DOI: 10.24996/ijs.2022.63.10.23

ISSN: 0067-2904

Remarks on Ultrasemiprime algebras

Mohammed Th. Al-Neima. [1]*, Ruqayah N. Balo [2], Nadia Adnan Abdalrazaq[2]
[1]*Department of Civil Engineering, College of Engineering, University of Mosul, Iraq*
[2]*Department of Mathematics, College of Education for Pure Sciences, University of Mosul, Iraq*

Received: 4/7/2021 Accepted: 19/12/2021 Published: 30/10/2022

Abstract
Every finite dimensional normed algebra is isomorphic to the finite direct product of ℝ or ℂ, it is also proved these algebras are ultrasemiprime algebras. In this paper, the ultrasemiprime proof of the finite direct product of ℝ and ℂ is generalized to the finite direct product of any ultrasemiprime algebras.

Keywords: Ultraprime algebra, Ultrasemiprime algebra, Direct product of algebras.

ملاحظات حول الجبور شبه أولية المميزة

محمد ذنون النعمة[1]*, رقية نافع بلو[2], نادية عدنان عبدالرازق [2]
[1]قسم الهندسة المدنية, كلية الهندسة, جامعة الموصل, نينوى, العراق
[2]قسم الرياضيات, كلية التربية للعلوم الصرفة, , جامعة الموصل, نينوى, العراق

الخلاصة
كل جبر منتهي البعد معياري يتشاكل تشاكلاً متقابلاً مع الضرب المباشر المنتهي من ℝ أو ℂ ، كذلك تم برهان ان هذه الجبور هي جبور شبه أولية مميزة. في هذا البحث برهان الشبه أولية المميزة للضرب المباشر المنتهي لـ ℝ و ℂ تم تعميمها الى الضرب المباشر المنتهي لأي جبور شبه أولية مميزة.

1. Introduction

Throughout this paper, all algebras are associative unless otherwise stated. Mathieu [1] introduced the ultraprime algebra by defining a norm on the algebra of quotients. The normed algebra A is ultraprime if there exists $c > 0$ such that $c\|a\|\|b\| \leq \|M_{a,b}\|$ for all $a, b \in A$, where $M_{a,b}: A \longrightarrow A$ is a linear operator defined by $M_{a,b}(x) = axb$. An example of ultraprime algebra is that every prime $\mathbb{C}^*$-algebra A is ultraprime and $\|M_{a,b}\| = \|a\|\|b\|$ for all $a, b \in A$ [2]. Ultraprime algebras were studied by many researchers[3][4][5]. If $b = a$ for all $a \in A$, then the definition of ultraprime algebra A transfers to the definition of ultrasemiprime algebra. Every ultraprime algebra is ultrasemiprime, however, the converse is not true, Mohammed [6, Theorem 5] proved that every finite dimensional normed algebra isomorphic to $\mathbb{R}^n$ and $\mathbb{C}^n$, where $n \in \mathbb{N}$ are ultrasemiprime, but $\mathbb{R}^n$ and $\mathbb{C}^n$ are not prime so that they are not ultraprime algebras.

*Email: mohammedmth@uomosul.edu.iq

Mohammed [6, Theorem 5] also proved that the finite direct product of $\mathbb{R}$ or $\mathbb{C}$ is an ultrasemiprime which is a special case in the finite direct product of ultrasemiprime algebras, because $\mathbb{R}$ and $\mathbb{C}$ are ultrasemiprime. In this paper, the generalization of Theorem 5 has been given for the finite direct product of ultrasemiprime algebras.

1. The ultrasemiprime algebras

Mathieu [1] studied the ultraprime algebras and gave an analytical adjective for algebra of quotients. He also defined an ultrasemiprime algebra which is given as follows:
The normed algebra A is an ultrasemiprime if there exists $c > 0$ such that $c\|a\|^2 \leq \|M_{a,a}\|$ for all $a \in A$. Every $\mathbb{C}^*$-algebra is an ultrasemiprime[7].

An ultrafilter is a subset of a partially ordered set that is maximal among all proper filters. Let I be an index set and $(A_i)_{i\in I}$ a family of normed spaces denoted by $\ell^\infty(I, A_i)$, the space of all bounded families $(x_i)_{i\in I}$ with $x_i \in A_i$. Let u is an ultrafilter on I, define $n_u = \{(x_i)_{i\in I} \in \ell^\infty(I, A_i): \lim_u\|x_i\| = 0\}$. The quotient $\ell^\infty(I, A_i)/n_u$ is called the ultraproduct of the normed spaces A_i with respect to the ultrafilter u. When one takes $I = \mathbb{N}$, $A_i = A$ for all $i \in I$, where A is normed space and an ultrafilter u on $\mathbb{N}$. Then $(A)_u$ is called ultrapower of A with respect to u and is denoted by $\hat{A}_u$ [1].

Mathieu [1, Lemma 1.1] gave two equivalent conditions for ultraprime algebras, they are shown in the next lemma.

Lemma 2.1 [1]

The following conditions are equivalent for the normed algebra A.
1. For any pair $(x_n), (z_n)$, $n \in \mathbb{N}$ of sequences in A such that $\|x_n\| = \|z_n\| = 1$ for all $n \in \mathbb{N}$, there exists a bounded sequence, (y_n), $n \in \mathbb{N}$ such that $(x_n y_n z_n)$, $n \in \mathbb{N}$ does not tend to zero.
2. A is an ultraprime algebra.
3. The algebra of ultrapower of A on u, $\hat{A}_u$ is prime, where u the free ultrafilter on $\mathbb{N}$.

In the following theorem, a condition that is similar to the first equivalent condition in Lemma 2.1 for the ultraprime is used. This gives an equivalent condition to ultrasemiprime algebra.

Theorem 2.2

The following statements are equivalent to a normed algebra A.
1. For any sequence $(x_n), n \in \mathbb{N}$ in A with $\|x_n\| = 1$, for all $n \in \mathbb{N}$, there exists a bounded sequence (y_n), $n \in \mathbb{N}$ in A such that the sequence $(x_n y_n x_n), n \in \mathbb{N}$ does not converge to zero.
2. There exists a positive number c, such that $c\|x\|^2 \leq \|M_{x,x}\|$ for all x in A.

Proof:

Let condition (1) be true and a positive number c for the sequence (a_n) in $A\backslash\{0\}$ to satisfy (2) for all $n \in \mathbb{N}$ does not exist. When $n = 1$, $\|M_{a_1,a_1}\| \leq \|a_1\|^2$, when $n = 2$, a positive number c that satisfies (2) does not exist. That means $\|M_{a_2,a_2}\| \leq \frac{1}{2}\|a_2\|^2$, when $c = \frac{1}{2}$. For n=3 and $c = \frac{1}{3}$ get $\|M_{a_3,a_3}\| \leq \frac{1}{3}\|a_3\|^2$, similarly $\|M_{a_n,a_n}\| \leq \frac{1}{n}\|a_n\|^2$, put $x_n = \frac{a_n}{\|a_n\|}$, we get (x_n) sequence in A such that $\|x_n\| = 1$, using (1) there exists abounded sequence (y_n) in A such that $(x_n y_n x_n)$ does not converge to zero, without losing generality assume $\|y_n\| \neq 0$ for all $n \in \mathbb{N}$ are taken.
Now $\lim_{n\to\infty}\|M_{x_n,x_n}\| \leq \lim_{n\to\infty}\frac{\|x_n\|^2}{n} = 0$. So $\lim\limits_{n\to\infty}\frac{\|x_n y_n x_n\|}{\|y_n\|} \leq \lim_{n\to\infty}\|M_{x_n,x_n}\|$.

We get the sequence $(x_n y_n x_n)$ converges to zero, then we get a contradiction, that means (2) is true.

Conversely, let (2) be true and (x_n) be any sequence in A with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, when $n = 1, x_1 \in A$, using (2), $c\|x_1\|^2 \leq \|M_{x_1,x_1}\| = \sup_{x\in A}\{\|x_1 x x_1\|, \|x\| = 1\}$, for any $\epsilon \geq 0$ there exists at least one element $y_1 \in A$ such that
$\|x_1 y_1 x_1\| \geq \sup_{x\in A}\{\|x_1 x x_1\|, \|x\| = 1\} - \epsilon = \|M_{x_1,x_1}\| - \epsilon \geq c\|x_1\|^2 - \epsilon$
In a special case, we take that $\|y_1\| = 1$ and $\epsilon < c$ in the term above for $y_1 \in A$. Now, when $n = 2$, $x_2 \in A$, by using (2), we get $c\|x_2\|^2 \leq \|M_{x_2,x_2}\| = \sup_{x\in A}\{\|x_2 x x_2\|, \|x\| = 1\}$, for $\epsilon < c$ there exists at least one element $y_2 \in A$ with $\|y_2\| = 1$ such that
$\|x_2 y_2 x_2\| \geq \sup_{x\in A}\{\|x_2 x x_2\|, \|x\| = 1\} - \epsilon = \|M_{x_2,x_2}\| - \epsilon \geq c\|x_2\|^2 - \epsilon$
Similarly, we get a bounded sequence (y_n) in A, so $c\|x_n\|^2 - \epsilon \leq \|x_n y_n x_n\|$ since $\|x_n\| = 1$, for all $n \in \mathbb{N}$. Therefore, $c - \epsilon \leq \|x_n y_n x_n\|$ for all $n \in \mathbb{N}$, since $0 < c - \epsilon$ that means the sequence $(x_n y_n x_n)$ does not convergent to zero.

3. Finite direct product of ultrasemiprime algebras

The direct product of prime algebras needs not to be prime [8, Example 2.33], and the direct product of ultraprime algebras needs not to be ultraprime. We show that it is different when the algebras are ultrasemiprime.

In [6], the authors proved that every finite dimensional normed algebra is an ultrasemiprime. The finite dimensional normed algebras are isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ [9, Theorem 2.3.1]. That means the finite direct product of $\mathbb{R}$ or $\mathbb{C}$ are ultrasemiprime.
The following Theorem gives the ultrasemiprime of a finite direct product of any ultrasemiprime algebras, which does not satisfy ultraprime algebra. In another way, it is a generalization to the ultrasemiprime of finite direct product of the fields $\mathbb{R}$ and $\mathbb{C}$, which are proved by Mohammed[6]. The direct product of algebra has taken with usual addition, scalar multiplication and multiplication.

Theorem 3.1

Let A, B be any ultrasemiprime algebras. Then $A \times B$ is an ultrasemiprime with norm $\|(a,b)\| = \max_{\substack{a\in A \\ b\in B}}\{\|a\|, \|b\|\}$

Proof:

Let A, B be ultrasemiprime normed algebras, put c_A is the constant of ultrasemiprime A, c_B is the constant of ultrasemiprime B and $c = \min\{c_A, c_B\}$. Let $(a,b) \in A \times B$, $\|(a,b)\| = \max\{\|a\|_A, \|b\|_B\}$. In general, either $\|a\| \geq \|b\|$ or $\|a\| \leq \|b\|$, the equality can be written in both cases without affecting the proof. Now, either $\|(a,b)\| = \|a\|$ or $\|(a,b)\| = \|b\|$.
In the first case, If $\|a\| \geq \|b\|$, then $\|(a,b)\| = \|a\|$. Hence,

$$\|M_{(a,b),(a,b)}\| = \sup_{(x,y)\in A\times B}\{\|(a,b)(x,y)(a,b)\|, \|(x,y)\| = 1\}$$
$$= \sup_{(x,y)\in A\times B}\{\|(axa, byb)\|, \|(x,y)\| = 1\}$$

Since $\|(axa, byb)\| = \max\{\|axa\|, \|byb\|\}$ and $\|(x,y)\| = \max\{\|x\|, \|y\|\}$, so

$$\|M_{(a,b),(a,b)}\| = \sup_{(x,y)\in A\times B}\left\{\max\{\|axa\|, \|byb\|\}, \max\{\|x\|, \|y\|\} = 1\right\}$$

Accordingly, we have four possibilities for $\|M_{(a,b),(a,b)}\|$

i. When $\|axa\| \geq \|byb\|$ and $\|x\| \geq \|y\|$
The $\|(x,y)\| = \max\{\|x\|, \|y\|\} = 1 = \|x\|$, by hypothesis $\|y\| \leq \|x\| = 1$, we have

$$\|M_{(a,b),(a,b)}\| = \sup_{(x,y)\in A\times B}\left\{\max\{\|axa\|, \|byb\|\}, \max\{\|x\|, \|y\|\} = 1\right\}$$

$= \sup_{x \in A}\{\|axa\|, \|x\| = 1\} = \|M_{a,a}\|$
$\geq c_A\|a\|^2$ since A is an ultrasemiprime
$= c_A\|(a,b)\|^2$ since $\|(a,b)\| = \|a\|$

Therefore, $\|M_{(a,b),(a,b)}\| \geq c_A\|(a,b)\|^2$

ii. When $\|axa\| \geq \|byb\|$ and $\|x\| \leq \|y\|$

The probability of equalization has been written in both cases without affecting the proof

The $\|(x,y)\| = \max \{\|x\|, \|y\|\} = 1 = \|y\|$, by hypothesis $\|x\| \leq \|y\| = 1$,

$\|M_{(a,b),(a,b)}\| = \sup_{(x,y) \in A \times B} \left\{ \begin{matrix} \max \{\|axa\|, \|byb\|\}, \max \{\|x\|, \|y\|\} = 1 \end{matrix} \right\}$
$= \sup_{x \in A}\{\|axa\|, \|x\| \leq 1\} = \|M_{a,a}\|$
$\geq c_A\|a\|^2 = c_A\|(a,b)\|^2$

Therefore, $\|M_{(a,b),(a,b)}\| \geq c_A\|(a,b)\|^2$

iii. When $\|axa\| \leq \|byb\|$ &$\|x\| \geq \|y\|$

The $\|(x,y)\| = \max \{\|x\|, \|y\|\} = 1 = \|x\|$, by hypothesis $\|y\| \leq \|x\| = 1$,

$\|M_{(a,b),(a,b)}\| = \sup_{(x,y) \in A \times B} \left\{ \begin{matrix} \max \{\|axa\|, \|byb\|\}, \max \{\|x\|, \|y\|\} = 1 \end{matrix} \right\}$
$= \sup_{y \in B}\{\|byb\|, \|y\| \leq 1\}$
$\geq \sup_{x \in A}\{\|axa\|, \|x\| = 1\}$ since $\|byb\| \geq \|axa\|$
$= \|M_{a,a}\|$
$\geq c_A\|a\|^2 = c_A\|(a,b)\|^2$

Therefore, $\|M_{(a,b),(a,b)}\| \geq c_A\|(a,b)\|^2$

iv. When $\|axa\| \leq \|byb\|$ &$\|x\| \leq \|y\|$

The $\|(x,y)\| = \max \{\|x\|, \|y\|\} = 1 = \|y\|$, by hypothesis $\|x\| \leq \|y\| = 1$,

$\|M_{(a,b),(a,b)}\| = \sup_{(x,y) \in A \times B} \left\{ \begin{matrix} \max \{\|axa\|, \|byb\|\}, \max \{\|x\|, \|y\|\} = 1 \end{matrix} \right\}$
$= \sup_{y \in B}\{\|byb\|, \|y\| = 1\}$
$\geq \sup_{x \in A}\{\|axa\|, \|x\| \leq 1\}$ since $\|byb\| \geq \|axa\|$
$= \|M_{a,a}\| \geq c_A\|a\|^2 = c_A\|(a,b)\|^2$

$\|M_{(a,b),(a,b)}\| \geq c_A\|(a,b)\|^2$. For the four possibilities, $\|M_{(a,b),(a,b)}\| \geq c_A\|(a,b)\|^2$ when $\|a\| \geq \|b\|$.

The second case, when $\|a\| \leq \|b\|$, then $\|(a,b)\| = \|b\|$, we also have four possibilities for $\|M_{(a,b),(a,b)}\|$

i. When $\|axa\| \geq \|byb\|$ & $\|x\| \geq \|y\|$

The $\|(x,y)\| = \max \{\|x\|, \|y\|\} = 1 = \|x\|$, by hypothesis $\|y\| \leq \|x\| = 1$, we have

$\|M_{(a,b),(a,b)}\| = \sup_{(x,y) \in A \times B} \left\{ \begin{matrix} \max \{\|axa\|, \|byb\|\}, \max \{\|x\|, \|y\|\} = 1 \end{matrix} \right\}$
$= \sup_{x \in A}\{\|axa\|, \|x\| = 1\}$
$\geq \sup_{y \in B}\{\|byb\|, \|y\| \leq 1\}$ since $\|axa\| \geq \|byb\|$
$= \|M_{b,b}\| \geq c_B\|b\|^2$ since B is an ultrasemiprime
$= c_B\|(a,b)\|^2$ since $\|(a,b)\| = \|b\|$

$\|M_{(a,b),(a,b)}\| \geq c_B\|(a,b)\|^2$. The proof of the rest of the possibilities is similar to first case $\|M_{(a,b),(a,b)}\| \geq c_B\|(a,b)\|^2$. From the first and second case, we have $\|M_{(a,b),(a,b)}\| \geq c\|(a,b)\|^2$ for all $(a,b) \in A \times B$

In Theorem 3.1, we proved that the finite direct product of ultrasemiprime algebras depends on its definition using the maximum norm between the finite direct product of algebras. The following Theorem depends on the equivalent condition of ultrasemiprime given in Theorem 2.2(1) using the sum norm between the finite direct product of algebras.

Theorem 3.2

Let A, B be any ultrasemiprime algebras. Then $A \times B$ is an ultrasemiprime, with norm defined by $\|(a, b)\| = \|a\| + \|b\|$.

Proof:

Let $(w_n), n \in \mathbb{N}$ be a sequence in $A \times B$, such that $\|w_n\| = 1$ for all $n \in \mathbb{N}$, so there exists a sequence $(a_n), n \in \mathbb{N}$ in A and $(b_n), n \in \mathbb{N}$ in B such that $w_n = (a_n, b_n), n \in \mathbb{N}$.
Assume that $A \times B$ is not an ultrasemiprime algebra. From Theorem 2.2(1) , any bounded sequence $(v_n), n \in \mathbb{N}$ in $A \times B$ does not exist that makes the sequence $(w_n v_n w_n)$ not converge to zero. That means for any bounded sequence $(v_n), n \in \mathbb{N}$ in A the sequence $(w_n v_n w_n)$ is convergent to zero.
There are three possibilities, without losing generality; elements equal to zero are not considered.

i. When $a_n = 0$, and $b_n \neq 0$ for all $n \in \mathbb{N}$
ii. When $b_n = 0$, and $a_n \neq 0$ for all $n \in \mathbb{N}$
iii. When $a_n \neq 0$, and $b_n \neq 0$ for all $n \in \mathbb{N}$

The first possibility, when $a_n = 0$, and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $w_n = (0, b_n)$, now $\left(\frac{b_n}{\|b_n\|}\right)$, $n \in \mathbb{N}$ is a sequence in B. Since B is an ultrasemiprime algebra from Theorem 2.2(1), there exists a bounded sequence $(y_n), n \in \mathbb{N}$ in B with a bound c_y such that $\left(\frac{b_n}{\|b_n\|} y_n \frac{b_n}{\|b_n\|}\right), n \in \mathbb{N}$ does not converge to zero. Also, $\left(\frac{b_n}{\|b_n\|} y_n \frac{b_n}{\|b_n\|}\right) = \frac{1}{\|b_n\|^2}(b_n y_n b_n)$ does not converge to zero, that means $(b_n y_n b_n)$ does not converge to zero.
Define $(v_n), n \in \mathbb{N}$, by $v_n = (0, y_n)$ where $(y_n), n \in \mathbb{N}$ is a sequence in B. (v_n) is a bounded sequence because $\|v_n\| = \|y_n\| \leq c_y$
Now, $(w_n v_n w_n)$ are convergent to zero, so $(w_n v_n w_n) = (0, b_n)(0, y_n)(0, b_n) = (0, b_n y_n b_n)$, $(w_n v_n w_n)$ are convergent to zero, then must $(b_n y_n b_n)$ are convergent to zero, which is contradiction so $(w_n v_n w_n)$ does not convergent to zero.
The second possibility, when $b_n = 0$, and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $w_n = (a_n, 0)$, the proof of this possibility is similar to the first one.
The third possibility, when $a_n \neq 0$, and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $w_n = (a_n, b_n)$, now $\left(\frac{a_n}{\|a_n\|}\right)$, $n \in \mathbb{N}$ is a sequence in A, since A is an ultrasemiprime algebra from Theorem 2.2(1), there exists a bounded sequence $(x_n), n \in \mathbb{N}$ in A with a bound c_x such that $\left(\frac{a_n}{\|a_n\|} x_n \frac{a_n}{\|a_n\|}\right), n \in \mathbb{N}$ does not converge to zero. $\left(\frac{a_n}{\|a_n\|} x_n \frac{a_n}{\|a_n\|}\right) = \frac{1}{\|a_n\|^2}(a_n x_n a_n)$ does not converge to zero, that means $(a_n x_n a_n)$ does not converge to zero. In a similar way for the sequence (b_n) $n \in \mathbb{N}$ in B, there exists a bounded sequence $(y_n), n \in \mathbb{N}$ in B with a bound c_y and $(b_n y_n b_n)$ does not converge to zero.

Define $(v_n), n \in \mathbb{N}$, by $v_n = (x_n, y_n)$ where $(x_n), n \in \mathbb{N}$ is a sequence in A and $(y_n), n \in \mathbb{N}$ is a sequence in B. (v_n) is a bounded sequence, because $\|v_n\| = \|(x_n, y_n)\| = \|x_n\| + \|y_n\| \leq c_x + c_y$.

Now $(w_n v_n w_n)$ is convergent to zero, so
$(w_n v_n w_n) = (a_n, b_n)(x_n, y_n)(a_n, b_n) = (a_n x_n a_n, b_n y_n b_n)$,
$(w_n v_n w_n)$ is convergent to zero, then $(a_n x_n a_n)$ and $(b_n y_n b_n)$ must converge to zero, which is a contradiction, so that $(w_n v_n w_n)$ does not converge to zero. Therefore, $A \times B$ is an ultrasemiprime algebra.

Corollary 3.3

The finite direct product of ultrasemiprime algebras is ultrasemiprime with sum norm or maximum norm.

Corollary 3.4

The finite direct product of ultraprime algebras is ultrasemiprime with sum norm or maximum norm.

Corollary 3.5

Every finite dimensional normed algebras is ultrasemiprime.

Proof:

Every finite dimensional normed algebra is isomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$ [9, Theorem 2.3.1]. From corollary 3.3, $\mathbb{R}^n$ and $\mathbb{C}^n$ are ultrasemiprime with maximum or sum norm. But the norms under finite dimensional normed algebras are equivalent [10, Theorem 2.4.5]. Therefore, $\mathbb{R}^n$ and $\mathbb{C}^n$ are ultrasemiprime with any norm.

Conclusions

In this work, a generalization of the ultrasemiprime proof of the finite direct product of $\mathbb{R}$ and $\mathbb{C}$ to the finite direct product of any ultrasemiprime algebras is given. Some related results and properties are also given and discussed.

Reference

[1] M. Mathieu, "Applications of ultraprime Banach algebras in the theory of elementary operators," Ph.D. dissertation, University of Tübingen, Germany, 1986.

[2] M. Cabrera and Á. Rodríguez, *Non-associative normed algebras. Volume 2: Representation Theory and the Zel'manov Approach*, vol. 167. Cambridge University Press, 2018. [Online]. Available: https://www.cambridge.org/core/books/nonassociative-normed-algebras/frontmatter/54B902AB8D6BE32FD5C916BACE13EF6C

[3] P. Ara and M. Mathieu, "On ultraprime Banach algebras with non-zero socle," 1991, pp. 89–98. [Online]. Available: https://www.jstor.org/stable/20489379

[4] M. C. GARCÍA and A. R. PALACIOS, "Nonassociative ultraprime normed algebras," *The Quarterly Journal of Mathematics*, vol. 43, no. 1, pp. 1–7, 1992, doi: https://doi.org/10. 1093 /qmath/43.1.1.

[5] M. Mathieu, "The symmetric algebra of quotients of an ultraprime Banach algebra," *Journal of the Australian Mathematical Society*, vol. 50, no. 1, pp. 75–87, 1991, doi: https://doi.org/10.1017 /S1446788700032560.

[6] A. A. Mohammed, "On Ultrasemiprime Algebras (Research Note)," *Dirasat: Pure Sciences*, vol. 33, no. 1, Art. no. 1, Jun. 2010.

[7] M. Bresar, "On the distance of the composition of two derivations to the generalized derivations," *Glasgow Mathematical Journal*, vol. 33, no. 1, pp. 89–93, 1991, doi: https://doi.org/10.1017 /S0017089500008077.

[8] M. Brešar, *Introduction to noncommutative algebra*. Switzerland: Springer, 2014. [Online]. Available: https://www.springer.com/gp/book/9783319086927.

[9] R. Sen, *A first course in functional analysis: Theory and applications*. London, SE1, 8HA, UK and New Yourk, NY 10016,USA: Anthem Press, 2013. [Online]. Available: https://www.jstor.org/stable/j.ctt1gxpbqd.

[10] E. Kreyszig, *Introductory functional analysis with applications*, vol. 1. New York: John Wiley & Sons. Inc., 1978. [Online]. Available: https://www.wiley.com/en-us/Introductory+Functional +Analysis+with+Applications-p-9780471504597.