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On Large-Lifting and Large-Supplemented Modules

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Abstract

In this paper, we introduce the concepts of Large-lifting and Large-supplemented modules as a generalization of lifting and supplemented modules. We also give some results and properties of this new kind of modules.

Keywords. L-small, L-lifting module, L-supplemented module.

حول مقاسات الرفع الاساسية والمقاسات المكملة الاساسية

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الخلاصة

الغرض من هذا البحث هو تقديم مفاهيم حول مقاسات الرفع الاساسية والمقاسات المكملة الاساسية وهي تعميم لمقاسات الرفع والمقاسات المكملة , وسوف نقوم بأستعراض بعض الخواص والنتائج لهذا النوع الجديد من المقاسات.

1. Introduction

Throughout this paper, we assume that R is a commutative ring with identity. A submodule N of an R-module M is called Large (essential) submodule in M, $(N \leq_e M)$. if for every nonzero submodule K of M, then $N \cap K \neq 0$ [1]. A proper submodule N of an R-module M is called small $(N \ll M)$, if for any submodule K of M such that $N + K = M$ implies that $K = M$ [1].Assume that N and K are submodules of M, where M *is* R module, then N is called supplement of K in M, if N is minimal with respect to the property $M = N + K$. This is equivalent to $M = N + K$ and $N \cap K \ll N$, if every submodule of M has a supplement in M, then M is called supplemented module [2]. An R-module M is called lifting, if for every submodule N of M there exists a submodule K of N such that $M = K \oplus H$ and $N \cap H \ll H$ where H be a submodule of M ,equivalently M is called lifting, if and only if for every submodule N of M there exists a submodule K of N such that $M = K \oplus H$ and $N \cap H \ll M$ [2]. In [3], we give the concept of Large-small (L-small) submodule , it is given as follows; Let N be a proper submodule of M , then N is called L-small submodule of M $(N \ll_L M)$, if $N + K = M$ where $K \leq M$, then K is essential submodule of M ($K \leq_e M$). In [4], we also give the concept of Large-coessential (L-coessential) submodule . It is given as follows; Let M be an R-module and K, N are submodules of M such that $K \leq N \leq M$,then K is said to be Large-coessential submodule, if $\frac{N}{K} \ll_L \frac{M}{K}$. This paper consists two sections, in section one we

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give the concept of Large-lifting (L-lifting) modules and some of its properties, such that an R-module M is said to be L-lifting, if for every submodule N of M there exists a submodule K of N such that $M = K \oplus H$ and $N \cap H \ll_L M$ where H is a submodule of M. In section two we introduce the concept of Large-supplemented (L-supplemented) modules, such that an R-module M is called L-supplemented, if every submodule of M has L-supplement in M , where a submodule N is called L-supplement of K in M, if $M = N + K$ and $N \cap K \ll_L N$. In Lemma(1.1), Lemma(1.2) and Lemma(1.3) we give some properties in [3] and [4] that we need it in this paper.

Lemma1.1[3]: (1) Let M be an R-module and K, N be submodules of M such that $K \leq N \leq M$, if $N \ll_L M$ then $K \ll_L M$.

(2) Let $f: M \to M`$ be an epimorphism where $M$ and $M`$ are an R-modules such that $N \ll_L M`$ then $f^{-1}(N) \ll_L M$.

(3) Let M be an R-module and K, N be submodules of M where K is a closed in M such that $K \leq N \leq M$, if $N \ll_L M$ then $K \ll_L M$ and $\frac{N}{K} \ll_L \frac{M}{K}$.

(4) Let M be an R-module and K, N be submodules of M such that $K \leq N \leq M$, and N is direct summand of M , if $K \ll_L M$, then $K \ll_L N$.

Lemma1.2[3]: (1) Let $M = \oplus_{i \in I} M_i$ be fully stable module, if $N_i \ll_L M_i$ then $\oplus_{i \in I} N_i \ll_L \oplus_{i \in I} M_i$.

(2) Let M be an R-module such that M is faithful, finitely generated and multiplication module and let I be an ideal of R then $I \ll_L R$ if and only if $IM \ll_L M$.

Lemma1.3[4]: (1) Let M be an R-module and K, N be submodules of M such that $K \leq N \leq M$, if $\frac{N}{K} \ll_L \frac{M}{K}$ then $N \ll_L M$.

(2) Let M be an R-module and K, N , and U be submodules of M such that $K \leq N \leq U \leq M$, then $N \leq_{L.ce} U$ in M if and only if, $\frac{N}{K} \leq_{L.ce} \frac{U}{K}$ in $\frac{M}{K}$.

Now, we need to prove the following lemma.

Lemma1.4: Let $M = M_1 \oplus M_2$ then $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$ if and only if, $N_1 \oplus N_2 \ll_L M_1 \oplus M_2$.

Proof: ($\Longrightarrow$) Let $U_1 \oplus U_2$ be a submodule of $M_1 \oplus M_2$ such that $N_1 \oplus N_2 + U_1 \oplus U_2 = M_1 \oplus M_2$. So that $(N_1 + U_1) \oplus (N_2 + U_2) = M_1 \oplus M_2$ and hence $N_1 + U_1 = M_1$ and $N_2 + U_2 = M_2$. Since $N_1 \ll_L M_1$ and $N_2 \ll_L M_2$, then $U_1 \leq_e M_1$ and $U_2 \leq_e M_2$, this implies that $U_1 \oplus U_2 \leq_e M_1 \oplus M_2$ by [1], and therefore $N_1 \oplus N_2 \ll_L M_1 \oplus M_2$.

($\Longleftarrow$) Let $N_1 \oplus N_2 \ll_L M_1 \oplus M_2 = M$. Since $N_1 \leq N_1 \oplus N_2 \ll_L M_1 \oplus M_2 = M$ then by Lemma(1.1), we have $N_1 \ll_L M$ and since $N_1 \leq M_1 \leq M$ and M_1 is direct summand of M then by Lemma(1.1) we get $N_1 \ll_L M_1$. Similarly we have $N_2 \ll_L M_2$.

2. Large-Lifting modules.

In this section we introduce the concept of Large-lifting modules and some properties of it are considered.

Definition 2.1: An R-module M is called Large-lifting (L-lifting), if for every submodule N of M there exists a submodule K of N such that $M = K \oplus H$ and $N \cap H \ll_L M$ where H is a submodule of M.

Remarks and Examples 2.2:

(1) Every lifting is L-lifting.

Proof: Let M be a lifting module and $N \leq M$, then $M = K \oplus H$ where $K \leq N$ and $N \cap H \ll M$ so $N \cap H \ll_L M$ where $H \leq M$ by [3].

(2) The following example shows that the converse of (1) is not true.

Example: Z as Z- module is L-lifting since for $N = nZ \leq Z$, there exists $\{\bar{0}\}$ direct summand of nZ such that $M = Z = \{\bar{0}\} + Z$ and $nZ \cap Z = nZ \ll_L Z$ by [3], also if $N = Z \leq Z$, let $K = Z \leq N$ such that $Z = Z \oplus \{\bar{0}\}$ and $Z \cap \{\bar{0}\} = \{\bar{0}\} \ll_L Z$, but Z is not lifting since nZ no

t small submodule in Z.
(3) Z_{24} as Z-module is not L-lifting since, Let $N = Z_{24}$, the only direct summand of Z_{24} are $\{\bar{0}\}$ and $3Z_{24}$, $8Z_{24}$ such that $Z_{24} = K \oplus H$. If $K = \{\bar{0}\}$ thus $H = Z_{24}$ and $N \cap H = Z_{24} \cap Z_{24} = Z_{24}$ which is not L-small in Z_{24} and if $K = 3Z_{24}$ thus $H = 8Z_{24}$ and $Z_{24} \cap 8Z_{24} = 8Z_{24}$ which is not L-small in Z_{24} and if $K = 8Z_{24}$ thus $H = 3Z_{24}$ and $Z_{24} \cap 3Z_{24} = 3Z_{24}$ which is not L-small in Z_{24}.
(4) Every semisimple module is lifting [2], hence L-lifting by (1). Thus Z_6 as Z-module is L-lifting.
(5) Let M be a semisimple module, then M is lifting if and only if M is L-lifting.
(6) Every hollow module is lifting [2], hence L-lifting by (1). Thus Z_4 as Z-module is hollow, so it is L-lifting.

Recall that an R-module M is called L-hollow module if every proper submodule of M is L-small submodule in M [3].

Remark 2.3: Every L-hollow module is L-lifting.

Proof: Let M be L-hollow module and N be a proper submodule of M and let $M = \{\bar{0}\} \oplus M$ and $N \cap M = N \ll_L M$, so that M is L-lifting.

The converse of previous remark is not true, the following example: Z_6 as Z-module is L-lifting by (4) but not L-hollow by [3].

Remark 2.4: Every Local module is hollow so L-hollow [3], hence it is L-lifting by Remark(2.3), where an R-module M is called local if it is hollow and has a unique maximal submodule [5].

Proposition 2.5: Let M be an indecomposable, then M is L-hollow if and only if M is L-lifting.

Proof: ($\Rightarrow$) Clear from Remark (2.3).
($\Leftarrow$) Let M is L-lifting and N be a proper submodule of M and let $K \leq N$ such that $M = K \oplus H$ where $H \leq M$ and $N \cap H \ll_L M$, since M is indecomposable, then either $K = 0$ or $K = M$. If $K = M$ then $N = M$ and this is a contradiction, so that $K = 0$, and hence $M = H$, so $N = N \cap M = N \cap H \ll_L M$ hence $N \ll_L M$. Therefore M is L-hollow.

The characterization of L-lifting module is given by the next theorem.

Theorem 2.6: Let M be an R-module, then the following statements are equivalent:
1- M is L-lifting module .
2- Every submodule N of M can be written as $N = V \oplus W$ where V direct summand of M and $W \ll_L M$.
3- Every submodule N of M there exists a direct summand K of M such that $K \leq N$ and $\frac{N}{K} \ll_L \frac{M}{K}$.

Proof: (1) $\Rightarrow$ (2) Let N be a submodule of M then there exists a submodule K of N such that $M = K \oplus H$ and $N \cap H \ll_L M$ where H is a submodule of M. Now $N = N \cap M = N \cap (K \oplus H) = K \oplus (N \cap H)$by modular law. Let $V = K$ and $W = N \cap H$, so $N = V \oplus W$ where V direct summand of M and $W \ll_L M$.
(2) $\Rightarrow$ (3) Let N be a submodule of M and $N = V \oplus W$ where V direct summand of M and $W \ll_L M$. It is enough to show that $\frac{N}{V} \ll_L \frac{M}{V}$. Let $\frac{U}{V} \leq \frac{M}{V}$ such that $\frac{N}{V} + \frac{U}{V} = \frac{M}{V}$ so $\frac{V \oplus W}{V} + \frac{U}{V} = \frac{M}{V}$, hence $M = V + W + U = W + U$. Since $W \ll_L M$, then $U \leq_e M$,and since V direct summand of M then V is closed in M , from [6-10], we have $\frac{U}{V} \leq_e \frac{M}{V}$, so that $\frac{N}{V} \ll_L \frac{M}{V}$.
(3) $\Rightarrow$ (1) Let N be a submodule of M then there exists a submodule K of N such that $M = K \oplus H$ and $\frac{N}{K} \ll_L \frac{M}{K}$. By Lemma(1.3), we have $N \ll_L M$ by and since $N \cap H \leq N \leq M$ so we get $N \cap H \ll_L M$ by Lemma(1.1).

Proposition 2.7: Let M be an indecomposable module, then M is not L-lifting for every nontrivial submodule N of M.

Proof: Suppose that M is L-lifting and by theorem (2.6), let $N = K + H$ where K direct summand of M and $H \ll_L M$, since M be an indecomposable then $K = 0$, hence $N = H \ll_L M$ and this is contradiction , so M is not L-lifting for every nontrivial submodule N of M.

Proposition 2.8: Any direct summand of L-lifting module is L-lifting.

Proof: Let M be L-lifting and assume that $M = M_1 \oplus M_2$. In order to show M_1 is L-lifting, let $N \leq M_1$ so that $N \leq M$ and by theorem (2.6), let $N = V \oplus W$ where V direct summand of M and $W \ll_L M$ hence $W \ll_L M_1$ by Lemma(1.1). Now, $M = V \oplus H$ where $H \leq M$, since V direct summand of M, then we get the result if we prove V direct summand of M_1, so $M_1 = M_1 \cap M = M_1 \cap (V \oplus H) = V \oplus (M_1 \cap H)$ by modular law, hence V direct summand of M_1, so M_1 is L-lifting.

Theorem 2.9: Let M be an R-module, then the following statements are equivalent:

1- M is L-lifting module.

2- For each submodule N of M, there exists $\varphi \in \mathrm{End}(M)$ such that $\emptyset^2 = \emptyset$, $\emptyset(M) \leq N$ and $(1-\emptyset)(N) \ll_L M$.

Proof: $(1) \Rightarrow (2)$ Let N be a submodule of M then there exists a submodule K of N such that $M = K \oplus H$ and $N \cap H \ll_L M$ where H be a submodule of M. Let $\emptyset : M \to K$ be a projection map clearly $\emptyset^2 = \emptyset$ and $M = K \oplus H = \emptyset(M) \oplus (1-\emptyset)(M)$, $\emptyset(M) \leq N$. Now $(1-\emptyset)(N) = N \cap (1-\emptyset)(M) = N \cap H \ll_L M$, so $(1-\emptyset)(N) \ll_L M$.

$(2) \Rightarrow (1)$ Let N be a submodule of M then there exists $\emptyset \in \mathrm{End}(M)$ such that $\emptyset^2 = \emptyset$, $\emptyset(M) \leq N$ and $(1-\emptyset)(N) \ll_L M$. Clearly that $M = \emptyset(M) \oplus (1-\emptyset)(M)$, let $K = \emptyset(M)$ and $H = (1-\emptyset)(M)$, hence $N \cap H = N \cap (1-\emptyset)(M)$. To show that $N \cap (1-\emptyset)(M) = (1-\emptyset)(N)$, let $u = (1-\emptyset)(v) \in N \cap (1-\emptyset)(M)$, since $(1-\emptyset)^2 = (1-\emptyset)$ so $u = (1-\emptyset)^2(v) = (1-\emptyset)(v) \in (1-\emptyset)(N)$. Now let $u = (1-\emptyset)(v) \in (1-\emptyset)(N)$; $v \in N$, then $u \in (1-\emptyset)(M)$, $u = (1-\emptyset)(v) \in N$, hence $u \in N \cap (1-\emptyset)(M)$ so $N \cap H = N \cap (1-\emptyset)(M) = (1-\emptyset)(N) \ll_L M$, hence $N \cap H \ll_L M$, so M is L-lifting module.

Remark 2.10: The following example shows that if M is L-lifting module and N is a submodule of M, then $\frac{M}{N}$ need not to be L-lifting module.

Example: Let Z be L-lifting module and $24Z \leq Z$ but $\frac{Z}{24Z} \simeq Z_{24}$ which is not L-lifting by (2.2).

Now, we introduce the following proposition in which $\frac{M}{N}$ be L-lifting module.

Proposition 2.11: Let M be L-lifting module and W be a submodule of M such that for every direct summand K of M, $\frac{K+W}{W}$ direct summand of $\frac{M}{W}$, then $\frac{M}{W}$ is L-lifting.

Proof: Let $\frac{N}{W} \leq \frac{M}{W}$, since M is L-lifting, then by theorem (2.6), there exists $K \leq N$ such that $M = K \oplus H$; $H \leq M$ and $\frac{N}{K} \ll_L \frac{M}{K}$, because of $K + W$ is direct summand of M, we have $\frac{N}{K+W} \ll_L \frac{M}{K+W}$ so $K + W \leq_{L.ce} N$ in M and by Lemma(1.3), we get $\frac{K+W}{W} \leq_{L.ce} \frac{N}{W}$ in $\frac{M}{W}$, hence $\frac{N/W}{(K+W)/W} \ll_L \frac{M/W}{(K+W)/W}$, therefore $\frac{M}{W}$ is L-lifting.

An R-module M is called distributive, if for all submodules K, N and U of M, then $K \cap (N + U) = (K \cap N) + (K \cap U)$ [9].

Corollary 2.12: Let M be L-lifting and distributive module and let W be a submodule of M then $\frac{M}{W}$ is L-lifting.

Proof: Let K be a direct summand of M, such that $M = K \oplus U$ for some submodule U of M, hence $\frac{M}{W} = \frac{K \oplus U}{W} = \frac{K+W}{W} + \frac{U+W}{W}$ and since M is distribution module, then $(K + W) \cap$

$(U+W) = \big((K+W)\cap U\big) + \big((K+W)\cap W\big) = (K\cap U) + (W\cap U) + (K\cap W) + W = W$, hence $\frac{M}{W} = \frac{K+W}{W} \oplus \frac{U+W}{W}$ and by proposition (2.11), we get $\frac{M}{W}$ is L-lifting.

Lemma 2.13 [6]: Let $M = M_1 \oplus M_2$ be an R-module, then $\frac{M}{A} = \frac{A+M_1}{A} \oplus \frac{A+M_2}{A}$ for every fully invariant submodule *A* of *M*.

Corollary 2.14: Let *M* be L-lifting module if *W* is fully invariant submodule of *M* then $\frac{M}{W}$ is L-lifting.

Proof: It directly comes from Lemma (2.13) and proposition (2.11).

3. Large-Supplemented modules

In this section we introduce the concept of Large-supplemented modules. Some results are also given .

Definition 3.1: Let *M* be an R-module and *N*, *K* are submodules of *M*, then *N* is called Large-supplement (L-supplement) of *K* in *M*, if $M = N + K$ and $N \cap K \ll_L N$. If every submodule of *M* has L-supplement, then *M* is called L-supplemented module.

Remarks and Examples 3.2:

(1) Every supplemented module is L-supplemented.

Proof: Let *M* be a supplemented and *N* be a submodule of *M*, then *N* is a supplement of *K* in *M*, so $M = N + K$ and $N \cap K \ll N$ hence $N \cap K \ll_L N$ by [3], so *N* is L-supplement of *K* in *M*, hence *M* is L-supplemented.

(2) Next example indicates that the converse of (1) is not true.

Example: Z as Z-module is L-supplemented since let $n, m \in N$, nZ is L-supplement of mZ since $Z = nZ + mZ$ and $nZ \cap mZ = (nm)Z \ll_L nZ$, but Z is not supplemented since nZ is not supplement in Z since $Z = nZ + mZ$ and $nZ \cap mZ = (nm)Z$ but $(nm)Z$ is not small in nZ, since $\{\bar{0}\}$ is the only small submodule.

(3) Let *M* be a semisimple module, then *M* is supplemented if and only if, *M* is L-supplemented.

(4) Next example shows that if *N* and *K* are submodules of *M*, and *N* is L-supplement of *K* in *M*, then it is not necessary that *K* is L-supplement of *N* in *M*.

Example: In Z_4 as Z-module, Z_4 is L-supplement of $\{\bar{0},\bar{2}\}$ in Z_4 since $Z_4 = Z_4 + \{\bar{0},\bar{2}\}$ and $Z_4 \cap \{\bar{0},\bar{2}\} = \{\bar{0},\bar{2}\} \ll_L Z_4$ but $\{\bar{0},\bar{2}\}$ is not L-supplement of Z_4 in Z_4 since $Z_4 = \{\bar{0},\bar{2}\} + Z_4$ and $\{\bar{0},\bar{2}\} \cap Z_4 = \{\bar{0},\bar{2}\}$ but $\{\bar{0},\bar{2}\}$ is not L-small in $\{\bar{0},\bar{2}\}$.

(5) In Z_6 as Z-module where $Z_6 = \{\bar{0},\bar{3}\} \oplus \{\bar{0},\bar{2},\bar{4}\}$ then $\{\bar{0},\bar{3}\}$ is L-supplement of $\{\bar{0},\bar{2},\bar{4}\}$ since $Z_6 = \{\bar{0},\bar{3}\} + \{\bar{0},\bar{2},\bar{4}\}$ and $\{\bar{0},\bar{3}\} \cap \{\bar{0},\bar{2},\bar{4}\} = \{\bar{0}\} \ll_L \{\bar{0},\bar{3}\}$ also $\{\bar{0},\bar{2},\bar{4}\}$ is L-supplement of $\{\bar{0},\bar{3}\}$.

(6) Every semisimple module is L-supplemented.

(7) In [2], authors proved that every direct summand of *M* is supplement submodule of *M* , hence it is L-supplement by (1).

(8) Let *M* be an R-module and *N* be L-hollow of *M*, then *N* is L-supplement of each proper submodule *K* of *M* such that $M = N + K$.

Proof: Let *K* be a proper submodule of *M* such that $M = N + K$. It is clear that $N \cap K \neq N$, since if $N \cap K = N$, then $N \leq K$ hence $K = M$ and this is a contradiction. Since *N* is L-hollow then $N \cap K \ll_L N$, so *N* is L-supplement of *K* in *M*.

(9) Let *M* be an R-module, then every L-small submodule of *M* has L-supplement in *M*.

Proof: Let *N* be L-small submodule of *M*, so $that\, M = N + M$ and $N \cap M = N \ll_L M$, therefore *M* is L-supplement of *N* in *M*.

(10) The converse of (9) is not true, for example Z_6 as Z-module.

Proposition 3.3: Let *M* be an R-module and *N*, *K* be submodules of *M* such that $N \leq K \leq M$ and *N* is closed in *K*, if *K* is L-supplement of *H* in *M* then $\frac{K}{N}$ is L-supplement of $\frac{H+N}{N}$ in $\frac{M}{N}$.

Proof: Since *K* is L-supplement of *H* in *M*, then we have $M = K + H$ and $K \cap H \ll_L K$. Now

$\frac{M}{N}=\frac{K+H}{N}=\frac{K}{N}+\frac{H+N}{N}$, we have to show that $\frac{K}{N}\cap\frac{H+N}{N}\ll_L\frac{K}{N}$, so that $\frac{K}{N}\cap\frac{H+N}{N}=\frac{K\cap(H+N)}{N}=\frac{(K\cap H)+N}{N}$ by modular law. Let $\frac{U}{N}\leq\frac{K}{N}$ where $U\leq K$ and $N\leq U$ such that $\frac{(K\cap H)+N}{N}+\frac{U}{N}=\frac{K}{N}$, so $\frac{(K\cap H)+N+U}{N}=\frac{K}{N}$ hence $(K\cap H)+N+U=K$ and since $N\leq U$ we have $(K\cap H)+U=K$, since $K\cap H\ll_L K$ then $U\leq_e K$ but $N\leq U\leq K$ and *N* is closed in *K*. from [10-15], we get $\frac{U}{N}\leq_e\frac{K}{N}$, therefore $\frac{K}{N}$ is L-supplement of $\frac{H+N}{N}$ in $\frac{M}{N}$.

Proposition 3.4: Let $f:M\rightarrow M`$ be an epimorphism, if *M*` is L-supplemented module then *M* is L-supplemented.

Proof: Let $H\leq M$, then $f(H)\leq M`$, since *M*` is L-supplemented then there exists K is L-supplement of $f(H)$ in $M`$, so $M`=K+f(H)$ and $K\cap f(H)\ll_L K$. Now $f^{-1}\big(K+f(H)\big)=f^{-1}(M`)$ hence $f^{-1}(K)+H=M$ and since $K\cap f(H)\ll_L K$ then $f^{-1}\big(K\cap f(H)\big)\ll_L f^{-1}(K)$ by Lemma(1.1), hence $f^{-1}(K)\cap H\ll_L f^{-1}(K)$ so, $f^{-1}(K)$ is L-supplement of *H* in *M*, hence *M* is L-supplemented.

Proposition 3.5: Let *M* be an R-module and *N*, *K* are submodules of *M* such that *K* is L-supplement of *N* in *M*, if $M=H+K$ for some submodule *H* of *N*, then *K* is L-supplement of *H* in *M*.

Proof: Suppose $M=H+K$ for some submodule *H* of *N* and *K* is L-supplement of *N* in *M*, so we have $M=N+K$ and $N\cap K\ll_L K$, and since $H\cap K\leq N\cap K\ll_L K$, then $H\cap K\ll_L K$ by Lemma(1.1), hence *K* is L-supplement of *H* in *M*.

Proposition 3.6: Let *M* be an R-module and *N*, K and U are submodules of *M* such that $N\leq K$, if *N* is L-supplement of U in *M* then *N* is L-supplement of $U\cap K$ in *K*.

Proof: Since *N* is L-supplement of U in *M* then we have, $M=N+U$ and $N\cap U\ll_L N$. Now $K=M\cap K=(N+U)\cap K=N+(U\cap K)$by modular law, and since $\cap\,(U\cap K)\leq N\cap U\ll_L N$, so we get $N\cap(U\cap K)\ll_L N$ by Lemma(1.1), hence *N* is L-supplement of $U\cap K$ in *K*.

Proposition 3.7: Let $M=M_1\oplus M_2$, if N_1 is L-supplement of N_2 in M_1 and K_1 is L-supplement of K_2 in M_2, then $N_1\oplus K_1$ is L-supplement of $N_2\oplus K_2$ in *M*.

Proof: Since N_1 is L-supplement of N_2 in M_1 and K_1 is L-supplement of K_2 in M_2, then we have $M_1=N_1+N_2$ and $N_1\cap N_2\ll_L N_1$,we also have $M_2=K_1+K_2$ and $K_1\cap K_2\ll_L K_1$, so $M=M_1\oplus M_2=(N_1+N_2)\oplus(K_1+K_2)=(N_1\oplus K_1)+(N_2\oplus K_2)$, since $N_1\cap N_2\ll_L N_1$ and $K_1\cap K_2\ll_L K_1$ then by Lemma(1.4), we have $(N_1\cap N_2)\oplus(K_1\cap K_2)\ll_L N_1\oplus K_1$.Clearly $(N_1\oplus K_1)\cap(N_2\oplus K_2)=(N_1\cap N_2)\oplus(K_1\cap K_2)\ll_L N_1\oplus K_1$, hence $N_1\oplus K_1$ is L-supplement of $N_2\oplus K_2$ in *M*.

Proposition 3.8: Let *M* be faithful, finitely generated and multiplication module over commutative ring R and *N* be a submodule of *M*, if *N* is L-supplement of *IM* in *M*, then *J* is L-supplement of I in R, where *I* , *J* are ideals of R.

Proof: Since *N* is L-supplement of *IM* in *M*, then we have $M=N+IM$ and $N\cap IM\ll_L N$, since *M* is multiplication then $N=JM$. Now $M=RM=IM+JM=(I+J)M$,and since *M* is faithful, finitely generated and multiplication, then *M* is cancellation by [8], so $R=I+J$ also we have $IM\cap N=IM\cap JM=(I\cap J)M\ll_L N=JM$, hence $(I\cap J)M\ll_L JM$. To show $I\cap J\ll_L J$, let H be an ideal of R such that $(I\cap J)+H=J$, so $(I\cap J)M+HM=JM$ and since $(I\cap J)M\ll_L JM$, then $HM\leq_e JM$ so $H\leq_e J$ so we get the result, and hence *J* is L-supplement of I in R.

The characterization of L-supplement submodules is given in the next theorem.

Theorem 3.9: Let *M* be an R-module and *N*, *K* are submodules of *M*, then the following statements are equivalent:

1- *K* is L-supplement of *N* in *M*.

2- $M=N+K$ and for every non-essential submodule *H* of *K*, then $M\neq N+H$.

Proof: $(1) \Longrightarrow (2)$ Assume K is L-supplement of N in M, so we have $M = N + K$ and $N \cap K \ll_L K$ and suppose $M = N + H$ where H is non-essential submodule of K, so $K = K \cap M = K \cap (N + H) = H + (N \cap K)$ by modular law, and since $N \cap K \ll_L K$ so we have $H \leq_e K$ and this is a contradiction, so that $M \neq N + H$.
$(2) \Longrightarrow (1)$ From (2) $M = N + K$, we must show $N \cap K \ll_L K$. Let $U \leq K$ such that $(N \cap K) + U = K$, if U is non-essential submodule of K, then by assumption $M \neq N + U$, so $M = N + K = N + (N \cap K) + U = N + U$ and this is a contradiction, so that $U \leq_e K$, hence $N \cap K \ll_L K$, and we get K is L-supplement of N in M.
Proposition 3.10: Let M be an R-module and M_1, H are submodules of M, such that M_1 is L-supplemented module, if $M_1 + H$ has L-supplement in M then H has L-supplement in M.
Proof: By assumption $M_1 + H$ has L-supplement in M, so there exists $U \leq M$ such that $M_1 + H + U = M$ and $(M_1 + H) \cap U \ll_L U$, since M_1 is L-supplemented then $(H + U) \cap M_1 \leq M_1$ has L-supplement in M_1, so there exists $V \leq M_1$ such that $((H + U) \cap M_1) + V = M_1$ and $(H + U) \cap V \ll_L V$. Now $M = M_1 + H + U = \big((H + U) \cap M_1\big) + V + H + U = H + (V + U)$. One can easily show $H \cap (V + U) \leq \big((H + V) \cap U\big) + \big((H + U) \cap V\big) \leq \big((H + M_1) \cap U\big) + \big((H + U) \cap V\big) \ll_L U + V$ by Lemma(1.4), so $H \cap (V + U) \ll_L U + V$ and $V + U$ is L-supplement of H in M, hence H has L-supplement in M.
Proposition 3.11: Let $M = M_1 \oplus M_2$ such that M_1 and M_2 are L-supplemented modules then M is L-supplemented module.
Proof: Let $H \leq M$ and since $M_1 + M_2 + H = M$, so it is trivial has L-supplement in M. By proposition (3.10) and since M_1 is L-supplemented, then $M_2 + H$ has L-supplement in M, again by proposition (3.10) and since M_2 is L-supplemented, then H has L-supplement in M, and hence M is L-supplemented module.

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