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Generalized Commuting Mapping in Prime and Semiprime Rings

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Abstract
Let R be an associative ring. The essential purpose of the present paper is to introduce the concept of generalized commuting mapping of R. Let U be a non-empty subset of R, a mapping $\mathfrak{F}$: R⟶ R is called a generalized commuting mapping on U if there exist a mapping $\mathcal{G}$:R⟶R such that $[\mathfrak{F}(\omega), \mathcal{G}(\omega)]=0$, holds for all $\omega \in$ U. Some results concerning the new concept are presented.

Key words and phrases: Derivation, Semiderivations, Commuting mapping, Prime ring, Semiprime rings.

الدوال الأبدالية المعممة على الحلقات الأولية وشبة الأولية

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قسم الرياضيات, كلية التربية , الجامعة المستنصرية, بغداد, العراق

الخلاصة
لتكن R حلقة تجميعية . الغاية الأساسية من هذا البحث تقديم مفهوم الدوال الأبدالية المعممة على R .
لتكن U مجموعة غير خالية من الحلقة $\mathfrak{R}$, الدالة $\mathfrak{F}$: R ⟶ R تسمى دالة أبدالية معممة أذا وجدت دالة $\mathcal{G}$: R ⟶ Rتحقق أن $[\mathfrak{F}(\omega), \mathcal{G}(\omega)]=0$ ولكل $\omega \in$ U . بعض النتائج المرتبطة بالمفهوم الجديد قدمت في هذا العمل.

1-Introduction

Throughout this paper, R will represent to an associative ring with center $\mathcal{Z}$(R). By $\mathcal{Q}_{\mathrm{R}}$we will denote the Martindale ring of quotient of R. The extended centroid of R is refer to the center of $\mathcal{Q}_{\mathrm{R}}$, which denote by $\mathcal{C}$, and the extended centroid is a field, furthermore $\mathcal{Z}(\mathrm{R}) \subseteq \mathcal{C}$ [1]. Given $n \geq 2$, a ring R in which $nt=0$ implies that $t=0$ is called an n-torsion free ring [2]. We denote by $[\omega, t]$ to the commutator $\omega t - t\omega$. The commutator identities $[\omega t, s] = [\omega, s]t + \omega[t, s]$ and $[\omega, ts] = [\omega, t]s + t[\omega, s]$ are frequently used [3]. Recall that a ring R with a property that whenever ω R t = {0} implies either ω =0 or t =0 is said to be a prime ring and is semiprime if whenever ω R ω=(0) implies ω=0 [4]. An equivalent definition of this concept is given as follows: A ring R is said to be semiprime if whenever U is an ideal of R satisfies U^k= (0) implies U=(0) [5].

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An additive mapping δ: R $\longrightarrow$R in which $[(\tau), \tau] \in \mathcal{Z}(R)$ whenever $\tau \in$ R is said to be centralizing on R. In a special case, δ is called commuting on R if $[(\tau), \tau] = 0$, whenever $\tau \in$ R [6]. A mapping d: R$\longrightarrow$ R is called a derivation of R if for any τ, $t \in$ R we have $d(\tau + t) = d(\tau) + d(t)$ and $d(\tau t) = d(\tau) t + \tau d(t)$ [7]. By a semi derivation mapping of a ring R, we shall mean a mapping $\mathcal{D}$: R $\longrightarrow$ R with additive property together with a mapping $\mathcal{G}$: R$\longrightarrow$ R such that $\mathcal{D}(\tau t) = \mathcal{D}(\tau)\, \mathcal{G}(t) + \tau\, \mathcal{D}(t) = \mathcal{D}(\tau) t + \mathcal{G}(\tau)\, \mathcal{D}(t)$ and $\mathcal{D}(\mathcal{G}(\tau)) = \mathcal{G}(\mathcal{D}(\tau))$ [8].

In this work, the notions of generalized commuting mapping are introduced. Furthermore, we investigate some necessary conditions that make every $\mathcal{G}$- commuting mapping a commuting mapping on R.

2 . Preliminary results:

We begin this section by introducing the concept of generalized commuting mapping.

Definition2.1:

Let R be a ring and U be a non-trivial subset of R. An additive mapping $\mathfrak{F}$: R $\longrightarrow$ R is said to be a generalized commuting mapping (or $\mathcal{G}$- commuting mapping) on U if there exist a mapping $\mathcal{G}$: R $\longrightarrow$ R such that $[\mathfrak{F}(\omega), \mathcal{G}(\omega)] = 0$, fulfilled for all $\omega \in$ U.

Remark 2.2:

Every commuting mapping ρ is a generalized commuting mapping $\mathcal{I}$-commuting mapping, where $\mathcal{I}$ is the identity mapping, but the converse is not true in general as we see in the following illustrative example:

Example2.3:

Let R be a ring having a nilpotent $0 \neq a \in$ R with $a^2=0$ and let U be the right ideal of R generated by a. Define mappings $\mathfrak{F}, \mathcal{G}$: R $\longrightarrow$ R by $\mathcal{G}(t) = [a, t]$ and $\mathfrak{F}(t) = ta$. Then, $\mathfrak{F}$ is $\mathcal{G}$-commuting mapping on U but not commuting.

We list some remarks and Lemmas which will be needed throughout this paper.

Lemma 2.4: [9]

Let R be a 2-torsion free semiprime ring and let $a,b \in$ R. then the following conditions are equivalent:

i. $a\, u\, b = 0$, for all $u \in$ R.
ii. $b\, u\, a = 0$, for all $u \in$ R.
iii. $a\, u\, b + b u\, a = 0$, for all $u \in$ R.

Furthermore, $ab=ba=0$ whenever any one of these conditions is fulfilled.

Lemma 2.5: [10]

Let R be a prime ring and C be the extended centroid of R. Let a, b $\in$ R be such that axb=bxa fulfilled for all $x \in$ R . If $a \neq 0$, then there exists $\lambda \in C$ with b= λa.

Lemma 2.6: [11]

Let $\mathfrak{R}$ be a semiprime ring of characteristic $\neq 2$ and $\mathfrak{F}$: R $\longrightarrow$ R be a non-trivial derivation satisfies $[[\mathfrak{F}(u), u], \mathfrak{F}(u)] = 0$, for any $u \in \mathfrak{R}$, then $\mathfrak{F}$ is a commuting mapping on $\mathfrak{R}$.

3. Results on generalized commuting mapping:

Theorem 3.1: Let R be a prime ring of characteristic$\neq$2 and $\mathcal{F}, \mathcal{G}$: R $\longrightarrow$ R are nonzero derivations such that $\mathcal{F}$ is a $\mathcal{G}$- commuting mapping on R. Then either $\mathcal{F}$ or $\mathcal{G}$ forced to be commuting mapping on R.

Proof*:* By our hypothesis, we have:

$$[\mathcal{F}(\sigma), \mathcal{G}(\sigma)] = 0, \text{ for all } \sigma \in R. \quad (1)$$

A linearization of the relation (1) leads to:

$$[\mathcal{F}(\sigma), \mathcal{G}(\tau)] + [\mathcal{F}(\tau), \mathcal{G}(\sigma)] = 0, \text{ whenever } \sigma, \tau \in R. \quad (2)$$

Taking $\tau\sigma$ instead of τ in (2), we obtain:

$[\mathcal{F}(\sigma), \mathcal{G}(\tau)]\, \sigma + \mathcal{G}(\tau)\, [\mathcal{F}(\sigma), \sigma] + [\mathcal{F}(\sigma), \tau]\, \mathcal{G}(\sigma) + \tau\, [\mathcal{F}(\sigma), \mathcal{G}(\sigma)] + [\mathcal{F}(\tau), \mathcal{G}(\sigma)]\, \sigma + \mathcal{F}(\tau)\, [\sigma, \mathcal{G}(\sigma)] + \tau\, [\mathcal{F}(\sigma), \mathcal{G}(\sigma)] + [\tau, \mathcal{G}(\sigma)]\, \mathcal{F}(\sigma) = 0$, whenever $\sigma, \tau \in$ R.

Using (1) and (2), we get:

$\mathcal{G}(\tau)\,[\mathcal{F}(\sigma),\sigma]+[\mathcal{F}(\sigma),\tau]\,\mathcal{G}(\sigma)+\mathcal{F}(\tau)\,[\sigma,\mathcal{G}(\sigma)]+[\sigma,\mathcal{G}(\sigma)]\,\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$. (3)

If we replace σ by $\sigma\tau$ in (3), then it implies that:

$\mathcal{G}(\sigma)\,\tau\,[\mathcal{F}(\sigma),\sigma]+\sigma\,\mathcal{G}(\tau)\,[\mathcal{F}(\sigma),\sigma]+[\mathcal{F}(\sigma),\sigma]\,\tau\,\mathcal{G}(\sigma)+\sigma\,[\mathcal{F}(\sigma),\tau]\,\mathcal{G}(\sigma)+\mathcal{F}(\sigma)\,\tau\,[\sigma,\mathcal{G}(\sigma)]+\sigma\,\mathcal{F}(\tau)\,[\sigma,\mathcal{G}(\sigma)]+\sigma\,[\tau,\mathcal{G}(\sigma)]\,\mathcal{F}(\sigma)+[\sigma,\mathcal{G}(\sigma)]\,\tau\,\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$.

In view of (3), this relation reduces to:

$\mathcal{G}(\sigma)\,\tau\,[\mathcal{F}(\sigma),\sigma]+[\mathcal{F}(\sigma),\sigma]\,\tau\,\mathcal{G}(\sigma)-\mathcal{F}(\sigma)\,\tau\,[\mathcal{G}(\sigma),\sigma]-[\mathcal{G}(\sigma),\sigma]\,\tau\,\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$. (4)

The left multiplication of (4) by $\mathcal{F}(\sigma)$ gives:

$\mathcal{F}(\sigma)\,\mathcal{G}(\sigma)\,\tau\,[\mathcal{F}(\sigma),\sigma]+\mathcal{F}(\sigma)\,[\mathcal{F}(\sigma),\sigma]\,\tau\,\mathcal{G}(\sigma)-\mathcal{F}(\sigma)^2\,\tau\,[\mathcal{G}(\sigma),\sigma]-\mathcal{F}(\sigma)\,[\mathcal{G}(\sigma),\sigma]\,\tau\,\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$. (5)

Putting $\mathcal{F}(\sigma)\,\tau$ instead of in (4) we get:

$\mathcal{G}(\sigma)\,\mathcal{F}(\sigma)\,\tau\,[\mathcal{F}(\sigma),\sigma]+[\mathcal{F}(\sigma),\sigma]\,\mathcal{F}(\sigma)\,\tau\,\mathcal{G}(\sigma)-\mathcal{F}(\sigma)^2\,\tau\,[\mathcal{G}(\sigma),\sigma]-[\mathcal{G}(\sigma),\sigma]\,\mathcal{F}(\sigma)\,\tau\,\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$. (6)

The subtracting of (5) from (6) leads to:

$\{\mathcal{G}(\sigma)\,\mathcal{F}(\sigma)\,\tau\,[\mathcal{F}(\sigma),\sigma]-\mathcal{F}(\sigma)\,\mathcal{G}(\sigma)\,\tau\,[\mathcal{F}(\sigma),\sigma]\}+\{[\mathcal{F}(\sigma),\sigma]\,\mathcal{F}(\sigma)\,\tau\,\mathcal{G}(\sigma)-\mathcal{F}(\sigma)\,[\mathcal{F}(\sigma),\sigma]\,\tau\,\mathcal{G}(\sigma)\}+\{\mathcal{F}(\sigma)^2\,\tau\,[\mathcal{G}(\sigma),\sigma]-\mathcal{F}(\sigma)^2\,\tau\,[\mathcal{G}(\sigma),\sigma]\}+\{\mathcal{F}(\sigma)\,[\mathcal{G}(\sigma),\sigma]\,\tau\,\mathcal{F}(\sigma)-[\mathcal{G}(\sigma),\sigma]\,\mathcal{F}(\sigma)\,\tau\,\mathcal{F}(\sigma)\}$

In view of (1), the last relation reduces to:

$[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]\,\mathcal{G}(\sigma)+[\mathcal{F}(\sigma),[\mathcal{G}(\sigma),\sigma]]\,\tau\,\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$.

Now, in the above relation, the substitution $[\mathcal{F}(\sigma),\sigma]$ instead of ones and right multiplication by $[\mathcal{F}(\sigma),\sigma]$ in another gives:

$[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]\,\tau\,[\mathcal{F}(\sigma),\sigma]\,\mathcal{G}(\sigma)+[\mathcal{F}(\sigma),[\mathcal{G}(\sigma),\sigma]]\,\tau\,[\mathcal{F}(\sigma),\sigma]\mathcal{F}(\sigma)=0$, whenever $\sigma,\tau\in R$.

$[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]\,\mathcal{G}(\sigma)\,[\mathcal{F}(\sigma),\sigma]+[\mathcal{F}(\sigma),[\mathcal{G}(\sigma),\sigma]]\,\tau\,\mathcal{F}(\sigma)\,[\mathcal{F}(\sigma),\sigma]=0$, whenever $\sigma,\tau\in R$.

Comparing the above expressions, we get:

$[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]\,[[\mathcal{F}(\sigma),\sigma],\mathcal{G}(\sigma)]+[[\mathcal{G}(\sigma),\sigma],\mathcal{F}(\sigma)]\,\tau\,[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]=0$, for all $\sigma,\tau\in R$.

According to Jacobi identity and the commutator properties, the last relation can be written as:

$[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]\quad[[\mathcal{F}(\sigma),\ ],\mathcal{G}(\sigma)]+[[\mathcal{F}(\sigma),\sigma],\mathcal{G}(\sigma)]\,\tau\,[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]=0$, whenever $\sigma,\tau\in R$.

An application of lemma (2.4) it follows that:

$[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]\quad[[\mathcal{F}(\sigma),\sigma],\mathcal{G}(\sigma)]=0$, whenever $\sigma,\tau\in R$.

However, since R is a ring having primeness property, we get either $[[\mathcal{F}(\sigma),\sigma],\mathcal{F}(\sigma)]=0$, whenever $\sigma\in R$, consequently $\mathcal{F}$ is a commuting mapping on R, or

$[[\mathcal{F}(\sigma),\sigma],\mathcal{G}(\sigma)]=0$, whenever $\sigma\in R$. (7)

Taking $z+\sigma$ instead of σ in (7), we see that

$[[\mathcal{F}(\sigma),\sigma],\mathcal{G}(z)]+[[\mathcal{F}(\sigma),z],\mathcal{G}(t)]+[[\mathcal{F}(z),\sigma],\mathcal{G}(t)]+[[\mathcal{F}(\sigma),z],\mathcal{G}(z)]+[[\mathcal{F}(z),\sigma],\mathcal{G}(z)]+[[\mathcal{F}(z),z],\mathcal{G}(\sigma)]=0$, whenever $\sigma,z\in R$.

The substitution $2z$ instead of z in above relation and comparing the new relation with the above, we get by 2-torsionity free of R:

$[[\mathcal{F}(\sigma),z],\mathcal{G}(z)]+[[\mathcal{F}(z),\sigma],\mathcal{G}(z)]+[[\mathcal{F}(z),z],\mathcal{G}(\sigma)]=0$, whenever $\sigma,z\in R$. (8)

Now, we use a same procedure that used to get (4) from (2), we arrive at:

$2[\mathcal{F}(z),z]\,\sigma\,[z,\mathcal{G}(z)]+[z,\mathcal{G}(z)]\,\sigma\,[\mathcal{F}(z),z]+\mathcal{F}(z)[\sigma,z]\,[z,\mathcal{G}(z)]+\mathcal{G}(z)\,\sigma\,[[\mathcal{F}(z),z],z]+[[\mathcal{F}(z),z],z]\,\sigma\,\mathcal{G}(z)=0$, whenever $\sigma,z\in R$.

Putting $\mathcal{G}(z)\,\sigma$ instead of σ in the last relation, we obtain:

$2[\mathcal{F}(z), z]\, \mathcal{G}(z)\, \sigma\, [z, \mathcal{G}(z)] + [z, \mathcal{G}(z)]\, \mathcal{G}(z)\, \sigma\, [\mathcal{F}(z), z] + \mathcal{F}(z)\mathcal{G}(z)[\sigma, z]\, [z, \mathcal{G}(z)] + \mathcal{F}(z)$ $[\mathcal{G}(z), z]\, \sigma [z, \mathcal{G}(z)] + \mathcal{G}(z)^2\, \sigma\, [[\mathcal{F}(z), z], z] + [[\mathcal{F}(z), z], z]\, \mathcal{G}(z)\, \sigma\, \mathcal{G}(z) = 0$, whenever $\sigma, z \in$ R. (9)

Multiply (9) from the left by $\mathcal{G}(z)$, we see

$2\, \mathcal{G}(z)[\mathcal{F}(z), z]\, \sigma\, [z, \mathcal{G}(z)] + \mathcal{G}(z)[z, \mathcal{G}(z)]\, \sigma\, [\mathcal{F}(z), z] + \mathcal{G}(z)\mathcal{F}(z)[\sigma, z]\, [z, \mathcal{G}(z)] + \mathcal{G}(z)^2\, \sigma$ $[[\mathcal{F}(z), z], z] + \mathcal{G}(z)[[\mathcal{F}(z), z], z]\, \sigma\, \mathcal{G}(z) = 0$, whenever $\sigma, z \in$ R. (10)

Subtracting (10) from (9) leads to:

$2[[\mathcal{F}(z), z], \mathcal{G}(z)]\, \sigma\, [z, \mathcal{G}(z)] + [[z, \mathcal{G}(z)], \mathcal{G}(z)]\, \sigma\, [\mathcal{F}(z), z] + [\mathcal{F}(z), \mathcal{G}(z)][\sigma, z]\, [z, \mathcal{G}(z)] +$ $[[[\mathcal{F}(z), z], z], \mathcal{G}(z)]\, \sigma\, \mathcal{G}(z) + \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma\, [z, \mathcal{G}(z)] = 0$, whenever $\sigma, z \in$ R.

By (7) and (1) the first and the third term are zeros respectively. So for any $\sigma, z \in$ R, we have $[[z, \mathcal{G}(z)], \mathcal{G}(z)]\, \sigma\, [\mathcal{F}(z), \,] + [[[\mathcal{F}(z), z], z], \mathcal{G}(z)]\, \sigma\, \mathcal{G}(z) + \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma\, [z, \mathcal{G}(z)] = 0$.

Now, in the above relation the substitution $\mathcal{F}(z)$ for t ones and right multiplication by $\mathcal{F}(z)$ in another gives:

$[[z, \mathcal{G}(z)], \mathcal{G}(z)]\sigma\, \mathcal{F}(z)[\mathcal{F}(z), z] + [[[\mathcal{F}(z), z], z], \mathcal{G}(z)]\, \sigma\, \mathcal{F}(z)\, \mathcal{G}(z) + \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma$ $\mathcal{F}(z)\, [z, \mathcal{G}(z)] = 0$, whenever $t, z \in$ R. (11)

$[[z, \mathcal{G}(z)], \mathcal{G}(z)]\, \sigma\, [\mathcal{F}(z), z]\mathcal{F}(z) + [[[\mathcal{F}(z), z], z], \mathcal{G}(z)]\, \sigma\, \mathcal{G}(z)\mathcal{F}(z) + \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma\, [z,$ $\mathcal{G}(z)]\, \mathcal{F}(z) = 0$, whenever $\sigma, z \in$ R. (12)

Again, Subtracting (11) from (12) we get because of (1) and the commutator identity that:

$[[, \mathcal{G}(z), z], \mathcal{G}(z)]\, \sigma\, [[\mathcal{F}(z), z], \mathcal{F}(z)] + \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma\, [[\mathcal{G}(z), z], \mathcal{F}(z)] = 0$, whenever $\sigma, z \in$ R.

The last relation can be written because of Jacobi identity as:

$[[, \mathcal{G}(z), z], \mathcal{G}(z)]\, \sigma\, [[\mathcal{F}(z), z], \mathcal{F}(z)] + \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma\, [[\mathcal{F}(z), z], \mathcal{G}(z)] - \mathcal{F}(z)\, [\mathcal{G}(z), z]\, \sigma\, [z,$ $[\mathcal{G}(z), \mathcal{F}(z)]] = 0$, whenever $\sigma, z \in$ R.

In view of (7) and (1), the above relation reduces to:

$[[, \mathcal{G}(z), z], \mathcal{G}(z)]\, \sigma\, [[\mathcal{F}(z), z], \mathcal{F}(z)] = 0$, whenever $\sigma, z \in$ R.

Using the primeness of R, since $[[\mathcal{F}(z), z], \mathcal{F}(z)] \neq 0$, for all $z \in$ R, then

$[[, \mathcal{G}(z), z], \mathcal{G}(z)] = 0$, whenever $z \in$ R.

Finally, an application of Lemma (2.6) forced $\mathcal{G}$ to be commuting on R.

Theorem 3.2: Let R be a prime ring and $\mathcal{D}:$ R $\longrightarrow$R be a nonzero semiderivations with associated homomorphism $\mathcal{G}:$ R $\longrightarrow$ R . If $\mathcal{D}$ is a $\mathcal{G}$-commuting mapping on R, then either $\mathcal{D}$ or $\mathcal{G}$ is a commuting mapping on R.

Proof :

For any $t \in$ R, we have:

$$[\mathcal{D}(t), \mathcal{G}(t)] = 0 \quad (13)$$

Setting $t = t + \omega$ in (13) and using (13), we get:

$$[\mathcal{D}(t), \mathcal{G}(\omega)] + [\mathcal{D}(\omega), \mathcal{G}(t)] = 0, \forall\, \omega, t \in \text{R}. \quad (14)$$

Putting $t\omega$ instead of ω in above relation, we find:

$[\mathcal{D}(t), \mathcal{G}(t)]\, \mathcal{G}(\omega) + \mathcal{G}(t)[\mathcal{D}(t), \mathcal{G}(\omega)] + [\mathcal{D}(t), \mathcal{G}(t)]\, \omega + \mathcal{D}(t)\, [\omega, \mathcal{G}(t)] + \mathcal{G}(t)\, [\mathcal{D}(\omega), \mathcal{G}(t)] + [\mathcal{D}(t), \mathcal{G}(t)]\, \mathcal{D}(\omega) = 0$, $\forall\, \omega, t \in$ R.

In view of (13) and (14), the last relation reduces to:

$$\mathcal{D}(t)\, [\omega, \mathcal{G}(t)] = 0, \forall\, \omega, t \in \text{R}. \quad (15)$$

The substitution ωt for ω in (15) leads to:

$$\mathcal{D}(t)\, \omega\, [\mathcal{G}(t), t] = 0, \forall\, \omega, t \in \text{R}. \quad (16)$$

Now, in (16), the left multiplication by t ones and putting $t\omega$ instead of ω in another gives:

$$t\mathcal{D}(t)\, \omega\, [\mathcal{G}(t), t] = 0, \forall\, \omega, t \in \text{R}. \quad (17)$$

$$\mathcal{D}(t)\, t\omega\, [\mathcal{G}(t), t] = 0, \forall\, \omega, t \in \text{R}. \quad (18)$$

Subtracting (17) from (18) implies that:

$[\mathcal{D}(t), t]\, \omega\, [\mathcal{G}(t), t] = 0$, $\forall\, \omega, t \in$ R.

Since R is prime, then we have either $[\mathcal{D}(t), t] = 0$, for all $t \in$ R, that is $\mathcal{D}$ is a commuting mapping on R or
$[\mathcal{G}(t), t] = 0$, for all $t \in$ R.
Hence, $\mathcal{G}$ is a commuting mapping on R.

Theorem 3.3: Let R be a prime ring of characteristic$\neq 2$ and $\mathcal{H}, \mathcal{K}$: R $\longrightarrow$ R be a nonzero left centralizer such that $\mathcal{H}$ is a $\mathcal{K}$- commuting mapping on R. Then, $[\mathcal{H}(t), t] = \lambda[\mathcal{K}(t), t]$, for all $t \in$ R and some $\lambda \in \mathcal{C}$.

Proof :

From our hypothesis, we have:

$$[\mathcal{H}(t), \mathcal{K}(t)]= 0, \text{ whenever } t \in R. \quad (19)$$

Taking $t+z$ instead of t in (19) gives

$$[\mathcal{H}(t), \mathcal{K}(z)]+ [\mathcal{H}(z), \mathcal{K}(t)]= 0, \text{ whenever } t, z \in R.$$

Replacing z by zt in the last relation, we arrive because (19) that

$$\mathcal{K}(z) [\mathcal{H}(t), t]+ \mathcal{H}(z) [t, \mathcal{K}(t)]= 0, \text{ whenever } t, z \in R. \quad (20)$$

Substituting tz for in (20), we get:

$$\mathcal{K}(t)z [\mathcal{H}(t), t]+ \mathcal{H}(t) z [t, \mathcal{K}(t)]= 0, \text{ whenever } t, z \in R . \quad (21)$$

Now, the left multiplication of (21) by in ones and putting instead of in another gives

$$t \mathcal{K}(t) z [\mathcal{H}(t), t] + t\mathcal{H}(t) z [t, \mathcal{K}(t)]= 0, \text{ whenever } t, z \in R. \quad (22)$$

$$\mathcal{K}(t) tz [\mathcal{H}(t), t] + \mathcal{H}(t) tz [t, \mathcal{K}(t)]= 0, \text{ whenever } t, z \in R. \quad (23)$$

From relations (22) and (23), we obtain
$[\mathcal{K}(t), t]z [\mathcal{H}(t), t] - [\mathcal{H}(t), t] z [\mathcal{K}(t), t]= 0$, whenever $t, z \in$ R.
In view of Lemma (2.5), it follows that for some λ in the extended centroid of R, we have:
$[\mathcal{H}(t), t]= \lambda[\mathcal{K}(t), t]$, whenever $t \in$ R.

Conclusions

In this article, we introduce the concept of generalized commuting mapping of a ring R. Some basic properties of this concept have been given and discussed. Furthermore, many results concerning with generalized commuting mapping have been investigated.

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