}$, which exists when

$$ralk\beta_2(s - qE) > s\,(\beta_0 - \beta_2 k). \tag{5}$$

(7) The second free species equilibrium point $I_7 = (n_1, 0, n_3, n_4)$,

here $n_1 = \frac{k}{r}\left(\frac{r\gamma_2-\alpha\beta_1}{\gamma_2}\right)$, $n_3 = \frac{\alpha}{\gamma_2}$ and $n_4 = \frac{(r\gamma_2-\alpha\beta_1)\beta_2 k-\beta_0 r\gamma_2}{r\gamma_1\gamma_2}$, which exists when

$$k\beta_2(r\gamma_2 - \alpha\beta_1) > \beta_0 r\gamma_2. \tag{6}$$

(8) The positive equilibrium point $I_8 = (n_1^*, n_2^*, n_3^*, n_4^*)$, here $n_1^* = \frac{k[rs(\gamma_2-\alpha\beta_1)+al\gamma_2(s-qE)]}{\gamma_2 sr}$,$n_2^* = \frac{L}{s}(s-qE)$, $n_3^* = \frac{\alpha}{\gamma_2}$ and $n_4^* = \frac{\beta_2 n_1^*-\beta_0}{\gamma_1}$, which exists when

$$al\gamma_2(s-qE) > rs(\alpha\beta_1 - \gamma_2) \text{ and } \beta_2 n_1^* > \beta_0. \tag{7}$$

3. The stability analysis

In this section, the conditions to guarantee the local behaviour of system (1) around each of the above equilibrium points are found. First, the Jacobian matrix of the system (1) at each point is determined, and then, the eigenvalues of the resulting matrix are computed.

The Jacobian matrix of the system (1) at the vanishing fixed point $I_1 = (0,0,0,0)$ can be written as:

$$J(I_1) = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & s-qE & 0 & 0 \\ 0 & 0 & -\beta_0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Then, the eigenvalues of $J(I_1)$ are given by $\lambda_{11} = r$, $\lambda_{12} = s-qE$, $\lambda_{13} = -\beta_{01}$ and $\lambda_{14} = -\alpha$. That means $I_1 = (0,0,0,0)$ is a saddle point in R_+^4.

The Jacobian matrix of the system at $I_2 = (\tilde{n}_1, 0,0,0)$ can be written as:

$$J(I_2) = \begin{bmatrix} -r & ak & -\beta_1 k & 0 \\ 0 & s-qE & 0 & 0 \\ 0 & 0 & \beta_2 k - \beta_0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Then, the eigenvalues of $J(I_2)$ are given by $\lambda_{21} = -r$, $\lambda_{22} = s-qE$, $\lambda_{23} = \beta_2 k - \beta_0$ and $\lambda_{24} = -\alpha$. That means I_2 is a locally asymptotical stable point if and only if

$$\text{s} < qE \text{ and } \beta_2 k < \beta_0. \tag{8}$$

Otherwise I_2 is a saddle point.

The Jacobian matrix of the system at $I_3 = (0, \dot{n}_2, 0,0)$ can be written as:

$$J(I_3) = \begin{bmatrix} r + a\dot{n}_2 & 0 & 0 & 0 \\ 0 & -(s-qE) & 0 & 0 \\ 0 & 0 & -\beta_0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Then, the eigenvalues of $J(I_3)$ are given by $\lambda_{31} = r + a\dot{n}_1 > 0$, $\lambda_{32} = -(s-qE)$, $\lambda_{33} = -\beta_0$ and $\lambda_{34} = -\alpha$. That means I_3 is a saddle point.

The Jacobian matrix of the system at $I_4 = (\bar{n}_1, \bar{n}_2, 0,0)$ can be written as:

$$J(I_4) = \begin{bmatrix} -(r + a\bar{n}_2) & a\bar{n}_1 & -\beta_1\bar{n}_1 & 0 \\ 0 & -(s-qE) & 0 & 0 \\ 0 & 0 & \beta_2\bar{n}_1 - \beta_0 & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

Then, the eigenvalues of $J(I_4)$ are given by $\lambda_{41} = -(\mathrm{r} + \mathrm{a}\bar{\mathrm{n}}_2) < 0$, $\lambda_{42} = -(s-qE)$, $\lambda_{43} = \beta_2\bar{n}_1 - \beta_0$ and $\lambda_{44} = -\alpha$. That means I_4 is a locally asymptotical stable point if and only if

$$\beta_2\bar{n}_1 < \beta_0. \tag{9}$$

Otherwise $I_4 = (\bar{n}_1, \bar{n}_2, 0,0)$ is a saddle point.

The Jacobian matrix of the system at $I_5 = (\ddot{n}_1, 0, \ddot{n}_3, 0)$ can be written as:

$$J(I_5)=\begin{bmatrix} -\frac{r\ddot{n}_1}{k} & a\ddot{n}_1 & -\beta_1\ddot{n}_1 & 0 \\ 0 & s-qE & 0 & 0 \\ \beta_2\ddot{n}_3 & 0 & 0 & -\gamma_1\ddot{n}_3 \\ 0 & 0 & 0 & \gamma_2\ddot{n}_3-\alpha \end{bmatrix}.$$

Then, it is easy to verify that the eigenvalues of $J(I_5)$ satisfy the following relations:

$\lambda_{51}+\lambda_{53}=\frac{-r\ddot{n}_1}{k}<0\,,$

$\lambda_{51}\cdot\lambda_{53}=\beta_1\ddot{n}_1\beta_2\ddot{n}_3>0,$

$\lambda_{52}=s-qE>0,$

$\lambda_{54}=\gamma_2\ddot{n}_3-\alpha.$

That means I_5 is a locally asymptotical stable point fixed point when

$$\gamma_2\ddot{n}_3<\alpha. \tag{10}$$

Otherwise I_5 is a saddle point.

The Jacobian matrix of the system at the equilibrium point $I_6=(\hat{n}_1,\hat{n}_2,\hat{n}_3,0)$ can be written as:

$$J(I_6)=\begin{bmatrix} -\frac{r\hat{n}_1}{k} & a\hat{n}_1 & -\beta_1\hat{n}_1 & 0 \\ 0 & -(s-qE) & 0 & 0 \\ \beta_2\hat{n}_3 & 0 & 0 & -\gamma_1\hat{n}_3 \\ 0 & 0 & 0 & \gamma_2\hat{n}_3-\alpha \end{bmatrix}$$

Then, it is easy to verify that the eigenvalues of $J(I_6)$ satisfy the following relations:

$\lambda_{61}+\lambda_{63}=-\frac{r\hat{n}_1}{k}<0,$

$\lambda_{61}\cdot\lambda_{63}=\beta_1\hat{n}_1\beta_2\hat{n}_3>0\,,$

$\lambda_{62}=-(s-qE)<0\,,$

$\lambda_{64}=\gamma_2\hat{n}_3-\alpha\,.$

That means I_6 is a locally asymptotical stable point if and only if

$$\gamma_2\hat{n}_3<\alpha. \tag{11}$$

Otherwise I_6 is a saddle point.

The Jacobian matrix of the system at the equilibrium point $I_7=(n_1,0,n_3,n_4)$ can be written as:

$$J(I_7)=\begin{bmatrix} -(r-\beta_1 n_2) & an_1 & -\beta_1 n_1 & 0 \\ 0 & s-qE & 0 & 0 \\ \beta_2 n_3 & 0 & 0 & -\gamma_1 n_3 \\ 0 & 0 & \gamma_2 n_4 & 0 \end{bmatrix}.$$

The first root of the characteristic equation of $J(I_7)$ is $s-qE$ and the other three roots are given by:

$$\lambda_1^3+A_1\lambda^2+A_2\lambda+A_3=0.$$

Where,

$\mathrm{A}_1=-(r-\beta_1 n_3),$

$\mathrm{A}_2=\gamma_1\gamma_2 n_3 n_4+\beta_1\beta_2 n_1 n_3,$

$\mathrm{A}_3=-\gamma_1\gamma_2 n_3 n_4(r-\beta_1 n_3),$

$\Delta=A_1A_2-A_3=-\beta_1\beta_2 n_1 n_3(r-\beta_1 n_3).$

Now, according to the Routh-Hurwitz criteria [14], all the eigenvalues of $J(I_7)$ have roots with negative real parts, provided that $\mathrm{A}_1>0, \mathrm{A}_3>0$ and $\Delta=\mathrm{A}_1>0$. Therefore, I_7 is locally asymptotically stable, if

$$s<qE \tag{12}$$

holds.

The Jacobian matrix of the system at the equilibrium point $I_8 = (n_1^*, n_2^*, n_3^*, n_4^*)$ can be written as:

$$J(I_8) = \begin{bmatrix} r - \frac{2rn_1^*}{k} - \beta_1 n_3^* + an_2^* & an_1^* & -\beta_1 n_1^* & 0 \\ 0 & -(s-qE) & 0 & 0 \\ \beta_2 n_3^* & 0 & 0 & -\gamma_1 n_3^* \\ 0 & 0 & \gamma_2 n_4^* & 0 \end{bmatrix}.$$

The first root of the characteristic equation of $J(I_8)$ is $qE - s$ and the other three roots are given by:

$\lambda^3 + \hat{A}\lambda^2 + \hat{B}\lambda + \hat{C} = 0.$

Where,

$\widehat{\mathrm{A}} = -\left[r - \left(\frac{2rn_1^*}{k}\right) - \beta_1 n_3^* + an_2^*\right] > 0,$

$\widehat{\mathrm{B}} = \gamma_1\gamma_2 n_3^* n_4^* + \beta_1\beta_2 n_1^* n_3^*,$

$\widehat{\mathrm{C}} = \gamma_1\gamma_2 n_3^* n_4^* \left(r - \frac{2rn_1^*}{k} - \beta_1 n_3^* + an_2^*\right) > 0,$

$\Delta_1 = A_1 A_2 - A_3 = -\beta_1\beta_2 n_1 n_3 (r - \beta_1 n_3) > 0.$

Now, according to the Routh-Hurwitz criteria, all the eigenvalues of $J(I_8)$ have roots with negative real parts, provided that $\widehat{\mathrm{A}} > 0, \widehat{\mathrm{C}} > 0$ and $\Delta_1 > 0$. Therefore, I_8 is locally asymptotically stable, if

$$s > qE \tag{13}$$

hold.

In the following theorem, the global stability condition for the positive equilibrium points is studied with the help of the Lyapunov method [15].

Theorem (2) Assume that the equilibrium point $I_8 = (n_1^*, n_2^*, n_3^*, n_4^*)$ is locally asymptotically stable in R_+^4. Then it is globally asymptotically stable in R_+^4 provided that

$$\frac{a^2\beta_2\gamma_2}{\beta_1\gamma_1} \le 4\left(\frac{rs}{kl}\right), \tag{14}$$

holds.

Proof: Consider the following positive definite function

$$W = c_1\left(n_1 - n_1^* - n_1^* \ln\frac{n_1}{n_1^*}\right) + c_2\left(n_2 - n_2^* - n_2^* \ln\frac{n_2}{n_2^*}\right)$$
$$+ c_3\left(n_3 - n_3^* - n_3^* \ln\frac{n_3}{n_3^*}\right) + c_4\left(n_4 - n_4^* - n_4^* \ln\frac{n_4}{n_4^*}\right),$$

where c_1, c_2 , c_3 and c_4 are positive constants to be determined.

Now the derivative of W along the trajectory of the system can be written as:

$$\frac{dW}{dt} = c_1\left(\frac{n_1 - n_1^*}{n_1}\right)\frac{dn_1}{dt} + c_2\left(\frac{n_2 - n_2^*}{n_2}\right)\frac{dn_2}{dt} + c_3\left(\frac{n_3 - n_3^*}{n_3}\right)\frac{dn_3}{dt} + c_4\left(\frac{n_4 - n_4^*}{n_4}\right)\frac{dn_4}{dt},$$

$$\frac{dW}{dt} = c_1(n_1 - n_1^*)\left(r\left(1 - \frac{n_1}{k}\right) - \beta_1 n_3 + an_2\right) + c_2\,(n_2 - n_2^*)\left(s\left(1 - \frac{n_2}{l}\right) - qE\right) + c_3(n_3 - n_3^*)(\beta_2 n_1 - \beta_0 - \gamma_1 n_4) + c_4(n_4 - n_4^*)(\gamma_2 n_3 - \alpha).$$

Therefore,

$$\frac{dW}{dt} = -c_1\frac{r}{k}(n_1 - n_1^*)^2 - c_1 a(n_1 - n_1^*)(n_2 - n_2^*) - c_2\frac{s}{l}(n_2 - n_2^*)^2 - (c_1\beta_1 - c_3\beta_2)(n_1 - n_1^*)(n_3 - n_3^*) - (c_3\gamma_1 - c_4\gamma_2)(n_3 - n_3^*)(n_4 - n_4^*).$$

By choosing the positive constants as:

$c_1 = \frac{\gamma_2\beta_2}{\gamma_1\beta_1};\ \ c_3 = \frac{\gamma_2}{\gamma_1};\ \ c_2 = c_4 = 1.$

We get:

$$\frac{dW}{dt} = -\left(\frac{r\gamma_2\beta_2}{k\gamma_1\beta_1}(n_1 - n_1^*)^2 - \frac{a\gamma_2\beta_2}{\gamma_1\beta_1}(n_1 - n_1^*)(n_2 - n_2^*) + \frac{s}{l}(n_2 - n_2^*)^2\right).$$

Thus, $\frac{dW}{dt} \le -\left(\sqrt{\frac{r\gamma_2\beta_2}{k\gamma_1\beta_1}}(n_1 - n_1^*) - \sqrt{\frac{s}{l}}(n_2 - n_2^*)\right)^2.$

Then $\frac{dW}{dt} \le 0$ under the condition (14), hence I_8 is a globally asymptotically stable point in the R_+^4.

4. Numerical analysis

This section aims to find the system's critical parameters that affect the behaviour of the proposed system by using numerical simulations. The dynamics of system (1) is obtained by solving system (1) numerically by using the predictor-corrector method with the six order Range Kutta method. The time series of the solution of system (1) is drawn using MATLAB for different sets of parameters. Now, for the following set of parameters:

$$r = 1, k = 5, a = 0.4, \beta_1 = 2, a = 0.4, \alpha = 0.2, qE = 0.4, s = 0.9, l = 4,$$
$$\beta_2 = 1.25, \gamma_1 = 0.6, \gamma_2 = 0.54, \beta_0 = 1. \qquad (15)$$

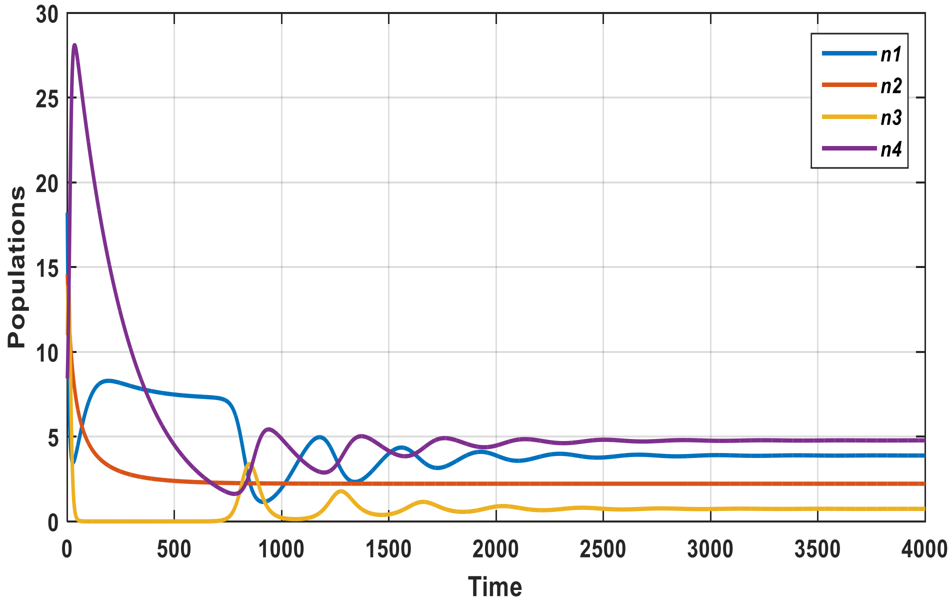

Figure 1-Dynamics of the four species with the data given by Eq. (15).

the condition (13) is satisfied. This shows that I_8 exists, and it is given by $(n_1^*, n_2^*, n_3^*, n_4^*)$ = (3.88, 2.22, 0.74, 4.76). (See Figure 2).

Figure 2 presents the dynamics of the four species with the data given by Eq. (15) with $\alpha = 0.89$. It shows that the condition (11) is satisfied and that I_6 exists and it is given by $(\hat{n}_1, \hat{n}_2, \hat{n}_3, 0) = (0.06, 2.22, 1.58, 0)$.

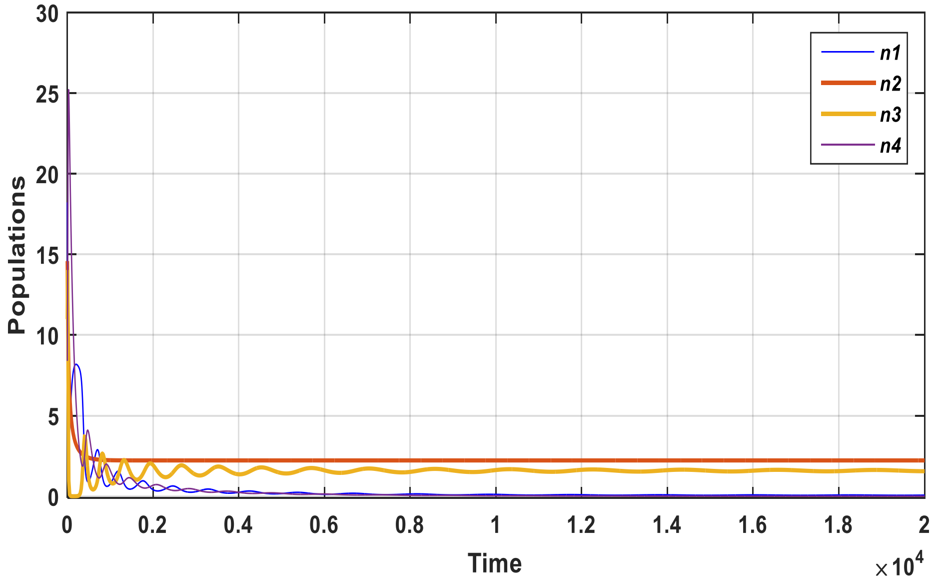

Figure 2-Dynamics of the four species with the data given by Eq. (15) with $\alpha = 0.89$.

Figure 3 presents the dynamics of the four species with the data given by Eq. (15) with $qE = 0.91$. It shows that the condition (12) is satisfied and that I_7 exists and it is given by $(n_1, 0, n_3, n_4) = (1.66, 0, 0.74, 2)$.

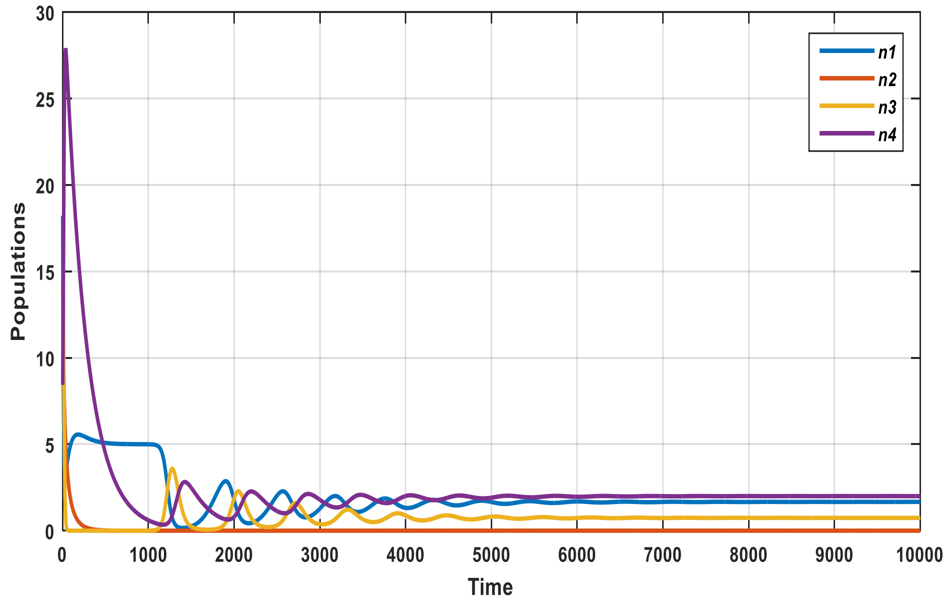

Figure 3-Dynamics of the four species with the data given by Eq. (15) with qE=0.91.

5. Discussions and Conclusions

In the proposed model, it is observed different eight equilibrium points at which the system is stable. The system's stability is determined based on the conditions at which the equilibrium points of the model equations exist. For example, at the equilibrium point I_7 only the second prey could become extinct. Since there is a harvesting effect for the latter, it can stay for a long time by controlling the parameter qE. This result has been shown in Figure 3. Moreover, Figure 2 shows the first three species can live together if condition (12) holds. Finally, figure 1 describes the interaction among the fourth species. The fourth species could survive for a long time when the state (12) is satisfied.

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